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Math Strategies

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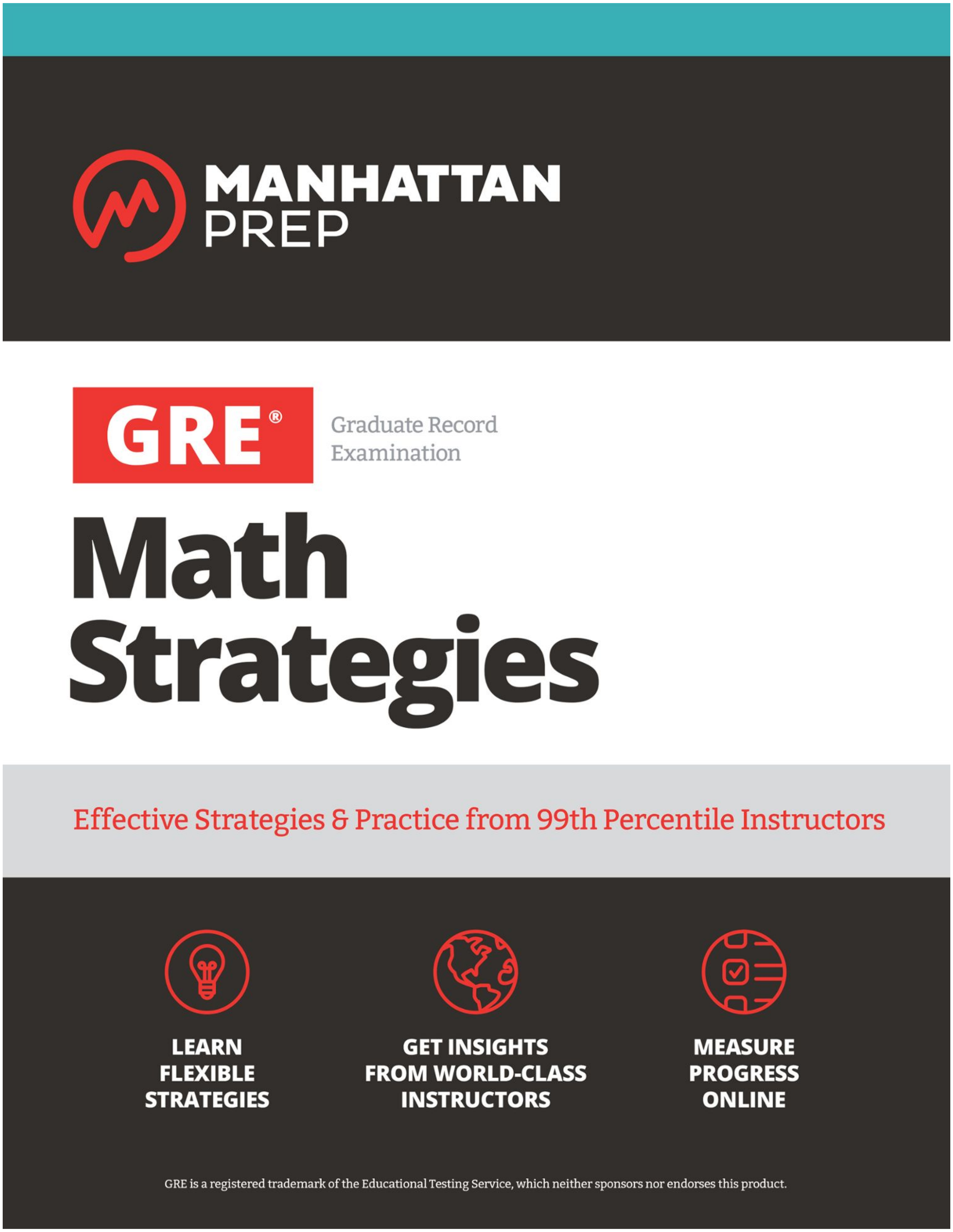

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Table of Contents

- Manhattan Prep GRE Math Strategies
  - Cover
  - Title Page
  - Copyright
  - GRE Strategy Guides
  - Letter
  - How to Access Your Online Resources
- Introduction
  - In This Chapter...
  - Introduction
  - The GRE Exam
  - Math Formats in Detail
- Unit One: Algebra
  - In This Unit
  - Chapter 1: Equations
    - In This Chapter...
    - Chapter 1 Equations
    - The Order of Operations (PEMDAS)
    - Solving for a Variable with One Equation
    - Solving for Variables with Two Equations
    - Subtraction of Expressions
    - Fraction Bars as Grouping Symbols
    - Check Your Skills Answer Key
    - Problem Set
    - Solutions
  - Chapter 2: Quadratic Equations
    - In This Chapter...
    - Identifying Quadratic Equations
    - Distributing
    - Factoring
    - Applying This to Quadratics
    - Factoring Quadratic Equations
    - Solving Quadratic Equations
    - Using FOIL with Square Roots
    - One-Solution Quadratics
    - Zero in the Denominator: Undefined
    - The Three Special Products

    - Check Your Skills Answer Key
    - Problem Set
    - Solutions
  - Chapter 3: Inequalities and Absolute Value
    - In This Chapter...
    - Inequalities
    - Solving Inequalities
    - Absolute Value—Distance on the Number Line
    - Putting Them Together: Inequalities and Absolute Values
    - Manipulating Compound Inequalities
    - Using Extreme Values
    - Optimization Problems
    - Check Your Skills Answer Key
    - Problem Set
    - Solutions
  - Chapter 4: Formulas and Functions
    - In This Chapter...
    - Plug In Formulas
    - Strange Symbol Formulas
    - Formulas with Unspecified Amounts
    - Sequence Formulas: Direct and Recursive
    - Sequence Problems
    - Sequences and Patterns
    - Functions
    - Check Your Skills Answer Key
    - Problem Set
    - Solutions
- Unit Two: Fractions, Decimals, and Percents
  - In This Unit
  - Chapter 5: Fractions
    - In This Chapter...
    - Chapter 5 Fractions
    - Manipulating Fractions
    - Switching Between Improper Fractions and Mixed Numbers
    - Division in Disguise
    - Fraction Operations: Know What to Expect
    - Comparing Fractions: Cross–Multiply

- NEVER Split the Denominator!
- Benchmark Values
- Picking Smart Numbers: Multiples of the Denominators
- When NOT to Use Smart Numbers
- Check Your Skills Answer Key
- Problem Set
- Solutions

Chapter 6: Digits and Decimals
- In This Chapter...
- Digits
- Decimals
- Place Value
- Adding Zeroes to Decimals
- Powers of 10: Shifting the Decimal
- The Heavy Division Shortcut
- Decimal Operations
- Terminating vs. Non-Terminating Decimals
- Units Digit Problems
- Check Your Skills Answer Key
- Problem Set
- Solutions

Chapter 7: Percents
- In This Chapter...
- Chapter 7 Percents
- Percents as Fractions: The Percent Table
- Benchmark Values: 10% and 5%
- Percent Increase and Decrease
- Percent Change vs. Percent of Original
- Successive Percents
- Smart Numbers: Pick 100
- Interest Formulas: Simple and Compound
- Check Your Skills Answer Key
- Problem Set
- Solutions

Chapter 8: FDP Connections
- In This Chapter...
- Chapter 8 FDP Connections

        - Converting Among Fractions, Decimals, and Percents (FDPs)
        - Common FDP Equivalents
        - When to Use Which Form
        - FDPs and Algebraic Translations
        - FDP Word Problems
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
- Unit Three: Geometry
    - In This Unit
    - Chapter 9: Geometry Problem Solving
        - In This Chapter...
        - Using Equations to Solve Geometry Problems
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
    - Chapter 10: Triangles and Diagonals
        - In This Chapter...
        - The Basic Properties of a Triangle
        - Perimeter and Area
        - Right Triangles
        - Pythagorean Triples
        - Isosceles Triangles and the 45–45–90 Triangle
        - Equilateral Triangles and the 30–60–90 Triangle
        - Diagonals of Other Polygons
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
    - Chapter 11: Polygons
        - In This Chapter...
        - Chapter 11 Polygons
        - Quadrilaterals: An Overview
        - Polygons and Interior Angles
        - Polygons and Perimeter
        - Polygons and Area
        - 3 Dimensions: Surface Area

  - 3 Dimensions: Volume
  - Quadrilaterals
  - Maximum Area of Polygons
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 12: Circles and Cylinders
  - In This Chapter...
  - The Basic Elements of a Circle
  - Sectors
  - Inscribed vs. Central Angles
  - Inscribed Triangles
  - Cylinders and Surface Area
  - Cylinders and Volume
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 13: Lines and Angles
  - In This Chapter...
  - Chapter 13 Lines and Angles
  - Intersecting Lines
  - Exterior Angles of a Triangle
  - Parallel Lines Cut By a Transversal
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 14: The Coordinate Plane
  - In This Chapter...
  - Chapter 14 The Coordinate Plane
  - Knowing Just One Coordinate
  - Knowing Ranges
  - Reading a Graph
  - Plotting a Relationship
  - Lines in the Plane
  - The Intercepts of a Line
  - The Intersection of Two Lines
  - The Distance Between Two Points

        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
- Unit Four: Number Properties
    - In This Unit
    - Chapter 15: Divisibility and Primes
        - In This Chapter...
        - Chapter 15 Divisibility and Primes
        - Divisibility Rules
        - Factors
        - Prime Numbers
        - Prime Factorization
        - The Factor Foundation Rule
        - The Factor/Prime Factorization Connection
        - Unknown Numbers and Divisibility
        - Fewer Factors, More Multiples
        - Divisibility and Addition/Subtraction
        - Remainders
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
    - Chapter 16: Odds and Evens
        - In This Chapter...
        - Chapter 16 Odds and Evens
        - Arithmetic Rules of Odds and Evens
        - The Sum of Two Primes
        - Testing Odd and Even Cases
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
    - Chapter 17: Positives and Negatives
        - In This Chapter...
        - Chapter 17 Positives and Negatives
        - Absolute Value: Absolutely Positive
        - A Double Negative = A Positive
        - Multiplying and Dividing Signed Numbers
        - Testing Positive and Negative Cases

  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 18: Exponents
  - In This Chapter...
  - Chapter 18 Exponents
  - Wow, That Increased Exponentially!
  - All About the Base
  - All About the Exponent
  - Combining Exponential Terms
  - Rewriting Bases
  - Simplifying Exponential Expressions
  - Rules of Exponents
  - Common Exponent Errors
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 19: Roots
  - In This Chapter...
  - Chapter 19 Roots
  - Multiplication and Division of Roots
  - Simplifying Roots
  - Solving Algebraic Equations Involving Exponential Terms
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 20: Consecutive Integers
  - In This Chapter...
  - Chapter 20 Consecutive Integers
  - Evenly Spaced Sequences
  - Properties of Evenly Spaced Sequences
  - Counting Integers: Add 1 Before You Are Done
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 21: Number Lines
  - In This Chapter...

        - Chapter 21 Number Lines
        - Relative Position and Relative Distance
        - Line Segments
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
- Unit Five: Word Problems
    - In This Unit
    - Chapter 22: Algebraic Translations
        - In This Chapter...
        - Chapter 22 Algebraic Translations
        - Decoding the GRE Word Problem
        - Common Word Problem Phrases
        - Translating Words Correctly
        - Hidden Constraints
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
    - Chapter 23: Rates and Work
        - In This Chapter...
        - Chapter 23 Rates and Work
        - Basic Motion: The RTD Chart
        - Matching Units in the RTD Chart
        - Multiple Rates
        - Multiple Rates Check Your Skills
        - Average Rate: Don’t Just Add and Divide
        - Basic Work Problems
        - Working Together: Add the Rates
        - Population Problems
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
    - Chapter 24: Ratios
        - In This Chapter...
        - Chapter 24 Ratios
        - Label Each Part of the Ratio with Units
        - Proportions

  - The Unknown Multiplier
  - Multiple Ratios: Make a Common Term
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 25: Statistics
  - In This Chapter...
  - Chapter 25 Statistics
  - Using the Average Formula
  - Evenly Spaced Sets: Take the Middle
  - Weighted Averages
  - Median: The Middle Number
  - Standard Deviation
  - Range
  - Quartiles and Percentiles
  - The Normal Distribution
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 26: Combinatorics
  - In This Chapter...
  - Chapter 26 Combinatorics
  - The Fundamental Counting Principle
  - Simple Factorials
  - Repeated Labels
  - Multiple Arrangements
  - Check Your Skills Answer Key
  - Problem Set
  - Solutions
- Chapter 27: Probability
  - In This Chapter...
  - Chapter 27 Probability
  - “1” Is the Greatest Probability
  - More Than One Event: “AND” vs. “OR”
  - The “1 – x” Probability Trick
  - More Forks in the Road
  - Check Your Skills Answer Key

        - Problem Set
        - Solutions
    - Chapter 28: Minor Problem Types
        - In This Chapter...
        - Chapter 28 Minor Problem Types
        - Optimization
        - Grouping
        - Overlapping Sets
        - Check Your Skills Answer Key
        - Problem Set
        - Solutions
- Unit Six: Quantitative Comparisons
    - In This Unit
    - Chapter 29: QC Basics & Game Plan
        - In This Chapter...
        - Chapter 29 QC Basics & Game Plan
        - Chapter 29 QC Basics & Game Plan: Answers and Explanations
        - Game Plan, Round 1
        - Game Plan, Round 2
        - Game Plan, Round 3
        - Game Plan, Round 4
    - Chapter 30: Algebra
        - In This Chapter...
        - Chapter 30 Algebra
        - Equations
        - Quadratics
        - Formulas
        - Inequalities and Absolute Values
        - Problem Set
        - Solutions
    - Chapter 31: Fractions, Decimals, and Percents
        - In This Chapter...
        - Chapter 31 Fractions, Decimals, and Percents
        - Quick Elimination: Less Than 1 vs. Greater Than 1
        - Simplifying Complex Fractions
        - Fractions with Exponents—Plug In 0 and 1

    - Percents
    - Problem Set
    - Solutions
  - Chapter 32: Geometry
    - In This Chapter...
    - Chapter 32 Geometry
    - Shape Geometry
    - Variable Creation
    - Word Geometry
    - Using Numbers
    - Problem Set
    - Solutions
  - Chapter 33: Number Properties
    - In This Chapter...
    - Chapter 33 Number Properties
    - Positives and Negatives
    - Exponents
    - Consecutive Integers
    - Problem Set
    - Solutions
  - Chapter 34: Word Problems
    - In This Chapter...
    - Chapter 34 Word Problems
    - Ratios
    - Statistics
    - Problem Set
    - Solutions
- Unit Seven: Data Interpretation
  - In This Unit
  - Chapter 35: Data Interpretation
    - In This Chapter...
    - Chapter 35 Data Interpretation
    - The Basic Process of Solving a Data Interpretation Question
    - Data Interpretation Graphs and Charts—aka—The Graph Zoo
    - A Straightforward Data Interpretation Problem Set
    - An Example Mixing Percents and Absolute Quantities

  - A Very Challenging Example, Requiring Excellent Technique
  - Solutions
- Acknowledgments
- Study Anywhere!

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Math Strategies
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September 4, 2018

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If you have any questions or comments in general, please email our Student Services team at gre@manhattanprep.com.

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I look forward to hearing from you. Thanks again, and best of luck preparing for the GRE!

Sincerely,

Chris Ryan
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INTRODUCTION

In This Chapter…

The GRE Exam

Math Formats in Detail

Introduction

We know that you're looking to succeed on the GRE so that you can go to graduate school and do the things you want to do in life.

We also know that you may not have done a lot of math since high school. It's going to take hard work on your part to get a top GRE score. That's why we've put together the only set of books that will take you from the basics all the way up to the material you need to master for a near-perfect score, or whatever your goal score may be. You've taken the first step. Now it's time to get to work!

HOW TO USE THIS BOOK

Manhattan Prep's GRE materials are comprehensive. Nevertheless. keep in mind, that, depending on your score goal, it may not be necessary to get absolutely everything. Grad schools only see your overall Quantitative, Verbal, and Writing scores—they don't see exactly which strengths and weaknesses went into creating those scores.

You may be enrolled in one of our courses, in which case you already have a syllabus telling you in what order you should approach this book. But if you bought this book online or at a bookstore, feel free to approach the units—and even the chapters within the units—in whatever order works best for you. For the most part, the units, and the chapters within them, are independent. You don't have to master one section before moving on to the next. So if you're having a hard time with one thing, you can make a note to come back to it later and move on to another section. Similarly, it may not be necessary to solve every practice problem for every section. As you go through the material, continually assess whether you understand and can apply the principles in each individual section and chapter. The best way to do this is to solve the Check Your Skills and Problem Sets throughout. If you're confident you have a concept or method down, feel free to move on. If you struggle with something, make note of it for further review. Stay active in your learning and stay oriented toward the test—it's easy to read something and think you understand it, only to have trouble applying it in the 1–2 minutes you have to solve a problem.

STUDY SKILLS

As you're studying for the GRE, try to integrate your learning into your everyday life. You're going to want to do at least a little bit of math every day. Try to learn and internalize a little bit at a time, switching up topics often to help keep things interesting.

Keep in mind that, while many of your study materials are on paper (including Education Testing Service's [ETS's] most recent source of official GRE questions, *The Official Guide to the GRE revised General Test, Third Edition*), your exam will be administered on a computer. Because this test is computer-based, you will *not* be able to write on diagrams of geometry figures or otherwise physically mark up problems. Get used to this now.

Solve the problems in these books on scratch paper. (Each of our books talks specifically about what to write down for different problem types.)

Again, as you study, stay focused on the test-day experience. As you progress, work on timed drills and sets of questions. Eventually, you should be taking full practice tests (available at www.manhattanprep.com/gre) under actual timed conditions.

The GRE Exam

EXAM STRUCTURE

The revised test has six sections. You will get a 10-minute break between the third and fourth sections and a 1-minute break between the others. The Analytical Writing section, also known as the Essay, is always first. The other five sections can be seen in any order and will include:

- Two Verbal Reasoning sections (20 questions each in 30 minutes per section). We'll call these sections Verbal for short.
- Two Quantitative Reasoning sections (20 questions each in 35 minutes per section). We'll call these sections Math for short.
- Either an unscored section or a research section.

An unscored section will look just like a third Verbal or Math section. You will not be told which of the three sections doesn't count. If you get a research section, it will be identified as such, and it will be the last section you get.

Section #	Section Type	# Questions	Time	Scored?
1	Essay	2 essays	30 minutes each	Yes
2	Verbal #1 Math #1 *(order can vary)*	Approx. 20	30 minutes	Yes
3		Approx. 20	35 minutes	Yes

Section #	**Section Type**	**# Questions**	**Time**	**Scored?**
		10-Minute Break		
4	Verbal #2 Math #2 *(order can vary)*	Approx. 20	30 minutes	Yes
5		Approx. 20	35 minutes	Yes
?	Unscored Section *(Verbal or Math, order can vary)*	Approx. 20	30 or 35 minutes	No
Last	Research Section	Varies	Varies	No

USING THE CALCULATOR

The small, four-function calculator with a square root button found on the GRE means that re-memorizing times tables or square roots is less important than it used to be. However, the calculator is not a cure-all; in many problems, the difficulty is in figuring out what numbers to put into the calculator in the first place. In some cases, using a calculator will actually be less helpful than doing the problem some other way. Take a look at an example:

If x is the remainder when (11)(7) is divided by 4 and y is the remainder when (14)(6) is divided by 13, what is the value of $x + y$?

This problem is designed so that the calculator won't tell the whole story. Certainly, the calculator will tell you that 11 × 7 = 77. When you divide 77 by 4, however, the calculator yields an answer of 19.25. The remainder is not 0.25 (a remainder is always a whole number).

You might just go back to your pencil and paper, and find the largest multiple of 4 that is less than 77. Because 4 does go into 76, you can conclude that 4 would leave a remainder of 1 when dividing into 77. (Notice that you don't even need to know how many times 4 goes into 76, just that it goes in. One way to mentally "jump" to 76 is to say that 4 goes into 40, so it goes into 80…that's a bit too big, so subtract 4 to get 76.)

However, it is also possible to use the calculator to find a remainder. Divide 77 by 4 to get 19.25. Thus, 4 goes into 77 nineteen times, with a remainder left over. Now use your calculator to multiply 19 (JUST 19, not 19.25) by 4. You will get 76. The remainder is 77 − 76, which is 1. Therefore, $x = 1$. You could also multiply the leftover 0.25 times 4 (the divisor) to find the remainder of 1.

Use the same technique to find y. Multiply 14 by 6 to get 84. Divide 84 by 13 to get 6.46. Ignore everything after the decimal, and just multiply 6 by 13 to get 78. The remainder is therefore 84 − 78, which is 6. Therefore, $y = 6$.

You are looking for $x + y$, and $1 + 6 = 7$, so the answer is 7.

You can see that blind faith in the calculator can be dangerous. Use it responsibly! And this leads us to …

PRACTICE USING THE CALCULATOR!

The on-screen calculator will slow you down or lead to incorrect answers if you’re not careful! If you plan to use it on test day (which you should), you’ll want to practice first.

We have created an online practice calculator for you to use. To access this calculator, go to www.manhattanprep.com/gre and sign in to the student center using the instructions on the “How to Access Your Online Resources” page found at the front of this book.

Throughout this book, you will see the 🖩 symbol. This symbol means “Use the calculator here!” As much as possible, have the online practice calculator up and running during your review of our math books. You’ll have the chance to use the on-screen calculator when you take our practice exams as well.

NAVIGATING THE QUESTIONS IN A SECTION

The GRE offers you the ability to move freely around the questions in a section. You can go forward and backward one-by-one and can even jump directly to any question from the “review list.” The review list provides a snapshot of which questions you have answered, which ones you have tagged for “mark and review,” and which ones are incomplete.

You should double-check the review list for completion if you finish the section early. Using the review list feature will take some practice as well, which is why we’ve built it into our online practice exams.

The majority of test-takers will be pressed for time. Some people won't be able to go back to multiple problems at the end of the section. Generally, if you can't get a question the first time, you won't be able to get it the second time around either. With these points in mind, here's what we recommend.

Do the questions in the order in which they appear.

When you encounter a difficult question, do your best to eliminate answer choices that you know are wrong.

If you're not sure of an answer, take an educated guess from the choices remaining. Do NOT skip it and hope to return to it later.

Using the "mark" button at the top of the screen, mark up to three questions per section that you think you might be able to solve with more time. Mark a question only after you have taken an educated guess.

Always click on the review list at the end of a section. This way, you can quickly make sure you have neither skipped nor incompletely answered any questions.

If you have time, identify any questions that you marked for review and return to them. If you do not have any time remaining, you will have already taken good guesses at the tough ones.

What you want to avoid is surfing—clicking forward and backward through the questions searching for "easy" ones. This will eat up valuable time. Of course, you'll want to move through the tough ones quickly if you can't get them, but try to avoid skipping around.

Following this guidance will take practice. Use our practice exams to fine-tune your approach.

Math Formats in Detail

The 20 questions in each Math section can be broken down by format as follows:

- 7 Quantitative Comparison questions—These ask you to compare two quantities and pick one of four choices (A, B, C, or D).
- 10 Discrete Quant questions—Most of these are standard multiple-choice questions, asking you to pick one of five choices (A, B, C, D, or E). A few will ask you to pick one *or more* choices from a list. Others will ask you to "fill in the blank," essentially.
- 3 Data Interpretation questions—All three questions will be related to a single set of data given in graph or table form. The formats of the questions themselves are the same as for Discrete Quant: most are standard multiple-choice, and the rest ask you to pick one or more choices or to fill in the blank.

QUANTITATIVE COMPARISON

The format of every Quantitative Comparison or "QC" question is the same. All QC questions contain a “Quantity A” and a “Quantity B.” Some also contain common information that applies to both quantities.

Your job is to, well, compare the two quantities (surprise!) and decide which one of the following four statements is true:

Quantity A is *always* greater than Quantity B, in every possible case.
Quantity B is *always* greater than Quantity A.
Quantity A is *always* equal to Quantity B.
None of the above is *always* true.

For instance, it might be that most of the time, Quantity A is greater, but in just one case, Quantity B is greater, or the two quantities are equal. Then, as the GRE puts it, "the relationship cannot be determined."

On the actual GRE, these four choices are worded exactly as shown in the following example:

$x \geq 0$

Quantity A	**Quantity B**
x	x^2

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

If $x = 0$, then the two quantities are equal. If $x = 2$, then Quantity (B) is greater. Thus, the relationship between the quantities can't be determined once and for all.

The answer is **(D)**.

Unit 6 in this book is all about Quantitative Comparisons.

SELECT ONE OR MORE ANSWER CHOICES

These are the Discrete Quant and Data Interpretation questions that ask you to pick one *or more* choices. According to the *Official Guide to the GRE revised General Test*, the official directions for “Select One or More Answer Choices” read as follows:

Directions:

Select one or more answer choices according to the specific question directions.

If the question does not specify how many answer choices to indicate, indicate all that apply.

The correct answer may be just one of the choices or as many as all of the choices, depending on the question.

No credit is given unless you indicate all of the correct choices and no others.

If the question specifies how many answer choices to indicate, indicate exactly that number of choices.

There is no partial credit. If three of six choices are correct and you indicate two of the three, no credit is given. If you are told to indicate two choices and you indicate three, no credit is given. Read the directions carefully.

On your screen, the answer choices for "Select One or More" will be *boxes*, not circles (as with standard "pick just one" multiple-choice questions). The boxes are a good visual reminder that you should be ready to pick more than one choice on these questions, just as you might check more than one box on a checklist.

Here’s a sample question:

> If $ab = |a| \times |b|$ and $ab \neq 0$, which of the following must be true?
>
> Indicate all such statements.
>
> $\boxed{A}$ $a = b$
> $\boxed{B}$ $a > 0$ and $b > 0$
> $\boxed{C}$ $ab > 0$
>
> Note that only one, only two, or all three of the choices may be correct. (Also note the word “must” in the question stem!)

If $ab = |a| \times |b|$, then you know *ab* is positive, since the right side of the equation must be positive. If *ab* is positive, however, that doesn’t necessarily mean that *a* and *b* are each positive; it simply means that they have the same sign.

Answer choice (A) is not correct because it is not true that *a* must equal *b*; for instance, *a* could be 2 and *b* could be 3.

Answer choice (B) is not correct because it is not true that *a* and *b* must each be positive; for instance, *a* could be −3 and *b* could be −4.

Now look at choice (C). Because $|a| \times |b|$ must be positive, *ab* must be positive as well; that is, because two sides of an equation are, by definition, equal to one another, if one side of the equation is positive, the other side must be positive as well. Thus, answer **(C)** is correct.

Strategy Tip:

Make sure to process the statement in the question (simplify it or list the possible scenarios) fully before considering the answer choices. That is, don't just look at $ab = |a| \times |b|$—rather, it's your job to draw inferences about the statement before plowing ahead. This will save you time in the long run!

Note that "indicate all that apply" didn't make the problem fundamentally different. You just had to do a little more work by checking every answer choice.

NUMERIC ENTRY

These are the Discrete Quant and Data Interpretation questions that ask you to "fill in the blank." That is, you type a number into a box on the screen. You have to come up with that number on your own. You are not able to work backward from answer choices, and in many cases, it will be difficult to make a guess. However, the math principles being tested are the same as on the rest of the exam.

You'll be given one box if you are supposed to enter an integer (such as 12 or 0 or −3) or a decimal (such as 2.7 or −1.53). Click on the answer box and type your answer.

In contrast, you'll be given two boxes if you are supposed to enter a fraction. One box (the numerator) will be on top of a fraction line, while the other box (the denominator) will be underneath. Click on each box separately to enter your answer. You do not have to reduce to lowest terms.

Follow directions carefully. You may be asked to round your answer in a particular way, or the units of measurement may be different from what you assume.

Here is a sample question:

If $x\Delta y = 2xy - (x - \mathrm{y})$, what is the value of $3\Delta 4$?

You are given a function involving two variables, *x* and *y*, and asked to substitute 3 for *x* and 4 for *y*:

$$\begin{aligned}x\Delta y &= 2xy-(x-y)\\ 3\Delta 4 &= 2(3)(4)-(3-4)\\ 3\Delta 4 &= 24-(-1)\\ 3\Delta 4 &= 25\end{aligned}$$

The answer is **25**.

So you would type 25 into the box.

Now that you've seen the structure and the math question formats of the GRE, it’s time to begin fine-tuning your math skills.

Unit One: Algebra

This unit covers algebra in all its various forms (and disguises) on the GRE. Master fundamental techniques and nuanced strategies to help you solve for unknown variables of every type.

In This Unit...

Chapter 1: Equations
Chapter 2: Quadratic Equations
Chapter 3: Inequalities & Absolute Value
Chapter 4: Formulas & Functions

Chapter 1

EQUATIONS

In This Chapter...

The Order of Operations (PEMDAS)
Solving for a Variable with One Equation
Solving for Variables with Two Equations
Subtraction of Expressions
Fraction Bars as Grouping Symbols

Chapter 1

Equations

The GRE will expect you to be proficient at manipulating and solving algebraic equations. If you haven’t faced equations since you were last in school, this can be intimidating. In this chapter, the objective is to help you become comfortable setting up and solving equations. You’ll start with some basic equations (without the variables at first), and then work your way up to some pretty tricky problems. Time to dive in.

The Order of Operations (PEMDAS)

$$3 + 4(5 - 1) - 3^2 \times 2 = ?$$

Before you start dealing with variables, spend a moment looking at expressions that are comprised of only numbers, such as the preceding example. The GRE probably won't ask you to compute something like this directly, but learning to use order of operations on numerical expressions will help you manipulate algebraic expressions and equations. So you have a string of numbers with mathematical symbols in between them. Which part of the expression should you focus on first?

Intuitively, most people think of going in the direction they read, from left to right. When you read a book, moving left to right is a wise move (unless you're reading a language such as Chinese or Hebrew). However, when you perform basic arithmetic, there is an order that is of greater importance: **the order of operations**.

The order in which you perform the mathematical functions should primarily be determined by the functions themselves. In the correct order, the six operations are **P**arentheses, **E**xponents, **M**ultiplication/**D**ivision, and **A**ddition/**S**ubtraction (or **PEMDAS**).

Before you solve a problem that requires PEMDAS, here's a quick review of the basic operations:

Parentheses can be written as ( ) or [ ] or even { }.

Exponents are $5^{2\leftarrow \text{these numbers}}$. For example, 5^2 (“five squared”) can be expressed as 5 × 5. In other words, it is 5 times itself, or two 5’s multiplied together.

Likewise, 4^3 (“four cubed,” or “four to the third power”) can be expressed as 4 × 4 × 4 (or three 4’s multiplied together). The exponent 3 tells you how many 4’s are in the product.

Roots are very closely related to exponents. For example, $\sqrt[3]{64}$ is the third root of 64 (commonly called the cube root). The cube root, in this case $\sqrt[3]{64}$, is basically asking the question, “What multiplied by itself three times equals 64?” This is written as 4 × 4 × 4 = 64, so $\sqrt[3]{64} = 4$. The plain old square root $\sqrt{9}$ can be thought of as $\sqrt{9}$. “What times itself equals 9?” The answer is 3 × 3 = 9, so $\sqrt{9} = 3$.

Exponents and roots can also undo each other: $\sqrt{5^2} = 5$ and $\left(\sqrt[3]{7^3}\right) = 7$.

Multiplication and **Division** can also undo each other: 2 × 3 ÷ 3 = 2 and 10 ÷ 5 × 5 = 10.

Multiplication can be expressed with a multiplication sign (×) or with parentheses: (5)(4) = 5 × 4 = 20. Division can be expressed with a division sign (÷), a slash (/), or a fraction bar(—): $20 \div 5 = 20/5 = \frac{20}{5} = 4$. Also remember that multiplying or dividing by a negative number changes the sign:

$4 \times (-2) = -8$	$-8 \div (-2) = 4$

Addition and **Subtraction** can also undo each other: $8 + 7 - 7 = 8$ and $15 - 6 + 6 = 15$.

PEMDAS is a useful acronym you can use to remember the order in which operations should be performed. Some people find it useful to write PEMDAS like this:

$$\underrightarrow{\mathrm{PE}^{\mathrm{M}}/_{\mathrm{D}}{}^{\mathrm{A}}/_{\mathrm{S}}}$$

For Multiplication/Division and Addition/Subtraction, perform whichever comes first from left to right. The reason that Multiplication and Division are at the same level of importance is that any Multiplication can be expressed as Division, and vice versa; for example, $7 \div 2$ is equivalent to $7 \times \frac{1}{2}$. In a sense, Multiplication and Division are two sides of the same coin.

Addition and Subtraction have this same relationship: $3 - 4$ is equivalent to $3 + (-4)$.

The correct order of steps to simplify this sample expression is as follows:

	$3 + 4(5 - 1) - 3^2 \times 2$
Parentheses	$3 + 4(4) - 3^2 \times 2$
Exponents	$3 + 4(4) - 9 \times 2$
Multiplication or **D**ivision (left to right)	$3 + 16 - 18$
Addition or **S**ubtraction (left to right)	$3 + 16 - 18 = 19 - 18 = 1$

Remember: If you have two operations of equal importance, you should do them in left-to-right order: 3 − 2 + 3 = 1 + 3 = 4. The only instance in which you would override this order is when the operations are in parentheses: 3 − (2 + 3) = 3 − (5) = −2.

Next are two problems together. Try them first on your own, then an explanation will follow:

$$5 - 3 \times 4^3 \div (7 - 1)$$

P

E

M/D

A/S

Your work should have looked like this:

$$5-3\times4^3\div(7-1)$$

$$5-3\times\underbrace{4^3}\div6$$

$$4^3=4\times4\times4=64\rightarrow\begin{array}{r}\overset{2}{1}6\\ \times\,4\\ \hline 64\end{array}$$

$$5-\underbrace{3\times64}\div6$$

$$\begin{array}{r}\overset{1}{6}4\\ \times\,3\\ \hline 192\end{array}$$

$$5-\underbrace{192\div6}$$

$$\begin{array}{r}32\\ 6\overline{)192}\\ -18\\ \hline 12\\ -12\\ \hline 0\end{array}$$

$$\underbrace{5-32}$$

$$-27$$

Here’s one more:

$$32\div2^4\times(5-3^2)$$

P

E

M/D

A/S

Here’s the work you should have done:

$$\begin{gathered}
32 \div 2^4 \times (5 - 3^2) \\
32 \div 2^4 \times (5 - 9) \\
32 \div 2^4 \times (-4) \\
32 \div 16 \times (-4) \\
2 \times (-4) \\
-8
\end{gathered}$$

Check Your Skills

Evaluate the following expressions.

1. $-4 + 12/3 =$

2. $(5 - 8) \times 10 - 7 =$

3. $-3 \times 12 \div 4 \times 8 + (4 - 6) =$

4. $2^4 \times (8 \div 2 - 1)/(9 - 3) =$

Solving for a Variable with One Equation

EXPRESSIONS VERSUS EQUATIONS

So far, you've been dealing only with expressions. Now you're going to be dealing with equations. An important structural difference between equations and expressions is that an equation consists of two expressions separated by an equals sign, while an expression lacks an equals sign altogether. An equation is a sentence: "Something equals something else." The somethings are each expressions.

Pretty much everything you will be doing with equations is related to one basic principle: You can do anything you want to one side of the equation, *as long as you also do the same thing to the other side of the equation*. Take the equation 3 + 5 = 8. You want to subtract 5 from the left side of the equation, but you still want the equation to be true. All you have to do is subtract 5 from the right side as well, and you can be confident that your new equation will still be valid:

$$\begin{array}{rcr} 3+5 & = & 8 \\ -5 & & -5 \\ \hline 3 \phantom{{}+5} & = & 3 \end{array}$$

Note that this would also work if you had variables in your equation:

$$\begin{array}{rcr} x + 5 & = & 8 \\ -5 & & -5 \\ \hline x & = & 3 \end{array}$$

Next, you’re going to see some of the many ways you can apply this principle to solving algebra problems.

SOLVING EQUATIONS

What does it mean to solve an equation? What are you really doing when you manipulate algebraic equations?

A solution to an equation is a number that, when substituted in for the variable, makes the equation *true*. Remember, an equation is a sentence: “Something equals something else.” In general, a sentence like this can be true or false. You want a way to make it true.

Take the equation $2x + 7 = 15$. You are looking for the value of x that will make this equation true. What if you plugged in 3 for x? If you replaced x with the number 3, you would get $2(3) + 7 = 15$. This equation can be simplified to $6 + 7 = 15$, which further simplifies to $13 = 15$. However, 13 does *not* equal 15, so when $x = 3$, the equation is *not* true. Therefore, $x = 3$ is *not* a solution to the equation.

Now, if you replaced x with the number 4, you would get $2(4) + 7 = 15$. This equation can be simplified to $8 + 7 = 15$. Simplify it further, and you get

$15 = 15$, which is a true statement.

That means that when $x = 4$, the equation is true. So $x = 4$ is a solution to the equation.

Now the question becomes, what is the best way to find these solutions? If you had to use trial and error, or guessing, the process could take a very long time. The following sections will talk about how you can efficiently and accurately manipulate equations so that solutions become easier to find.

Isolating a Variable

You know that you can make a change to an equation as long as you make the same change to both sides. This is called the **Golden Rule.** Now, look at the various changes you can make. Try to solve the following problem:

$$\text{If } 5(x - 1)^3 - 30 = 10, \text{ then } x = ?$$

To solve for a variable, you need to get it by itself on one side of the equals sign. To do that, you need to change the appearance of the equation, without changing the fact that it's true. The good news is that all of the changes you need to make to this equation to solve for x will actually be very familiar to you—PEMDAS operations!

To get x by itself, you want to move every term that *doesn't include* the variable to the other side of the equation. The easiest thing to move at this stage is the 30, so start there. If 30 is being subtracted on the left side of the equation, and you want to move it to the other side, then you need to do the opposite operation to cancel it out. So you’re going to **add** 30 to both sides, like this:

$$\begin{array}{rcr} 5(x-1)^3 - \cancel{30} & = & 10 \\ +\cancel{30} & & +30 \\ \hline 5(x-1)^3 & = & 40 \end{array}$$

Now you’ve only got one term on the left side of the equation. The *x* is still inside the parentheses, and the expression in the parentheses is being multiplied by 5, so the next step will be to move that 5 over to the other side of the equation. Once again, you want to perform the opposite operation, so **divide** both sides of the equation by 5:

$$\frac{\cancel{5}\,(x-1)^3}{\cancel{5}} = \frac{40}{5} \qquad \leftarrow \qquad \begin{array}{l}\text{These horizonta}\\ \text{lines mean divis}\end{array}$$

$$(x-1)^3 = 8$$

At this point, you could cube ($x - 1$), but that is going to involve a whole lot of multiplication. Instead, you can get rid of the exponent by performing the opposite operation. Roots are the opposite of exponents. So if the left side of the equation is raised to the third power, you can undo that by taking the third root of both sides, also known as the cube root, as shown here:

$$\begin{aligned}\sqrt[3]{(x-1)^3} &= \sqrt[3]{8}\\ (x-1) &= 2\end{aligned}$$

Now that nothing else is being done to the parentheses, you can just get rid of them. The equation is:

$$x - 1 = 2$$

Then, add 1 to both sides, and you get $x = 3$. This would have been hard to guess! If you plug 3 back in for x in the original equation (a step often worth doing, especially as you're learning), you'll find that this value makes the original equation true.

Take a look at the steps you took to isolate x. Do you notice anything? You *added* 30, then you *divided* by 5, then you got rid of the *exponent*, and then you simplified the *parentheses*. You did PEMDAS backward! And, in fact, when you're isolating a variable, it turns out that the simplest way to do so is to reverse the order of PEMDAS when deciding what order you will perform your operations. Start with addition/subtraction, then multiplication/division, then exponents, and finish with terms in parentheses.

Now that you know the best way to isolate a variable, go through one more example. Try it on your own first, then an explanation will follow.

If $4\sqrt{(x-6)} + 7 = 19,$ then $x = ?$

A/S

M/D

E

P

The equation you're simplifying is $4\sqrt{(x-6)}+7=19$. If there's anything to add or subtract, that will be the easiest first step. There is, so first get rid of the 7 by subtracting 7 from both sides:

$$\begin{array}{r}
4\sqrt{(x-6)}+7=19 \\
-7\quad-7 \\
\hline
4\sqrt{(x-6)}\qquad=12
\end{array}$$

Now you want to see if there's anything being multiplied or divided by the term containing an *x*. The square root that contains the *x* is being multiplied by 4, so, your next step will be to get rid of the 4. You can do that by dividing both sides of the equation by 4:

$$\begin{aligned}\frac{\cancel{4}\sqrt{(x-6)}}{\cancel{4}} &= \frac{12}{4}\\ \sqrt{(x-6)} &= 3\end{aligned}$$

Now that you’ve taken care of multiplication and division, it’s time to check for exponents. That means you need to check for exponents and roots, because they’re so intimately related. There are no exponents in the equation, but the *x* is inside a square root. To cancel out a root, you can use an exponent. Squaring a square root will cancel it out, so your next step is to square both sides:

$$\begin{aligned}\sqrt{(x-6)} &= 3\\ \left(\sqrt{(x-6)}\right)^2 &= 3^2\\ x-6 &= 9\end{aligned}$$

The final step is to add 6 to both sides, and you end up with $x = 15$.

Check Your Skills

Solve for *x* in the following equations.

5. $3(x + 4)^3 - 5 = 19$

6. $\dfrac{3x - 7}{2} + 20 = 6$

7. $\sqrt[3]{(x + 5)} - 7 = -8$

EQUATION CLEAN-UP MOVES

You've covered the basic operations that you'll be dealing with when solving equations. But what would you do if you were asked to solve for *x* in the following equation?

$$\frac{5x - 3(4 - x)}{2x} = 10$$

Now *x* appears in multiple parts of the equation, and your job has become more complicated. In addition to your PEMDAS operations, you also need to be able to simplify, or clean up, your equation. There are different ways you can clean up this equation. First, notice how you have an *x* in the denominator (the bottom of the fraction) on the left side of the equation. You're trying to find the value of *x*, not of some number divided by *x*. So your first clean-up move is to **always get variables out of denominators**. The way to do that is to multiply both sides of the equation by the *entire* denominator. Watch what happens:

$$\cancel{2x} \times \frac{5x - 3(4 - x)}{\cancel{2x}} = 10 \times 2x$$

If you multiply a fraction by its denominator, you can cancel out the entire denominator. Now you're left with:

$$5x - 3(4 - x) = 20x$$

No more fractions! What should you do next? At some point, if you want the value of *x*, you're going to have to get all the terms that contain an *x* together. Right now, however, that *x* sitting inside the parentheses seems pretty tough to get to. To make that *x* more accessible, you should **simplify grouped terms within the equation**. That 3 on the outside of the parentheses wants to multiply the terms inside, so you need to **distribute** it. What that means is you're going to multiply the 3 by each term inside, one at a time: 3 times 4 is 12, and 3 times $-x$ is $-3x$. The equation becomes:

$$5x - (12 - 3x) = 20x$$

Now, if you subtract what's in the parentheses from 5*x*, you can get rid of the parentheses altogether. Just as you multiplied the 3 by *both* terms inside the parentheses, you also have to subtract both terms:

$$\begin{aligned} 5x - (12) - (-3x) &= 20x \\ 5x - 12 + 3x &= 20x \end{aligned}$$

Remember, *subtracting a negative number is the same as adding a positive number; the negative signs cancel out.*

You’re very close. You’re ready to make use of your final clean-up move—**combine like terms**. “Like terms” are terms that can be combined into one term. For example, “3*x*” and “5*x*” are like terms because they can be combined into “8*x*.” Ultimately, all the PEMDAS operations and clean-up moves have one goal: to get a variable by itself so you can determine its value. At this point, you have four terms in the equation: 5*x*, −12, 3*x*, and 20*x*. You want to get all the terms with an *x* on one side of the equation, and all the terms that only contain numbers on the other side.

First, combine 5*x* and 3*x*, because they’re on the same side of the equation. That gives you:

$$8x - 12 = 20x$$

Now you want to get the 8*x* together with the 20*x*, but which one should you move? The best move to make here is to move the 8*x* to the right side of the equation, because that way one side of the equation will have terms that contain only numbers (−12) and the right side will have terms that contain variables (8*x* and 20*x*). So now it’s time for the PEMDAS operations again. To find *x*:

$$\begin{array}{rcl} 8x - 12 & = & 20x \\ -\,8x \qquad & & -8x \\ \hline -\,12 & = & 12x \\ \dfrac{-12}{12} & = & \dfrac{12x}{12} \\ -\,1 & = & x \end{array}$$

Before moving on to the next topic, review what you’ve learned:

- **Whatever you do to one side of the equation, you must do to the other side at the same time.**
- **To isolate a variable, you should perform the PEMDAS operations in *reverse order*:**

  Addition/Subtraction

  Multiplication/Division

  Exponents/Roots

  Parentheses

- **To clean up an equation:**

  Get variables out of denominators by multiplying both sides by that entire denominator.

  Simplify grouped terms by multiplying or distributing.

Combine similar or like terms.

Check Your Skills

Solve for *x* in the following equations.

8. $\dfrac{11+3(x+4)}{x-3}=7$

9. $\dfrac{-6-5(3-x)}{2-x}=6$

10. $\dfrac{2x+6(9-2x)}{x-4}=-3$

Solving for Variables with Two Equations

Some GRE problems, including word problems, give you two equations, each of which has two variables. To solve such problems, you'll need to solve for one or each of those variables. At first glance, this problem may seem quite daunting:

If $3x + y = 10$ and $y = x - 2$, what is the value of y?

Maybe you've gotten pretty good at solving for one variable, but now you face two variables and two equations.

You might be tempted to test numbers, and indeed you could actually solve this problem that way. Could you do so in under two minutes? Maybe not. Fortunately, there is a much faster way.

SUBSTITUTION

One method for combining equations is called substitution. In substitution, you *insert the expression for one variable in one equation into that variable in the other equation.* The goal is to end up with one equation with one variable, because once you get a problem to that point, you know you can solve it.

There are four basic steps to substitution, which can be demonstrated with the previous question.

Step One is to isolate one of the variables in one of the equations. For this example, y is already isolated in the second equation: $y = x - 2$.

For **Step Two**, it is important to understand that the left and right sides of the equation are equivalent. This may sound obvious, but it has some interesting implications. If y equals $x - 2$, then that means you could replace the variable y with the expression $(x - 2)$ and the equation would have the same value. In fact, that's exactly what you're going to do. Step Two will be to go to the first equation and replace the variable y with its equivalent, $(x - 2)$. In other words, you're substituting $(x - 2)$ in for y. So:

$$3x + y = 10 \rightarrow 3x + (x - 2) = 10$$

Now for **Step Three**, you have one equation and one variable, so the next step is to solve for x:

$$\begin{aligned} 3x + x - 2 &= 10 \\ 4x &= 12 \\ x &= 3 \end{aligned}$$

Now that you have a value for x, **Step Four** is to substitute that value into either original equation to solve for your second variable, y:

$$y = x - 2 \rightarrow y = 3 - 2 = 1$$

So the answer to the question is *y* = 1. **It should be noted that Step Four will only be necessary if the variable you solve for in Step Three is not the variable the question asks for.** The question asked for *y*, but you found *x*, so Step Four was needed to answer the question.

Now that you've gotten the hang of substitution, try a new problem:

If $2x + 4y = 14$ and $x - y = -8$, what is the value of *x*?

As you learned, the first step is to isolate your variable. Because the question asks for *x*, you should manipulate the second equation to isolate *y*. Taking this approach will make Step Four unnecessary and save you time:

$$\begin{aligned} x - y &= -8 \\ x &= -8 + y \\ x + 8 &= y \end{aligned}$$

Then, for Step Two, you can substitute for *y* in the first equation:

$$\begin{aligned} 2x + 4y &= 14 \\ 2x + 4\left(x + 8\right) &= 14 \end{aligned}$$

Now, for Step Three, isolate *x*:

$$\begin{aligned} 2x + 4x + 32 &= 14 \\ 6x &= -18 \\ x &= -3 \end{aligned}$$

So the answer to the question is $x = -3$.

Check Your Skills

Solve for x and y in the following equations.

11. $x = 10$
    $x + 2y = 26$

12. $x + 4y = 10$
    $y - x = -5$

13. $6y + 15 = 3x$
    $x + y = 14$

Subtraction of Expressions

One of the most common errors involving order of operations occurs when an expression with multiple terms is subtracted. The subtraction must occur across *every* term within the expression. Each term in the subtracted part must have its sign reversed. Here are several examples:

$x - (y - z) = x - y + z$	(Note that the signs of both y and $-z$ have been reversed.)
$x - (y + z) = x - y - z$	(Note that the signs of both y and z have been reversed.)
$x - 2(y - 3z) = x - 2y + 6z$	(Note that the signs of both y and $-3z$ have been reversed.)

What is $5x - [y - (3x - 4y)]$?

Both expressions in parentheses must be subtracted, so the signs of each term must be reversed for *each* subtraction. Note that the square brackets are just fancy parentheses, used so that you avoid having parentheses right next to each other. Work from the innermost parentheses outward:

$$\begin{aligned} 5x - [y - (3x - 4y)] &= \\ 5x - [y - 3x + 4y] &= \\ 5x - y + 3x - 4y &= \mathbf{8x - 5y} \end{aligned}$$

Check Your Skills

14. Simplify: $3a - [2a - (3b - a)]$

Fraction Bars as Grouping Symbols

Even though fraction bars do not fit into the PEMDAS hierarchy, they do take precedence. In any expression with a fraction bar, you should **pretend that there are parentheses around the numerator and denominator of the fraction**. This may be obvious as long as the fraction bar remains in the expression, but it is easy to forget if you eliminate the fraction bar or add or subtract fractions, so put parentheses in to remind yourself:

$$\text{Simplify:} \frac{x-1}{2} - \frac{2x-1}{3} \rightarrow \text{Write on your paper as:} \frac{(x-1)}{2} - \frac{(2x-1)}{3}$$

The common denominator for the two fractions is 6, so multiply the numerator and denominator of the first fraction by 3, and those of the second fraction by 2:

$$\frac{(x-1)}{2}\left(\frac{3}{3}\right) - \frac{(2x-1)}{3}\left(\frac{2}{2}\right) = \frac{(3x-3)}{6} - \frac{(4x-2)}{6}$$

Once you put all numerators over the common denominator, the parentheses remind you to reverse the signs of both terms in the second numerator:

$$\frac{(3x-3)-(4x-2)}{6}=\frac{3x-3-4x+2}{6}=\frac{-x-1}{6}=-\frac{x+1}{6}$$

In that last step, the minus sign was pulled out from both the *x* and the 1 and put in front of the fraction. You can do this because −*x* − 1 is the same thing as −(*x* + 1). Again, the fraction bar is working as a grouping symbol.

Check Your Skills

15. Simplify: $\frac{a+4}{4}-\frac{2a-2}{3}$

Check Your Skills Answer Key

1. **0**

$-4 + 12/3 =$	Divide first.
$-4 + 4 = 0$	Then add the two numbers.

2. **−37**

$(5 - 8) \times 10 - 7 =$	
$(-3) \times 10 - 7 =$	First, combine what is inside the parentheses.
$-30 - 7 =$	Then multiply −3 by 10.
$-30 - 7 = -37$	Subtract the two numbers.

3. **−74**

$-3 \times 12 \div 4 \times 8 + (4 - 6)$	
$-3 \times 12 \div 4 \times 8 + (-2)$	First, combine what's in the parentheses.
$-36 \div 4 \times 8 + (-2)$	Multiply −3 by 12.
$-9 \times 8 - 2$	Divide −36 by 4.
$-72 + (-2) = -74$	Multiply −9 by 8 and then subtract 2.

4. 8

$2^4 \times (8 \div 2 - 1)/(9 - 3) =$	
$2^4 \times (4 - 1)/(6) =$	$8/2 = 4$ and $9 - 3 = 6$
$16 \times (3)/(6) =$	$4 - 1 = 3$ and $2^4 = 16$
$48/6 =$	Multiply 16 by 3.
$48/6 = 8$	Divide 48 by 6.

5. **$x = -2$**

$3(x + 4)^3 - 5 = 19$	
$3(x + 4)^3 = 24$	Add 5 to both sides.
$(x + 4)^3 = 8$	Divide both sides by 3.
$(x + 4) = 2$	Take the cube root of both sides.
$x = -2$	Remove the parentheses and subtract 4 from both sides.

6. **$x = -7$**

$\frac{3x - 7}{2} + 20 = 6$	
$\frac{3x - 7}{2} = -14$	Subtract 20 from both sides.
$3x - 7 = -28$	Multiply both sides by 2.
$3x = -21$	Add 7 to both sides.
$x = -7$	Divide both sides by 3.

7. **$x = -6$**

$\sqrt[3]{(x+5)} - 7 = -8$	
$\sqrt[3]{(x+5)} = -1$	Add 7 to both sides.
$x + 5 = -1$	Cube both sides, remove parentheses.
$x = -7$	Subtract 5 from both sides.

8. ***x* = 11**

$\dfrac{11 + 3(x+4)}{x-3} = 7$	
$11 + 3(x+4) = 7(x-3)$	Multiply both sides by the denominator (x − 3).
$11 + 3x + 12 = 7x - 21$	Simplify grouped terms by distributing.
$23 + 3x = 7x - 21$	Combine like terms (11 and 12.)
$23 = 4x - 21$	Subtract 3x from both sides.
$44 = 4x$	Add 21 to both sides.
$11 = x$	Divide both sides by 4.

9. ***x* = 3**

$\dfrac{-6 - 5(3-x)}{2-x} = 6$	
$-6 - 5(3-x) = 6(2-x)$	Multiply both sides by the denominator (2 − x).
$-6 - 15 + 5x = 12 - 6x$	Simplify grouped terms by distributing.
$-21 + 5x = 12 - 6x$	Combine like terms (−6 and −15).
$-21 + 11x = 12$	Add 6x to both sides.
$11x = 33$	Add 21 to both sides.

$x = 3$	Divide both sides by 11.

10. ***x* = 6**

$$\frac{2x + 6(9 - 2x)}{x - 4} = -3$$

$2x + 6(9 - 2x) = -3(x - 4)$	Multiply by the denominator ($x - 4$).
$2x + 54 - 12x = -3x + 12$	Simplify grouped terms by distributing.
$-10x + 54 = -3x + 12$	Combine like terms ($2x$ and $-12x$).
$54 = 7x + 12$	Add $10x$ to both sides.
$44 = 4x$	Subtract 12 from both sides.
$x = 3$	Divide both sides by 7.

11. ***x* = 10, *y* = 8**

$x = 10$
$x + 2y = 26$

$(10) + 2y = 26$	Substitute 10 for x in the second equation.
$2y = 16$	Subtract 10 from both sides.
$y = 8$	Divide both sides by 2.

12. ***x* = 6, *y* = 1**

$x + 4y = 10$
$y - x = -5$

$y = x - 5$	Isolate y in the second equation.

$x + 4(x - 5) = 10$	Substitute $(x - 5)$ for y in the first equation.
$x + 4x - 20 = 10$	Simplify grouped terms within the equation.
$5x - 20 = 10$	Combine like terms (x and $4x$).
$5x = 30$	Add 20 to both sides.
$x = 6$	Divide both sides by 5.
$y - 6 = -5$	Substitute 6 for x in the second equation to solve for y.
$y = 1$	Add 6 to both sides.

13. $x = 11, y = 3$

$$6y + 15 = 3x$$
$$x + y = 14$$

$2y + 5 = x$	Divide the first equation by 3.
$(2y + 5) + y = 14$	Substitute $(2y + 5)$ for x in the second equation.
$3y + 5 = 14$	Combine like terms ($2y$ and y).
$3y = 9$	Subtract 5 from both sides.
$y = 3$	Divide both sides by 3.
$x + 3 = 14$	Substitute 3 for y in the second equation to solve for x.
$x = 11$	

14. $3b$

$3a - [2a - (3b - a)]$	Rewrite the expression carefully.

$= 3a - [2a - 3b + a]$	Distribute the minus sign and drop interior parentheses.
$= 3a - [3a - 3b]$	Combine like terms ($2a$ and a).
$= 3a - 3a + 3b$	Distribute the minus sign and drop brackets.
$= 3b$	Perform the subtraction ($3a - 3a$) to cancel those terms.

15. $\dfrac{20-5a}{12}$

$\dfrac{a+4}{4} - \dfrac{2a-2}{3}$	Rewrite the expression and identify the common denominator (12).
$= \dfrac{3}{3} \times \dfrac{(a+4)}{4} - \dfrac{4}{4} \times \dfrac{(2a-2)}{3}$	Multiply the first fraction by $\dfrac{1}{2}$ and the second fraction by $\dfrac{1}{2}$.
$= \dfrac{3\,(a+4)}{12} - \dfrac{4\,(2a-2)}{12}$	Show the products on top and bottom of each fraction.
$= \dfrac{3\,(a+4) - 4\,(2a-2)}{12}$	Combine the two fractions into one fraction with the common denominator.
$= \dfrac{3a+12-8a+8}{12}$	Distribute the 3 and the −4 and drop parentheses.
$= \dfrac{20-5a}{12}$	Combine like terms ($3a$ and $-8a$), including constraints (12 and 8).

Problem Set

1. Evaluate $-3x^2$, $-3x^3$, $3x^2$, $(-3x)^2$, and $(-3x)^3$ if $x = 2$, and also if $x = -2$.

2. Evaluate $(4 + 12 \div 3 - 18) - [-11 - (-4)]$

3. Which of the parentheses in the following expressions are unnecessary and could thus be removed without any change in the value of the expression?

ı)

$$-(5^2) - (12 - 7)$$

)

$$(x + y) - (w + z) - (a \times b)$$

4. Evaluate $\left[\dfrac{4+8}{2-\ (-6)}\right] - \ [4 + 8 \div 2 - \ (-6)]$

5. Simplify: $x - (3 - x)$

6. Simplify: $(4 - y) - 2(2y - 3)$

7. Solve for x: $2(2 - 3x) - (4 + x) = 7$

8. Solve for z: $\frac{4z - 7}{3 - 2z} = -5$

9.

Quantity A	**Quantity B**
$3 \times (5 + 6) \div -1$	$3 \times 5 + 6 \div -1$

10.

$$(x - 4)^3 + 11 = -16$$

Quantity A	**Quantity B**
x	-4

11.

$$2x + y = 10$$

$$3x - 2y = 1$$

Quantity A	**Quantity B**
x	y

Solutions

1.

If $x = 2$:	**If $x = -2$:**
$-3x^2 = -3(4) = \mathbf{-12}$	$-3x^2 = -3(4) = \mathbf{-12}$
$-3x^3 = -3(8) = \mathbf{-24}$	$-3x^3 = -3(-8) = \mathbf{24}$
$3x^2 = 3(4) = \mathbf{12}$	$3x^2 = 3(4) = \mathbf{12}$
$(-3x)^2 = (-6)^2 = \mathbf{36}$	$(-3x)^2 = 6^2 = \mathbf{36}$
$(-3x)^3 = (-6)^3 = \mathbf{-216}$	$(-3x)^3 = 6^3 = \mathbf{216}$

Remember that exponents are evaluated *before* multiplication. Watch not only the order of operations, but also the signs in these problems.

2. **−3**

$$\begin{aligned} (4 + 12 \div 3 - 18) - [-11 - (-4)] &= \\ (4 + 4 - 18) - (-11 + 4) &= \\ (-10) - (-7) &= \\ -10 + 7 &= \mathbf{-3} \end{aligned}$$

3. **(a):** The parentheses around 5^2 are unnecessary, because this exponent is performed before the negation (which counts as multiplying by −1) and before the subtraction. The other parentheses are necessary because they cause the right-hand subtraction to be performed before the left-hand subtraction. Without them, the two subtractions would be performed from left to right.

(b): The first and last pairs of parentheses are unnecessary. The addition is performed before the neighboring subtraction by default, because addition and subtraction are performed from left to right. The multiplication is the first operation to be performed, so the right-hand parentheses are completely unnecessary. The middle parentheses are necessary to ensure that the addition is performed before the subtraction that comes to the left of it.

4. $\boldsymbol{-\frac{25}{2}}$

$$\begin{aligned}
\left[\frac{4+8}{2-(-6)}\right]-[4+8\div 2-(-6)] &= && \\
\left(\frac{4+8}{2+6}\right)-(4+8\div 2+6) &= && \text{Subtraction of negative} = \text{addition.}\\
\left(\frac{12}{8}\right)-\left(4+4+6\right) &= && \text{Fraction bar acts as a grouping symbol.}\\
\frac{3}{2}-14 &= && \text{Arithmetic}\\
\frac{3}{2}-\frac{28}{2} &= -\frac{\mathbf{25}}{\mathbf{2}} && \text{Arithmetic}
\end{aligned}$$

5. **2*x* − 3**

Do not forget to reverse the signs of every term in a subtracted expression: $x-(3-x)=x-3+x=2x-3$

6. **−5*y* + 10 (or 10 − 5*y*)**

Do not forget to reverse the signs of every term in a subtracted expression. $(4-y)-2(2y-3)=4-y-4y+6=-5y+10$ (or $10-5y$)

7. **−1**

$$\begin{aligned} 2\,(2-3x)-\ (4+x) &= 7 \\ 4-6x-4-x &= 7 \\ -7x &= 7 \\ x &= -1 \end{aligned}$$

8. $\frac{1}{2}$

$$\begin{aligned} \frac{4z-7}{3-2z} &= -5 \\ 4z-7 &= -5(3-2z) \\ 4z-7 &= -15+10z \\ 8 = 6z \\ z = \tfrac{8}{6} &= \tfrac{4}{3} \end{aligned}$$

9. **(B)**

Evaluate Quantity A first:

3 × (5 + 6) ÷ −1	
3 × (11) ÷ −1	Simplify the parentheses.
33 ÷ −1	Multiply and divide in order from left to right.
−33	
Now evaluate Quantity B:	
3 × 5 + 6 ÷ −1	
15 + −6	Multiply and divide in order from left to right.
9	Add.

Quantity A	Quantity B
−33	9

Quantity B is greater.

10. **(A)**

Simplify the given equation to solve for *x*:

$$\begin{aligned}(x-4)^3+11 &= -16\\ (x-4)^3 &= -27\\ x-4 &= -3\\ x &= 1\end{aligned}$$

Quantity A	Quantity B
$x = 1$	−4

Quantity A is greater.

11. **(B)**

Use substitution to solve for the values of *x* and *y*:

$2x + y = 10 \rightarrow y = 10 - 2x$	Isolate *y* in the first equation.
$3x - 2y = 1 \rightarrow 3x - 2(10 - 2x) = 1$	Substitute (10 − 2*x*) for *y* in the second equation.
$3x - 20 + 4x = 1$	Distribute
$7x = 21$	Group like terms (3*x* and 4*x*) and add 20 to both sides.
$x = 3$	Divide both sides by 7.
$2x + y = 10 \rightarrow 2(3) + y = 10$	Substitute 3 for *x* in the first equation.

$6 + y = 10$

$y = 4$

Quantity A	**Quantity B**
$x = 3$	$y = 4$

Quantity B is greater.

Chapter 2

QUADRATIC EQUATIONS

In This Chapter...

Identifying Quadratic Equations
Distributing
Factoring
Applying This to Quadratics
Factoring Quadratic Equations
Solving Quadratic Equations
Using FOIL with Square Roots
One-Solution Quadratics
Zero in the Denominator: Undefined
The Three Special Products

Chapter 2

Quadratic Equations

Identifying Quadratic Equations

This section begins with a question:

> If $x^2 = 4$, what is x?

You know what to do here. Simply take the square root of both sides:

$$\begin{aligned} \sqrt{x^2} &= \sqrt{4} \\ x &= 2 \end{aligned}$$

So $x = 2$. The question seems to be answered. But, what if x were equal to -2? What would be the result? Plug -2 in for x:

$$(-2)^2 = 4 \rightarrow 4 = 4$$

If plugging −2 in for x yields a true statement, then −2 must be a solution to the equation. But, from your initial work, you know that 2 is a solution to the equation. So which one is correct?

As it turns out, they both are. An interesting thing happens when you start raising variables to exponents. The number of possible solutions increases. When a variable is squared, as in the example here, it becomes possible that there will be two solutions to the equation.

What this means is that whenever you see an equation with a squared variable, you need to:

- Recognize that the equation may have two solutions.
- Know how to find both solutions.

A quadratic equation is any equation for which the highest power on a variable is the second power (e.g., x^2).

For an equation such as $x^2 = 25$ or $x^2 = 9$, finding both solutions shouldn't be too challenging. Take a minute to find both solutions for each equation.

You should have found that x equals either 5 or −5 in the first equation, and 3 or −3 in the second equation. However, what if you are asked to solve for x in the following equation?

$$x^2 + 3x - 10 = 0$$

Unfortunately, you don’t yet have the ability to deal with equations like this, which is why the next part of this chapter will deal with some more important tools for manipulating and solving **quadratic equations: distributing** and **factoring**.

Distributing

You first came across distributing when you were learning how to clean up equations and isolate a variable. Essentially, distributing is applying multiplication across a sum.

To review, if you are presented with the expression $3(x + 2)$, and you want to simplify it, you have to distribute the 3 so that it is multiplied by both the x and the 2:

$$3(x+2) \rightarrow (3 \times x) + (3 \times 2) \rightarrow 3x + 6$$

But what if the first part of the multiplication is more complicated? Suppose you need to simplify $(a + b)(x + y)$?

Simplifying this expression is really an extension of the principle of distribution—every term in the first part of the expression must multiply every term in the second part of the expression. In order to do so correctly every time, you can use a handy acronym to remember the steps necessary: FOIL. The letters stand for **F**irst, **O**uter, **I**nner, **L**ast.

In this case, it looks like this:

$(\boldsymbol{a} + b)(\boldsymbol{x} + y)$	F − multiply the first term in each of the parentheses: $a \times x = ax$.

$(\boldsymbol{a} + b)(x + \boldsymbol{y})$	O – multiply the outer term in each: $a \times y = ay$.
$(a + \boldsymbol{b})(\boldsymbol{x} + y)$	I – multiply the inner term in each: $b \times x = bx$.
$(a + \boldsymbol{b})(x + \boldsymbol{y})$	L – multiply the last term in each: $b \times y = by$.

So you have $(a + b)(x + y) = ax + ay + bx + by$.

You can verify this system with numbers. Take the expression (3 + 4)(10 + 20). This is no different than multiplying (7)(30), which gives you 210. See what happens when you FOIL the numbers:

$(\mathbf{3} + 4)(\mathbf{10} + 20)$	F – multiply the first term in each of the parentheses: $3 \times 10 = 30$.
$(\mathbf{3} + 4)(10 + \mathbf{20})$	O – multiply the outer term in each: $3 \times 20 = 60$.
$(3 + \mathbf{4})(\mathbf{10} + 20)$	I – multiply the inner term in each: $4 \times 10 = 40$.
$(3 + \mathbf{4})(10 + \mathbf{20})$	L – multiply the last term in each: $4 \times 20 = 80$.

Finally, sum the four products: 30 + 60 + 40 + 80 = 210.

Now that you have the basics down, go through a more GRE-like situation. Take the expression $(x + 2)(x + 3)$. Once again, begin by FOILing it:

$(\boldsymbol{x} + 2)(\boldsymbol{x} + 3)$	F – multiply the first term in each of the parentheses: $x \times x = x^2$.
$(\boldsymbol{x} + 2)(x + \mathbf{3})$	O – multiply the outer term in each: $x \times 3 = 3x$.
$(x + \mathbf{2})(\boldsymbol{x} + 3)$	I – multiply the inner term in each: $2 \times x = 2x$.
$(x + \mathbf{2})(x + \mathbf{3})$	L – multiply the last term in each: $2 \times 3 = 6$.

The expression becomes $x^2 + 3x + 2x + 6$. Combine like terms, and you are left with $x^2 + 5x + 6$. The next section will discuss the connection between distributing, factoring, and solving quadratic equations. But for the moment, practice FOILing expressions.

Check Your Skills

FOIL the following expressions.

1. $(x + 4)(x + 9)$

2. $(y + 3)(y - 6)$

3. $(x + 7)(3 + x)$

Factoring

What is factoring? *Factoring is the process of reversing the distribution of terms.*

For example, when you multiply y and $(5 - y)$, you get $5y - y^2$. Reversing this, if you're given $5y - y^2$, you can "factor out" a y to transform the expression into $y(5 - y)$. Another way of thinking about factoring is that you're *pulling out* a common factor that's in every term and rewriting the expression as a *product*.

You can factor out many different things on the GRE: variables, variables with exponents, numbers, and expressions with more than one term, such as $(y - 2)$ or $(x + w)$. Here are some examples:

$t^2 + t$ $= t(t + 1)$	Factor out a t. Notice that a 1 remains behind when you factor a t out of a t.
$5k^3 - 15k^2$ $= 5k^2(k - 3)$	Factor out a $5k^2$.
$21j + 35k$ $= 7(3j + 5k)$	Factor out a 7; because the variables are different, you can't factor out any variables.

If you ever doubt whether you've factored correctly, just distribute back. For instance, $t(t + 1) = t \times t + t \times 1 = t^2 + t$, so $t(t + 1)$ is the correct factored form of $t^2 + t$.

You might factor expressions for a variety of reasons. One common reason is to simplify an expression (the GRE complicates equations that are actually quite simple). The other reason, which will be discussed in more detail shortly, is to find possible values for a variable or combination of variables.

Check Your Skills

Factor the following expressions.

4. $4 + 8t$

5. $5x + 25y$

6. $2x^2 + 16x^3$

Applying This to Quadratics

If you were told that $7x = 0$, you would know that x must be 0. This is because the only way to make the product of two or more numbers equal 0 is to have at least one of those numbers equal 0. Clearly, 7 does not equal 0, which means that x must be 0.

What if you were told that $kj = 0$? Well, now you have two possibilities. If $k = 0$, then $0(j) = 0$, which is true, so $k = 0$ is a solution to the equation $kj = 0$. Likewise, if $j = 0$, then $k(0) = 0$, which is also true, so $j = 0$ is also a solution to $kj = 0$.

Either of these scenarios make the equation true, and are the only scenarios, in fact, that make the product $kj = 0$. (If this is not clear, try plugging in non-zero numbers for both k and j and see what happens.)

This is why you want to rewrite quadratic equations such as $x^2 + 3x - 10 = 0$ in factored form: $(x + 5)(x - 2) = 0$. The left side of the factored equation is a *product*, so it's really the same thing as $jk = 0$. Now you know that either $x + 5$ is 0, or $x - 2$ is 0. This means either $x = -5$ or $x = 2$. Once you've factored a quadratic equation, it's straightforward to find the solutions.

Check Your Skills

List all possible solutions to the following equations.

7. $(x - 2)(x - 1) = 0$

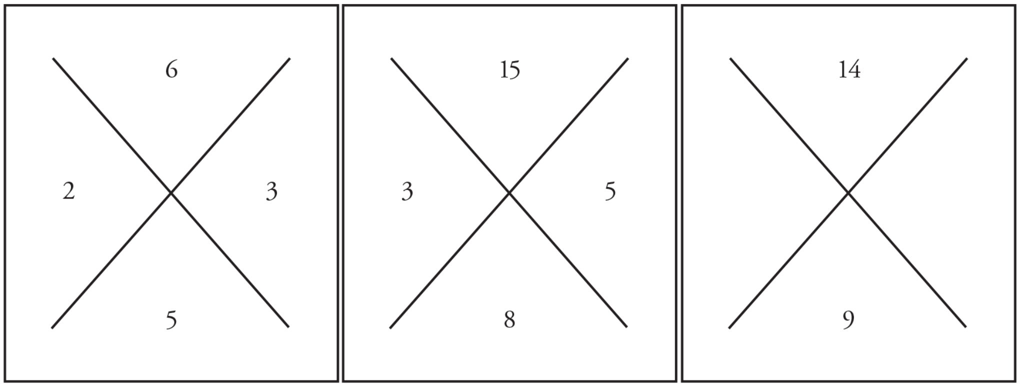
8. $(x + 4)(x + 5) = 0$

9. $(y - 3)(y + 6) = 0$

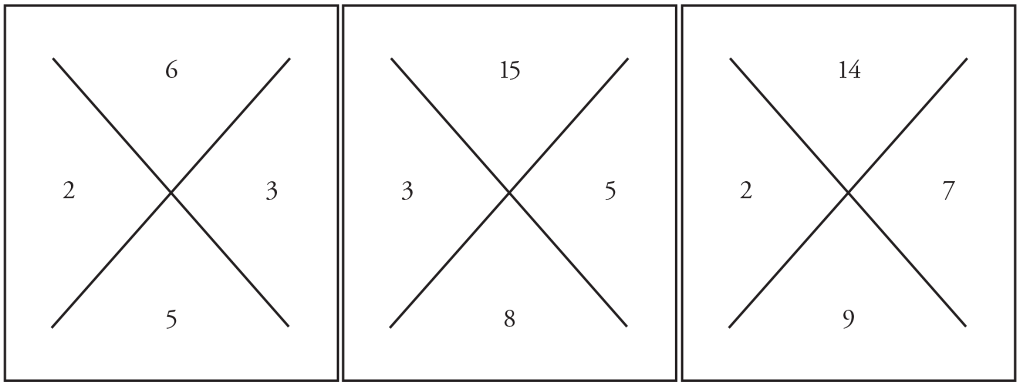
Factoring Quadratic Equations

Okay, so now you understand *why* you want to factor a quadratic expression, but *how* do you do it? It’s not easy to look at $x^2 + 3x - 10$ and see that it equals $(x + 5)(x - 2)$.

To get started, try solving the following puzzle. (Hint: It involves addition and multiplication.) The first two are done for you:

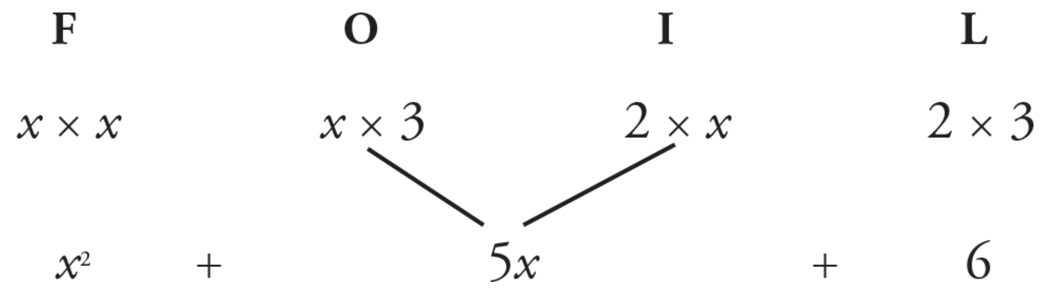


Have you figured out the trick to this puzzle? The answers are as follows:



The way the diamonds work is that you multiply the two numbers on the sides to obtain the top number, and you add them to arrive at the bottom number.

Take another look at the connection between $(x + 2)(x + 3)$ and $x^2 + 5x + 6$:



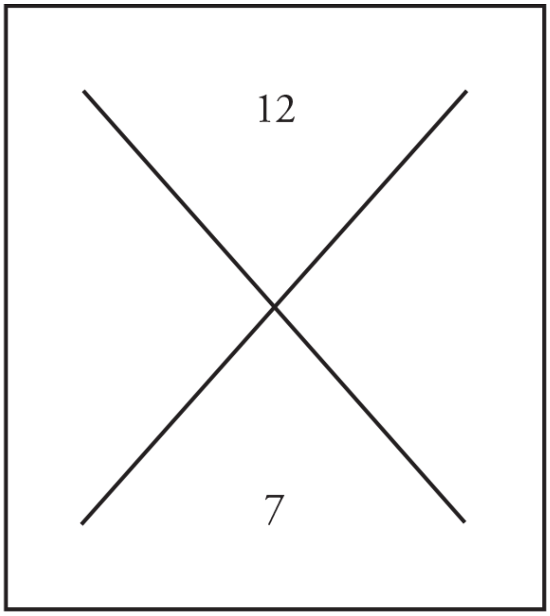
The 2 and the 3 play two important roles in building the quadratic expression:

- They multiply to give you 6, which is the final term in your quadratic expression.
- Multiplying the outer terms gives you $3x$, and multiplying the inner terms gives you $2x$. You can then add those terms to get $5x$, the middle term of your quadratic expression.

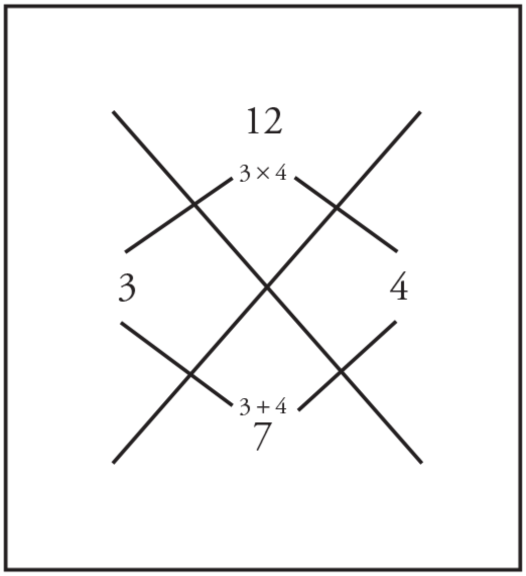
So when you are trying to factor a quadratic expression such as $x^2 + 5x + 6$, the key is to find the two numbers whose product equals the final term (6) and whose sum equals the coefficient of the middle term (the 5 in $5x$). In this case, the two numbers that multiply to 6 and add up to 5 are 2 and 3: $2 \times 3 = 6$ and $2 + 3 = 5$.

So the diamond puzzle is just a visual representation of this same goal. For any quadratic expression, take the final term (the **constant**) and place it in the top portion of the diamond. Take the **coefficient** of the middle term (in this case, the 5 in 5*x*) and place it in the lower portion of the diamond. For instance, if the middle term is 5*x*, take the 5 and place it at the bottom of the diamond. Now go through the entire process with a new example: $x^2 + 7x + 12$.

The final term is 12, and the coefficient of the middle term is 7, so the diamond will look like this:



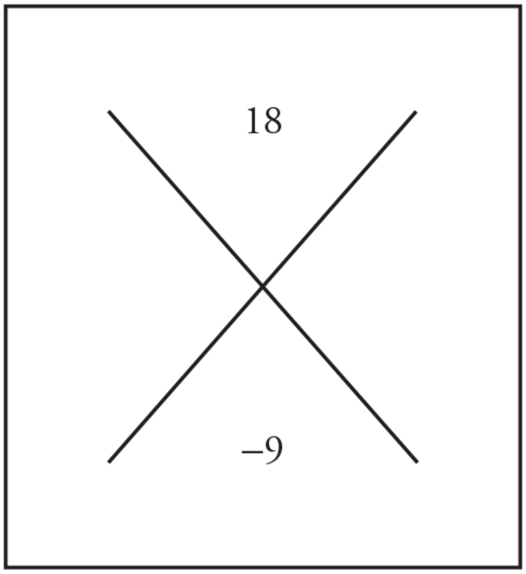
When factoring quadratics (or solving the diamond puzzle), it is better to focus first on determining which numbers could multiply to the final term. The reason is that these problems typically deal only with integers, and there are far fewer pairs of integers that will multiply to a certain product than will add to a certain sum. For instance, in this problem, there are literally an infinite number of integer pairs that can add to 7 (remember, negative numbers are also integers: −900,000 and 900,007 sum to 7, for instance). On the other hand, there are only a few integer pairs that multiply to 12. You can actually list them all out: 1 and 12, 2 and 6, and 3 and 4. Because 1 and 12 sum to 13, they don't work; 2 and 6 sum to 8, so they don't work either. However, 3 and 4 sum to 7, so this pair of numbers is the one you want. So your completed diamond looks like this:



Now, because your numbers are **3** and **4**, the factored form of your quadratic expression becomes $(x + \mathbf{3})(x + \mathbf{4})$.

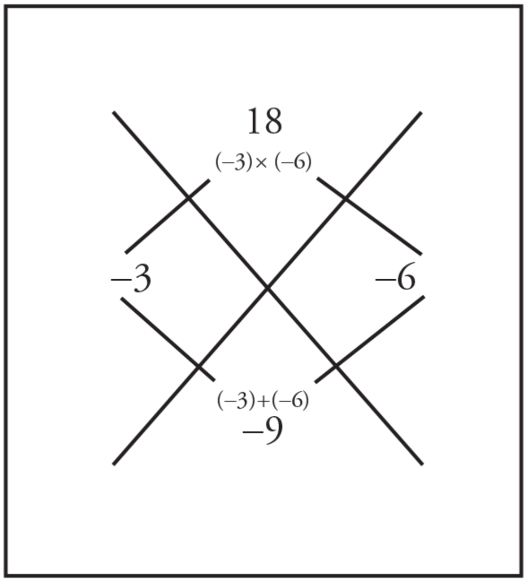
Note: if you are factoring $x^2 + 7x + 12 = 0$, you get $(x + 3)(x + 4) = 0$, so your solutions are *negative* 3 or *negative* 4, not 3 and 4 themselves. Remember, if you have $(x + 3)(x + 4) = 0$, then either $x + 3 = 0$ or $x + 4 = 0$.

Here's another example with one important difference. Solve the diamond puzzle for this quadratic expression: $x^2 - 9x + 18$. Your diamond looks like this:



You need two numbers that multiply to positive 18, but sum to −9. Here, you know the product is positive, but the sum is negative. So when the top number is positive and the bottom number is negative, the two numbers you are looking for will both be negative.

Once again, it will be easier to start by figuring out what pairs of numbers can multiply to 18. In this case, three different pairs all multiply to 18: −1 and −18, −2 and −9, and −3 and −6. The pair −3 and −6, however, is the only pair of numbers that also sums to −9, so this is the pair you want. Fill in the missing numbers, and your diamond becomes:



If the numbers on the left and right of the diamond are −**3** and −**6**, the factored form of the quadratic expression becomes (x − **3**)(x − **6**), so the solutions are *positive* 3 or *positive* 6.

To recap, when the final term of the quadratic is positive, the two numbers you are looking for will either both be positive or both be negative. If the middle term is positive, as in the case of $x^2 + 7x + 12$, the numbers will both be positive (3 and 4). If the middle term is negative, as in the case of $x^2 - 9x + 18$, the numbers will both be negative (−3 and −6).

Check Your Skills

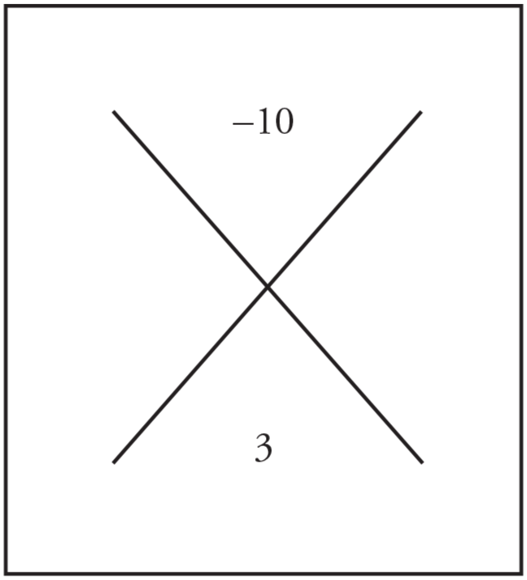
Factor the following quadratic expressions.

10. $x^2 + 14x + 33$

11. $x^2 - 14x + 45$

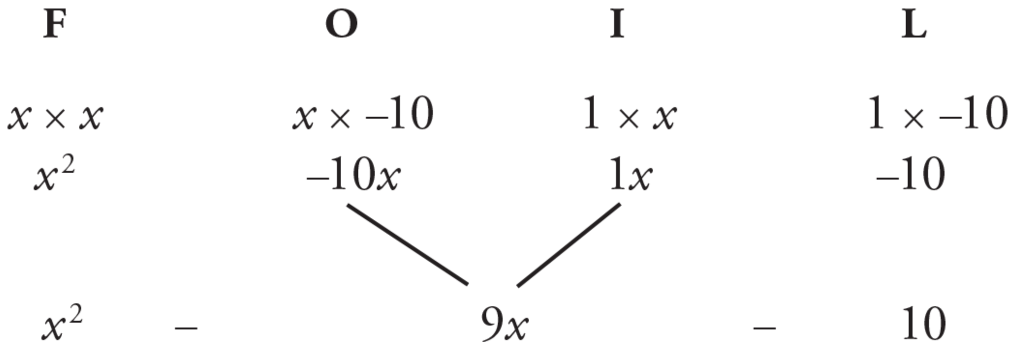
The previous section dealt with quadratic equations in which the final term was positive. This section discusses how to deal with quadratics in which the final term is negative. The basic method is the same, although there is one important twist.

Take a look at the quadratic expression $x^2 + 3x - 10$. Start by creating your diamond:

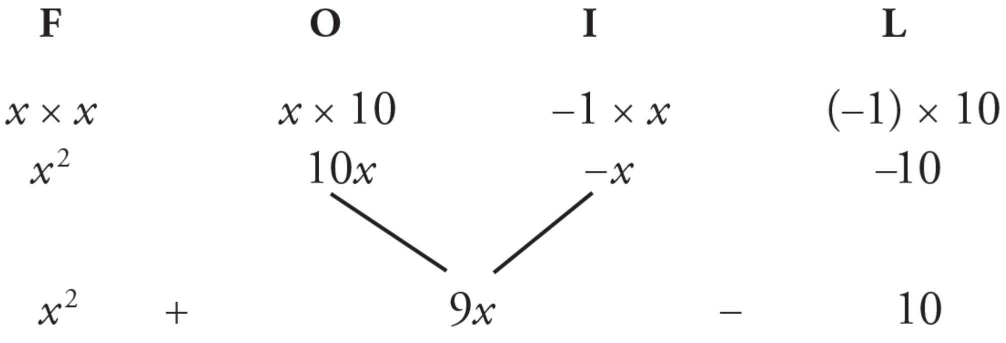


You are looking for two numbers that will multiply to −10. The only way for the product of two numbers to be negative is for one of them to be positive and one of them to be negative. That means that in addition to figuring out pairs of numbers that multiply to 10, you also need to worry about which number will be positive and which will be negative. For the moment, disregard the signs. There are only two pairs of integers that multiply to 10: 1 and 10 and 2 and 5. Start testing the pair 1 and 10, and see what you can learn.

Try making 1 positive and 10 negative. If that were the case, the factored form of the expression would be $(x + 1)(x - 10)$. FOIL it and see what it would look like:



The sum of 1 and −10 is −9, but you want 3. That's not correct, so try reversing the signs. Now see what happens if you make 1 negative and 10 positive. The factored form would now be $(x - 1)(x + 10)$. Once again, FOIL it out:

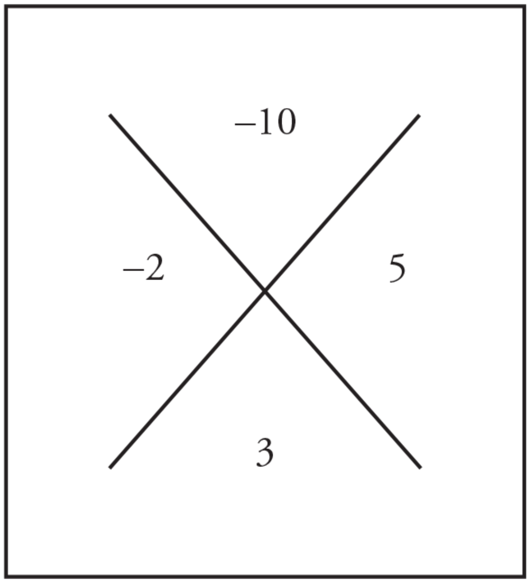


Again, this doesn't match your target. The sum of −1 and 10 is not 3. Compare these examples to the examples in the last section. Notice that, with the examples in the last section, the two numbers summed to the coefficient of the middle term (in the example $x^2 + 7x + 12$, the two numbers you wanted, 3 and 4, summed to 7, which is the coefficient of the middle term). In these two examples, however, because one number was positive and one number was negative, it is actually the *difference* of 1 and 10 that gave us the coefficient of the middle term.

This will be discussed further as the example continues. For now, to factor quadratics in which the final term is negative, you actually ignore the sign initially and look for two numbers that multiply to the coefficient of the final term (ignoring the sign) and whose *difference* is the coefficient of the middle term (ignoring the sign).

Going back to the example, the pair of numbers 1 and 10 did not work, so look at the pair 2 and 5. Notice that the coefficient of the middle term is 3, and the difference of 2 and 5 is 3. This has to be the correct pair, so all you need to do is determine whether your factored form is $(x + 2)(x - 5)$ *or* $(x - 2)(x + 5)$. Take some time now to FOIL both expressions and figure out which one is correct.

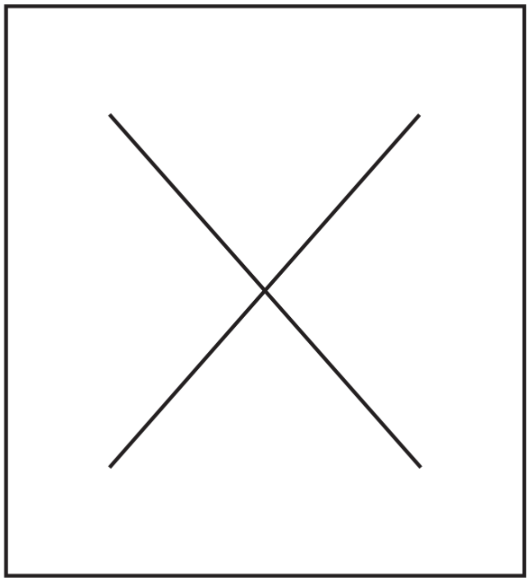
You should have come to the conclusion that $(x - 2)(x + 5)$ was the correctly factored form of the expression. That means your diamond looks like this:



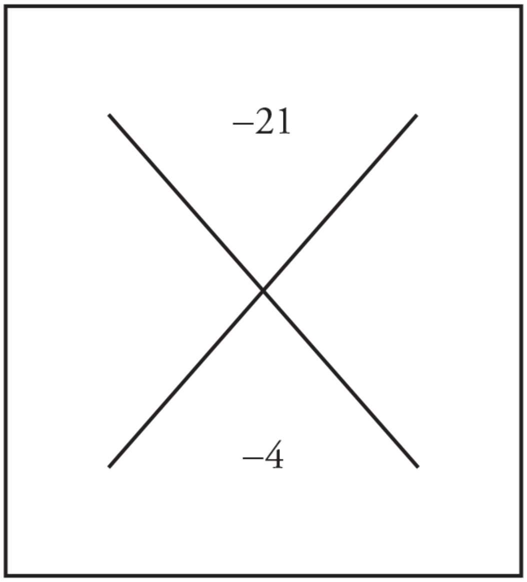
To recap, the way to factor *any* quadratic expression where the final term is negative is as follows:

> Ignore the signs initially. Find a pair of numbers that multiply to the coefficient of the final term and whose *difference* is the coefficient of the middle term (for $x^2 + 3x - 10$, the numbers 2 and 5 multiply to 10 and $5 - 2 = 3$).
>
> Now that you have the pair of numbers (2 and 5), you need to figure out which one will be positive and which one will be negative. As it turns out, this is straightforward to do. Pay attention to signs again. If the sign of the middle term is positive, then the greater of the two numbers will be positive and the smaller will be negative. This was the case in the previous example. The middle term was +3, so the pair of numbers was +5 and −2. On the other hand, when the middle term is negative, the greater number will be negative and the smaller number will be positive.

Work through one more example to see how this works. What is the factored form of $x^2 - 4x - 21$? Take some time to work through it for yourself before looking at the explanation.

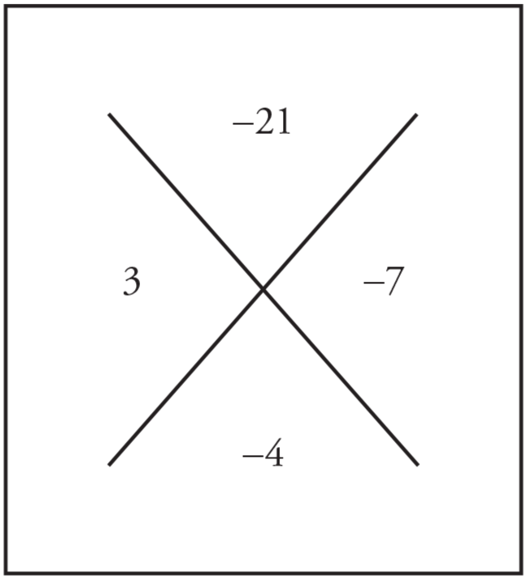


First, start your diamond. It looks like this:



Because the coefficient of the final term (−21) is negative, you're going to ignore the signs for the moment, and focus on finding pairs of integers that will multiply to 21. The only possible pairs are 1 and 21, and 3 and 7. Next, take the difference of both pairs: 21 − 1 = 20 and 7 − 3 = 4. The second pair matches the −4 on the bottom of the diamond (because you are ignoring the sign of the −4 at this stage), so 3 and 7 is the correct pair of numbers.

Now all that remains is to determine the sign of each. The coefficient of the middle term (−4) is negative, so you need to assign the negative sign to the greater of the two numbers, 7. That means that the 3 will be positive. Subsequently, the correctly factored form of the quadratic expression is $(x + 3)(x - 7)$:



Check Your Skills

Factor the following expressions.

12. $x^2 + 3x - 18$

13. $x^2 - 5x - 66$

Solving Quadratic Equations

Now that you know how to factor quadratic expressions, it's time to make that final jump to actually solving quadratic equations. When first discussing factoring, it was noted that when one side of the equation is equal to 0, you can make use of the rule that anything times 0 is 0. In the case of the equation $(x - 5)(x + 10) = 0$, you know that either $(x - 5) = 0$ or $(x + 10) = 0$, which means that $x = 5$ *or* $x = -10$.

The whole point of factoring quadratic equations is so that you can make use of this rule. Therefore, before you factor a quadratic expression, you *must* make sure that the other side of the equation equals 0.

Suppose you see an equation $x^2 + 10x = -21$, and you need to solve for x. The x^2 term in the equation should tell you that this is a quadratic equation, but it's not yet ready to be factored. Before it can be factored, you have to move everything to one side of the equation. In this equation, the easiest way to do that is to add 21 to both sides, giving you $x^2 + 10x + 21 = 0$. Now that one side equals 0, you are ready to factor.

The final term is positive, so you're looking for two numbers to multiply to 21 and sum to 10. The numbers 3 and 7 fit the bill, so your factored form is $(x + 3)(x + 7) = 0$. That means that $x = -3$ *or* $x = -7$.

Now you know all the steps to successfully factor and solve quadratic equations.

Check Your Skills

Solve the following quadratic equations.

14. $x^2 - 3x + 2 = 0$

15. $x^2 + 2x - 35 = 0$

16. $x^2 - 15x = -26$

Using FOIL with Square Roots

Some GRE problems ask you to solve factored expressions that involve roots. For example, the GRE might ask you to solve the following:

> **What is the value of $(\sqrt{8}-\sqrt{3})(\sqrt{8}+\sqrt{3})$?**

Even though these problems do not involve any variables, you can solve them as you would solve a pair of quadratic factors, using FOIL:

First: $\sqrt{8}\times\sqrt{8}=8$

Outer: $\sqrt{8}\times\sqrt{3}=\sqrt{24}$

Inner: $\sqrt{8}\times\left(-\sqrt{3}\right)=-\sqrt{24}$

Last: $\left(-\sqrt{3}\right)\left(\sqrt{3}\right)=-3$

The four terms are: $8+\sqrt{24}-\sqrt{24}-3.$

You can simplify this expression by removing the two middle terms (they cancel each other out) and subtracting: $8+\sqrt{24}-\sqrt{24}-3=8-3=5.$ Although the problem looks complex, using FOIL reduces the entire expression to 5.

Check Your Skills

17. FOIL $\left(\sqrt{8}-\sqrt{2}\right)\left(\sqrt{8}-\sqrt{2}\right)$

One-Solution Quadratics

Not all quadratic equations have two solutions. Some have only one solution. One-solution quadratics are also called **perfect square** quadratics, because both roots are the same. Consider the following examples:

$$
\begin{aligned}
x^2 + 8x + 16 &= 0 && \textbf{Here, the one solution for } x \textbf{ is } -4. \\
(x+4)(x+4) &= 0 \\
(x+4)^2 &= 0 \\
x^2 - 6x + 9 &= 0 && \textbf{Here, the one solution for } x \textbf{ is } 3. \\
(x-3)(x-3) &= 0 \\
(x-3)^2 &= 0
\end{aligned}
$$

Be careful not to assume that a quadratic equation always has two solutions. Always factor quadratic equations to determine their solutions. In doing so, you will see whether a quadratic equation has one or two solutions.

Check Your Skills

18. Solve for x: $x^2 - 10x + 25 = 0$

Zero in the Denominator: Undefined

Math convention does not allow division by 0. When 0 appears in the denominator of an expression, then that expression is undefined. How does this convention affect equations that contain quadratic expressions? Consider the following:

> What are the solutions to the following equation?
>
> $$\frac{x^2+x-12}{x-2}=0$$

Notice a quadratic expression in the numerator. It is a good idea to simplify quadratic expressions by factoring, so factor this numerator as follows:

$$\frac{x^2+x-12}{x-2}=0 \rightarrow \frac{(x-3)(x+4)}{x-2}=0$$

If either of the factors in the numerator is 0, then the entire expression becomes 0. Thus, the solutions to this equation are $x = 3$ or $x = -4$.

Note that making the denominator of the fraction equal to 0 would *not* make the entire expression equal to 0. Recall that if 0 appears in the denominator, the expression becomes undefined. Thus, $x = 2$ (which would make the denominator equal to 0) is *not* a solution to this equation. In fact, because setting *x* equal to 2 would make the denominator 0, the value 2 is not allowed: ***x* cannot equal 2**.

Check Your Skills

19. Solve for *x*: $\frac{(x+1)(x-2)}{x-4}=0$

The Three Special Products

Three quadratic expressions called *special products* come up so frequently on the GRE that it pays to memorize them. You should immediately recognize these three expressions and know how to factor (or distribute) each one automatically. This will usually put you on the path toward the solution to the problem:

Special Product #1: $x^2 - y^2 = (x + y)(x - y)$

Special Product #2: $x^2 + 2xy + y^2 = (x + y)(x + y) = (x + y)^2$

Special Product #3: $x^2 - 2xy + y^2 = (x - y)(x - y) = (x - y)^2$

You should be able to identify these products when they are presented in disguised form. For example, $a^2 - 1$ can be factored as $(a + 1)(a - 1)$. Similarly, $(a + 1)^2$ can be distributed as $a^2 + 2a + 1$.

Avoid the following common mistakes with special products:

Wrong: $(x + y)^2 = x^2 + y^2$?
$(x - y)^2 = x^2 - y^2$?

Right: $(x + y)^2 = x^2 + 2xy + y^2$
$(x - y)^2 = x^2 - 2xy + y^2$

Check Your Skills

Factor the following.

20. $4a^2 + 4ab + b^2 = 0$

21. $x^2 + 22xy + 121y^2 = 0$

Check Your Skills Answer Key

1. $x^2 + 13x + 36$

$(x + 4)(x + 9)$

$(\mathbf{x} + 4)(\mathbf{x} + 9)$	F – multiply the first term in each parentheses: $x \times$
$(\mathbf{x} + 4)(x + \mathbf{9})$	O – multiply the outer term in each: $x \times 9 = 9x$.
$(x + \mathbf{4})(\mathbf{x} + 9)$	I – multiply the inner term in each: $4 \times x = 4x$.
$(x + \mathbf{4})(x + \mathbf{9})$	L – multiply the last term in each: $4 \times 9 = 36$.

$x^2 + 9x + 4x + 36 \rightarrow x^2 + 13x + 36$

2. $y^2 - 3y - 18$

$(y + 3)(y - 6)$

$(\mathbf{y} + 3)(\ \mathbf{y} - 6)$	F – multiply the first term in each parentheses: $y \times$
$(\mathbf{y} + 3)(y - \mathbf{6})$	O – multiply the outer term in each: $y \times -6 = -6y$.
$(y + \mathbf{3})(\mathbf{y} - 6)$	I – multiply the inner term in each: $3 \times y = 3y$.
$(y + \mathbf{3})(y - \mathbf{6})$	L – multiply the last term in each: $3 \times -6 = -18$.

$y^2 - 6y + 3y - 18 \rightarrow y^2 - 3y - 18$

3. $x^2 + 10x + 21$

$(x + 7)(3 + x)$	
$(\boldsymbol{x} + 7)(\mathbf{3} + x)$	F – multiply the first term in each parentheses: $x \times$
$(\boldsymbol{x} + 7)(3 + \boldsymbol{x})$	O – multiply the outer term in each: $x \times x = x^2$.
$(x + \mathbf{7})(\mathbf{3} + x)$	I – multiply the inner term in each: $7 \times 3 = 21$.
$(x + \mathbf{7})(3 + \boldsymbol{x})$	L – multiply the last term in each: $7 \times x = 7x$.
$3x + x^2 + 21 + 7x \rightarrow x^2 + 10x + 21$	

4. $4 + 8t$

$\mathbf{4(1 + 2}\boldsymbol{t}\mathbf{)}$	Factor out a 4.

5. $5x + 25y$

$\mathbf{5(}\boldsymbol{x}\mathbf{ + 5}\boldsymbol{y}\mathbf{)}$	Factor out a 5.

6. $2x^2 + 16x^3$

$\mathbf{2}\boldsymbol{x}^{\mathbf{2}}\mathbf{(1 + 8}\boldsymbol{x}\mathbf{)}$	Factor out a $2x^2$.

7. $\boldsymbol{x}$ = 2 OR 1

$(x - 2)(x - 1) = 0$	
$(x - 2) = 0 \rightarrow x = 2$	Remove the parentheses and solve for x.
OR $(x - 1) = 0 \rightarrow x = 1$	Remove the parentheses and solve for x.

8. **$x = -4$ OR -5**

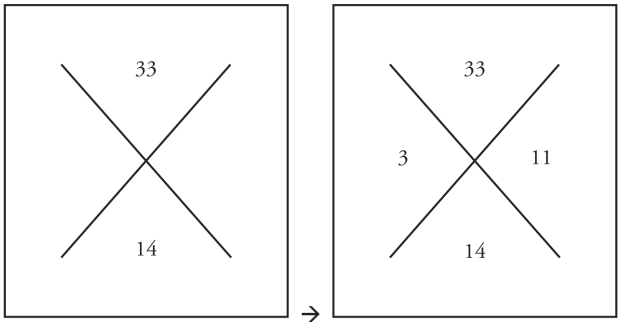
$(x + 4)(x + 5) = 0$	
$(x + 4) = 0 \rightarrow x = -4$	Remove the parentheses and solve for x.
OR $(x + 5) = 0 \rightarrow x = -5$	Remove the parentheses and solve for x.

9. **$y = 3$ OR -6**

$(y - 3)(y + 6) = 0$	
$(y - 3) = 0 \rightarrow y = 3$	Remove the parentheses and solve for y.
OR $(y + 6) = 0 \rightarrow y = -6$	Remove the parentheses and solve for y.

10. **$(x + 3)(x + 11)$**

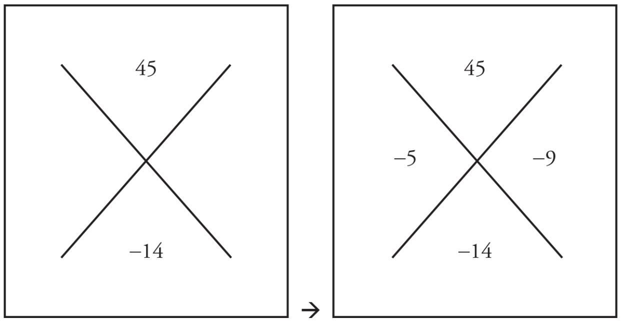
$x^2 + 14x + 33$



The numbers 1 and 33 and 3 and 11 multiply to 33, and the numbers 3 and 11 sum to 14 $(x + 3)(x + 11)$

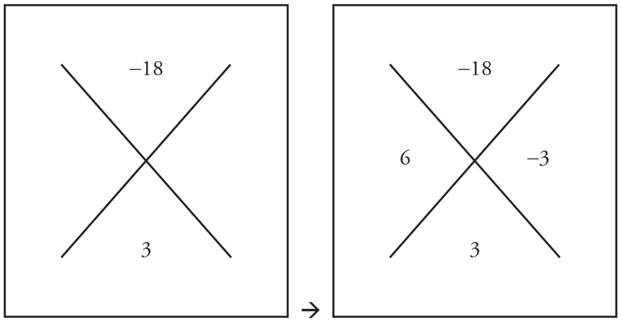
11. **$(x - 5)(x - 9)$**

$x^2 - 14x + 45$



The numbers 1 and 45, 3 and 15, and 5 and 9 multiply to 45. The numbers 5 and 9 sum to 14. $(x - 5)(x - 9)$

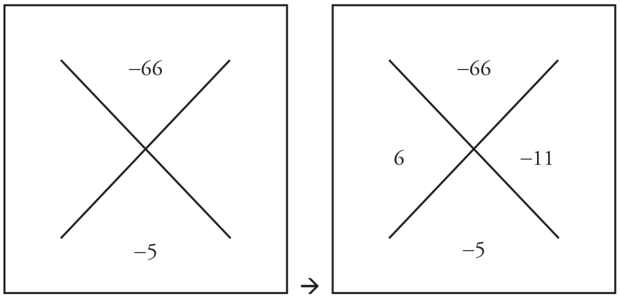
12. **$(x + 6)(x - 3)$**
$x^2 + 3x - 18$



The numbers 1 and 18, 2 and 9, and 3 and 6 multiply to 18. The difference of 3 and 6 is 3. The middle term is positive, so the greater of the two numbers (6) is positive.

$(x + 6)(x - 3)$

13. **$(x + 6)(x - 11)$**
$x^2 - 5x - 66$

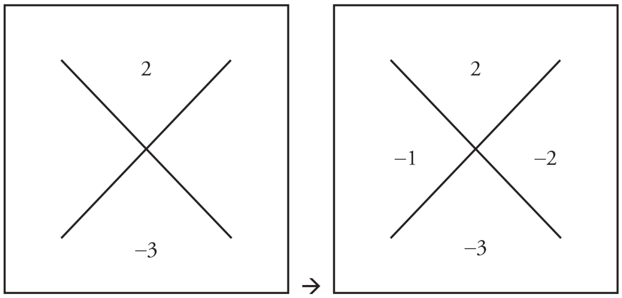


The numbers 1 and 66, 2 and 33, 3 and 22, and 6 and 11 multiply to 66. The difference of 6 and 11 is 5.

$(x + 6)(x - 11)$

14. ***x* = 1 OR 2**

$x^2 - 3x + 2 = 0$

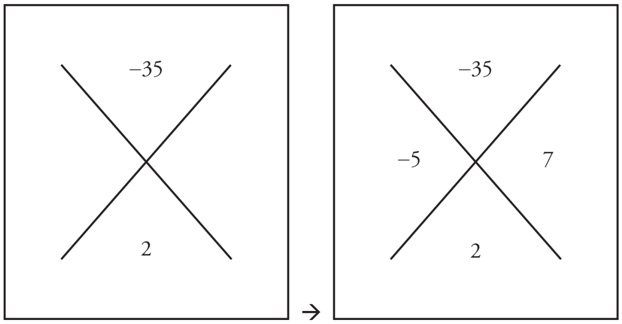


The numbers 1 and 2 multiply to 2 and add to 3.

$(x - 1)(x - 2) = 0$

15. ***x* = 5 OR −7**

$x^2 + 2x - 35 = 0$

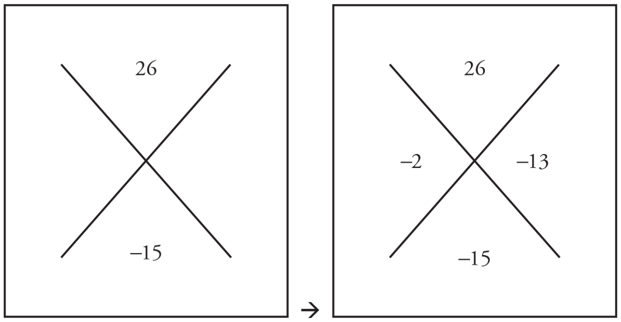


The numbers 5 and 7 multiply to 35 and their difference is 2. The middle term is positive, so the greater of the two numbers, 7, is positive. Thus, $(x - 5)(x + 7) = 0$.

16. ***x* = 2 OR 13**

$x^2 - 15x = -26$

$x^2 - 15x + 26 = 0$ Add 26 to both sides so that the expression equals 0.



The numbers 2 and 13 multiply to 26 and sum to 15.

$(x - 2)(x - 13) = 0$

17. **2**

$$\text{FOIL}\left(\sqrt{8}-\sqrt{2}\right)\left(\sqrt{8}-\sqrt{2}\right)$$

$$\begin{aligned}&\text{First: } \sqrt{8}\times\sqrt{8}=8\\&\text{Outer: } \sqrt{8}\times(-\sqrt{2})=-\sqrt{16}=-4\\&\text{Inner: } (-\sqrt{2})\times\sqrt{8}=-\sqrt{16}=-4\\&\text{Last: } (-\sqrt{2})\times(-\sqrt{2})=2\\&\text{Sum of FOIL terms: } 8-4-4+2=2\end{aligned}$$

18. ***x* = 5**

$$\begin{aligned}x^2-10x+25 &= 0\\(x-5)(x-5) &= 0\\x=5\end{aligned}$$

19. ***x* = −1, 2; *x* ≠ 4**

$$\frac{(x+1)(x-2)}{x-4}=0$$

The numerator is 0 if either $(x + 1)$ or $(x - 2)$ is 0. Thus, $x = -1$ or $x = 2$. However, $x \neq 4$ because $x = 4$ would make the fraction undefined.

20. **$(2a + b)^2 = 0$**

$4a^2 + 4ab + b^2 = 0 \rightarrow (2a)^2 + 2(2a)(b) + b^2 = 0 \rightarrow (2a + b)(2a + b) = 0$

21. **$(x + 11y)^2 = 0$**

$x^2 + 22xy + 121y^2 = 0 \rightarrow x^2 + 2x(11y) + (11y)^2 = 0 \rightarrow (x + 11y)(x + 11y) = 0$

Problem Set

Solve the following problems. Distribute and factor when needed.

1. If -4 is a solution for x in the equation $x^2 + kx + 8 = 0$, what is k?

2. If 8 and -4 are the solutions for x, which of the following could be the equation?

(A) $x^2 - 4x - 32 = 0$
(B) $x^2 - 4x + 32 = 0$
(C) $x^2 + 4x - 12 = 0$
(D) $x^2 + 4x + 32 = 0$
(E) $x^2 + 4x + 12 = 0$

3. If $16 - y^2 = 10(4 + y)$, what is y?

4. If $x^2 - 10 = -1$, what is x?

5. If $x^2 - 13x = 30$, what is x?

6. If the area of a certain square (expressed in square meters) is added to its perimeter (expressed in meters), the sum is 77. What is the length of a side of the square?

7. Hugo lies on top of a building, throwing pennies straight down to the street below. The formula for the height in meters, H, that a penny falls is $H = Vt + 5t^2$, where V is the original velocity of the penny (how fast Hugo throws it as it leaves his hand in meters per second) and t is equal to the time it takes to hit the ground in seconds. The building is 60 meters high, and Hugo throws the penny down at an initial speed of 20 meters per second. How long does it take for the penny to hit the ground?

8. $\left(3 - \sqrt{7}\right)\left(3 + \sqrt{7}\right) = \ ?$

9. If $x^2 - 6x - 27 = 0$ and $y^2 - 6y - 40 = 0$, what is the maximum value of $x + y$?

10. If $x^2 - 10x + 25 = 16$, what is x?

11. $x^2 - 2x - 15 = 0$

Quantity A	**Quantity B**
x	1

12. $x^2 - 12x + 36 = 0$

Quantity A	**Quantity B**
x	6

13. $xy > 0$

Quantity A	**Quantity B**
$(x + y)^2$	$(x - y)^2$

Solutions

1. ***k* = 6**

   If −4 is a solution, then you know that $(x + 4)$ must be one of the factors of the quadratic equation. The other factor is $(x + ?)$. You know that the product of 4 and ? must be equal to 8; thus, the other factor is $(x + 2)$. You know that the sum of 4 and 2 must be equal to k. Therefore, $k = 6$.

2. **(A)**

   If the solutions to the equation are 8 and −4, the factored form of the equation is:

   $(x - 8)(x + 4) = 0$

   Distributed, this equals: $x^2 - 4x - 32 = 0$.

3. ***y* = {−4, −6}**

   Simplify and factor to solve.

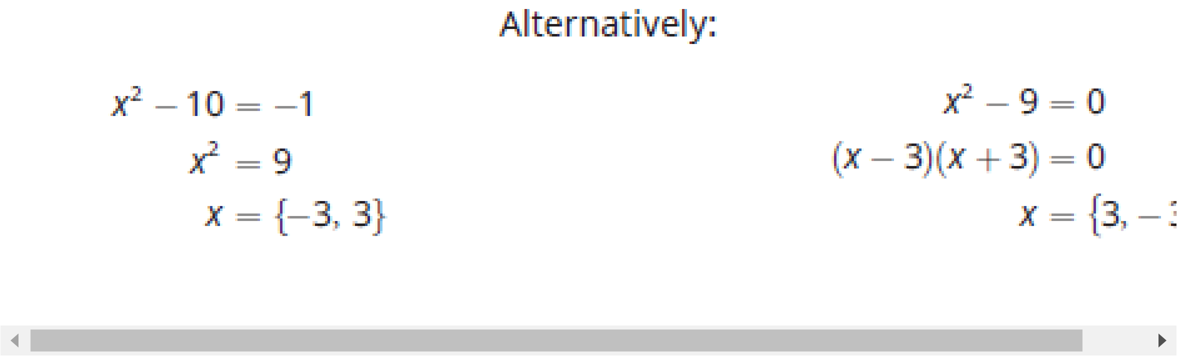
$$16 - y^2 = 10(4 + y)$$
$$16 - y^2 = 40 + 10y$$
$$y^2 + 10y + 24 = 0$$
$$(y + 4)(y + 6) = 0$$

$y + 4 = 0$ OR $y + 6 = 0$

$y = -4$ $y = -6$

Notice that it is possible to factor the left side of the equation first: $16 - y^2 = (4 + y)(4 - y)$. However, doing so is potentially dangerous: You may decide to then divide both sides of the equation by $(4 + y)$. You cannot do this, because it is possible that $(4 + y)$ equals 0 (and, in fact, for one solution of the equation, it does).

4. $\boldsymbol{x = \{-3, 3\}}$



Alternatively:

$$x^2 - 10 = -1$$
$$x^2 = 9$$
$$x = \{-3, 3\}$$

$$x^2 - 9 = 0$$
$$(x - 3)(x + 3) = 0$$
$$x = \{3, -$$

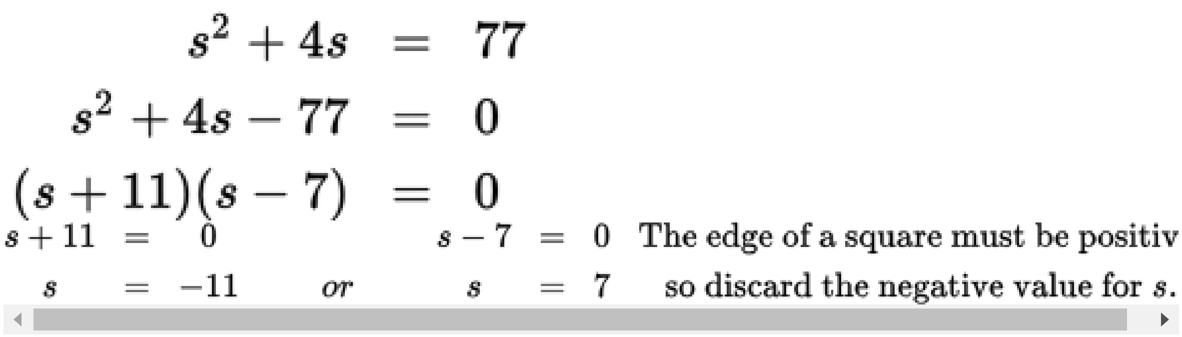
5. $\boldsymbol{x = \{15, -2\}}$

$$\begin{aligned} x^2 - 13x &= 30 \\ x^2 - 13x - 30 &= 0 \\ (x + 2)(x - 15) &= 0 \end{aligned}$$

$$\begin{array}{rclcrcl} x + 2 & = & 0 & or & x - 15 & = & 0 \\ x & = & -2 & or & x & = & 15 \end{array}$$

6. $\boldsymbol{s = 7}$

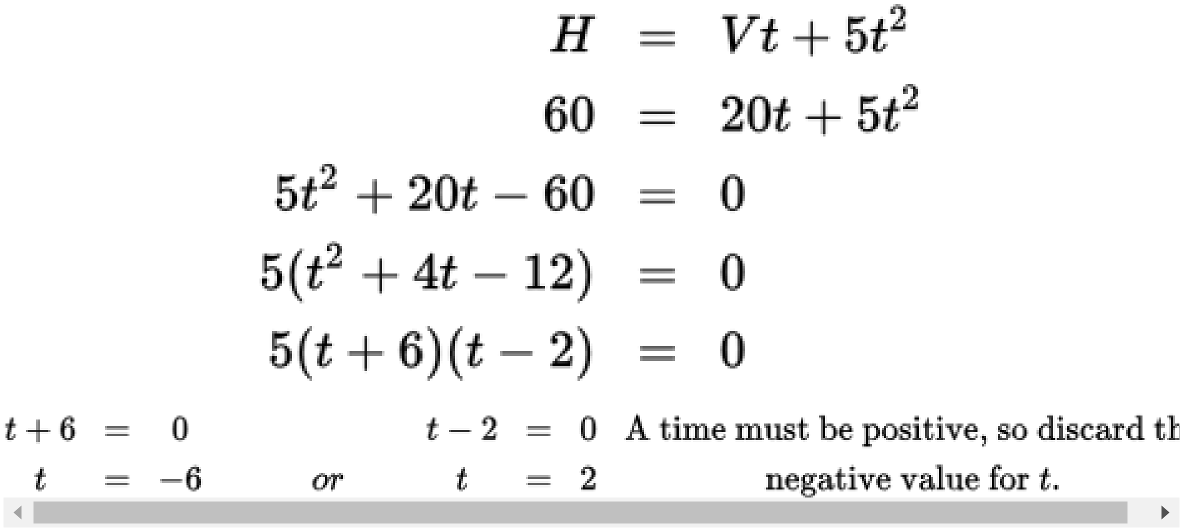
The area of the square = s^2. The perimeter of the square = $4s$:



$$\begin{aligned} s^2 + 4s &= 77 \\ s^2 + 4s - 77 &= 0 \\ (s+11)(s-7) &= 0 \end{aligned}$$

$$\begin{array}{lllllll} s+11 &=& 0 & & s-7 &=& 0 \quad \text{The edge of a square must be positiv} \\ s &=& -11 & or & s &=& 7 \quad \text{so discard the negative value for } s. \end{array}$$

7. ***t* = 2**



$$\begin{aligned} H &= Vt + 5t^2 \\ 60 &= 20t + 5t^2 \\ 5t^2 + 20t - 60 &= 0 \\ 5(t^2 + 4t - 12) &= 0 \\ 5(t+6)(t-2) &= 0 \end{aligned}$$

$$\begin{array}{lllllll} t+6 &=& 0 & & t-2 &=& 0 \quad \text{A time must be positive, so discard th} \\ t &=& -6 & or & t &=& 2 \quad \text{negative value for } t. \end{array}$$

8. **2**

   Use FOIL to simplify this product:

$$\begin{aligned} &\mathbf{F}:\ 3 \times 3 = 9 \\ &\mathbf{O}:\ 3 \times \sqrt{7} = 3\sqrt{7} \\ &\mathbf{I}: -\sqrt{7} \times 3 = -3\sqrt{7} \\ &\mathbf{L}: -\sqrt{7} \times \sqrt{7} = -7 \\ &9 + 3\sqrt{7} - 3\sqrt{7} - 7 = 2 \end{aligned}$$

9. **19**

Factor both quadratic equations. Then use the greatest possible values of x and y to find the maximum value of the sum $x + y$:

$$\begin{aligned} x^2 - 6x - 27 &= 0 \\ (x+3)(x-9) &= 0 \end{aligned} \qquad\qquad \begin{aligned} y^2 - 6y - 40 &= 0 \\ (y+4)(y-10) &= 0 \end{aligned}$$

$$\begin{aligned} x + 3 &= 0 \\ x &= -3 \end{aligned} \quad or \quad \begin{aligned} x - 9 &= 0 \\ x &= 9 \end{aligned} \quad \begin{aligned} y + 4 &= 0 \\ y &= -4 \end{aligned} \quad or \quad \begin{aligned} y - 10 &= \\ y &= \end{aligned}$$

The maximum possible value of $x + y = 9 + 10 = 19$.

10. ***x* = {1, 9}**

$$\begin{aligned} x^2 - 10x + 25 &= 16 \\ x^2 - 10x + 9 &= 0 \\ (x-9)(x-1) &= 0 \end{aligned}$$

$$\begin{aligned} x - 9 &= 0 \\ x &= 9 \end{aligned} \quad or \quad \begin{aligned} x - 9 &= 0 \\ x &= 9 \end{aligned}$$

11. **(D)**

First, factor the equation in the common information:

$x^2 - 2x - 15 = 0 \rightarrow (x - 5)(x + 3) = 0$

$x = 5$ *or* $x = -3$

$$x^2 - 2x - 15 = 0$$

Quantity A **Quantity B**

$x = 5 \text{ or } -3$	1

The value of x could be greater than or less than 1. **The relationship cannot be determined.**

12. **(C)**

First, factor the equation in the common information:

$x^2 - 12x + 36 = 0 \rightarrow (x - 6)(x - 6) = 0$
$x = 6$

$$x^2 - 12x + 36 = 0$$

Quantity A	**Quantity B**
$x = 6$	6

The two quantities are equal.

13. **(A)**

Expand the expressions in both columns:

$$xy > 0$$

Quantity A	**Quantity B**
$(x + y)^2 =$ $x^2 + 2xy + y^2$	$(x + y)^2 =$ $x^2 + 2xy + y^2$

Now subtract $x^2 + y^2$ from both columns:

$xy > 0$

Quantity A	**Quantity B**
$\begin{array}{r} x^2 + 2xy + y^2 \\ -\,(x^2 \quad + \quad y^2) \\ \hline 2xy \end{array}$	$\begin{array}{r} x^2 + 2xy + y^2 \\ -\,(x^2 \quad + \quad y^2) \\ \hline 2xy \end{array}$

Because *xy* is positive, Quantity A will be positive, regardless of the values of *x* and *y*. Similarly, Quantity B will always be negative, regardless of the values of *x* and *y*.

Quantity A is greater.

Chapter 3

INEQUALITIES & ABSOLUTE VALUE

In This Chapter…

Inequalities

Solving Inequalities

Absolute Value—Distance on the Number Line

Putting Them Together: Inequalities and Absolute Values

Manipulating Compound Inequalities

Using Extreme Values

Optimization Problems

Chapter 3

Inequalities & Absolute Value

Inequalities

Earlier you explored how to solve equations. Now look at how you can solve *inequalities*.

Inequalities are expressions that use <, >, ≤, or ≥ to describe the relationship between two values.

Examples of inequalities:

$5 > 4$ $y \leq 7$ $x < 5$ $2x + 3 \geq 0$

The following table illustrates how the various inequality symbols are translated. Notice that when inequalities are translated, you read from left to right:

$x < y$	x is less than y.	
$x > y$	x is greater than y.	
$x \leq y$	x is less than or equal to y.	x is at most y.

$x \geq y$	x is greater than or equal to y.	x is at least y.

You can also have two inequalities in one statement (sometimes called **compound inequalities**):

$9 < g < 200$	This translates to 9 is less than g, and g is less than 200.
$-3 < y \leq 5$	Now −3 is less than y, and y is less than or equal to 5.
$7 \geq x > 2$	Here 7 is greater than or equal to x, and x is greater than 2.

To visualize an inequality, it is helpful to represent it on a number line:

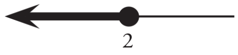
Example 1

$y > 5$

Note: The 5 is *not* included in the line (as shown by the circle *around* 5), because it is not a part of the solution—y is greater than 5, but not equal to 5.

Example 2

$b \leq 2$



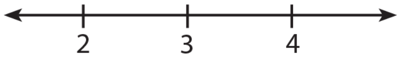
Here, the 2 is included in the solution, (as shown by the filled circle at 2) because b can equal 2.

Visually, any number covered by the black arrow will make the inequality true and so is a solution to the inequality. Conversely, any number not covered by the black arrow will make the inequality untrue and is not a solution.

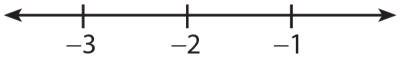
Check Your Skills

Represent the following equations on the number line provided.

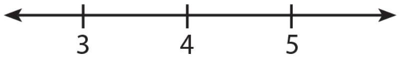
1. $x > 3$



2. $b \geq -2$



3. $y = 4$



Translate the following into inequality statements.

4. z is greater than v.

5. The total amount is greater than $2,000.

Solving Inequalities

What does it mean to "solve an inequality"?

You may be asking yourself, "I know what it means to solve an equation (such as $x = 2$), but what does it mean to solve an inequality?" Essentially, the principle is the same.

A solution is a number that makes an equation or inequality true. When you plug a solution back into the original equation or inequality, you get a *true statement*. This idea works the same for both equations and inequalities.

However, equations have only one, or just a few, values as solutions, but inequalities give a whole *range* of values as solutions—way too many to list individually.

Here's an example to help illustrate:

Equation: $x + 3 = 8$	**Inequality: $x + 3 < 8$**
The solution to $x + 3 = 8$ is $x = 5$. 5 is the *only* number that will	The solution to $x + 3 < 8$ is $x < 5$. Now, 5 itself is not a solution because $5 + 3 < 8$ is not a true statement. However, 4 is a solution because $4 + 3 < 8$ is true. For that matter, 4.99, 3, 2, 2.87, −5, and −100 are all also solutions. And the list goes on. Whichever of the correct answers you plug in, you need to arrive at something that looks like:

make the equation true.	
Plug back in to check:	
$5 + 3 = 8$. True.	(Any number less than 5) $+ 3 < 8$. True.

Check Your Skills

6. Which of the following numbers are solutions to the inequality $x < 10$?

   Indicate all that apply.

   A -3
   B 2.5
   C $-3/2$
   D 9.999

CLEANING UP INEQUALITIES

As with equations, your objective is to isolate the variable on one side of the inequality. When the variable is by itself, it is easiest to see what the solution (or range of solutions) really is. Although $2x + 6 < 12$ and $x < 3$ provide the same information (the second inequality is a simplified form of the first), you understand the full range of solutions much more easily when you look at the second inequality, which literally tells you that “*x* is less than 3.”

Fortunately, the similarities between equations and inequalities don’t end there—the techniques you will be using to clean up inequalities are the same that you used to clean up equations. (One important difference will be discussed shortly.)

INEQUALITY ADDITION AND SUBTRACTION

If you were told that $x = 5$, what would $x + 3$ equal? $x + 3 = (5) + 3$, or $x + 3 = 8$. In other words, if you add the same number to both sides of an equation, the equation is still true.

The same holds true for inequalities. If you add or subtract the same number from both sides of a true inequality, the inequality remains true:

<u>Example 1</u>

$$\begin{array}{rcl} a - 4 & > & 6 \\ +4 & & +4 \\ \hline a & > & 10 \end{array}$$

<u>Example 2</u>

$$\begin{array}{rcl} y + 7 & < & 3 \\ -7 & & -7 \\ \hline y & < & -4 \end{array}$$

You can also add or subtract variables from both sides of an inequality. There is no difference between adding/subtracting numbers and adding/subtracting variables:

$$\begin{array}{rcr} x+5 & = & 8 \\ -5 & & -5 \\ \hline x & = & 3 \end{array}$$

Check Your Skills

Isolate the variable in the following inequalities.

7. $x - 6 < 13$

8. $y + 11 \geq -13$

9. $x + 7 > 7$

INEQUALITY MULTIPLICATION AND DIVISION

You can also use multiplication and division to isolate the variables, as long as you recognize a very important distinction. *If you multiply or divide by a negative number, you must switch the direction of the inequality sign.* If you are multiplying or dividing by a positive number, the direction of the sign stays the same.

Here are a couple of examples to illustrate.

Multiplying or dividing by a *positive* number—the sign stays the same.

Example 1		Example 2	
$2x > 10$		$z/3 \leq 2$	
$2x/2 > 10/2$	Divide each side by 2.	$z/3 \times (3) \leq 2 \times (3)$	Multiply each side by 3.
$x > 5$		$z \leq 6$	

In both instances, the sign remains the same because you are multiplying or dividing by a positive number.

Multiplying or dividing by a *negative* number—switch the sign.

Example 1		Example 2	
$-2x > 10$		$-4b \geq -8$	
$-2x/-2 < 10/-2$	Divide each side by −2. Switch the sign.	$-4b/-4 <= -8/-4$	Divide each side by −4. Flip the inequality sign.
$x < -5$		$b \leq 2$	

Why do you do this? Take a look at the following example that illustrates why you need to switch the signs when multiplying or dividing by a negative number:

Start with a TRUE Statement: 5 < 7	
Incorrect if you *don't* switch	**Switch the sign—Correct!**

Start with a TRUE Statement: $5 < 7$			
Incorrect if you *don't* switch		**Switch the sign—Correct!**	
$(-1) \times 5 < (-1) \times 7$ $-5 < -7$	Multiply both sides by -1. NOT TRUE	$(-1) \times 5 > (-1) \times 7$ $-5 > -7$	Multiply both sides by -1 AND switch the sign. STILL TRUE

In each case, you begin with a true inequality statement, $5 < 7$, and then multiply by -1. You see that you have to switch the sign for the inequality statement to remain true.

What about multiplying or dividing an inequality by a *variable?* The short answer is…**try not to do it**! The issue is that you don't know the sign of the "hidden number" that the variable represents. If the variable logically can't be negative (e.g., it counts people or measures a length), then you can go ahead and multiply or divide.

If the variable must be negative, then you are also free to multiply or divide—just remember to flip the sign. However, if you don't know whether the variable is positive or negative, try to work through the problem with the inequality as is. (If the problem is a Quantitative Comparison, consider whether not knowing the sign of the variable you want to multiply or divide by means that the answer is (D).)

Check Your Skills

Isolate the variable in each equation.

10. $x + 3 \geq -2$

11. $-2y < 8$

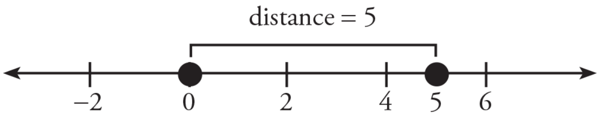
12. $a + 4 \geq 2a$

Absolute Value—Distance on the Number Line

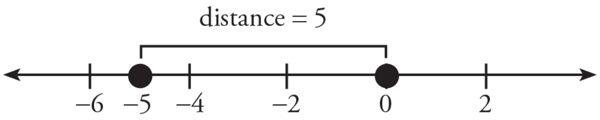
The GRE adds another level of difficulty to equations and inequalities in the form of *absolute value.*

The “absolute value” of a number describes how far that number is away from 0. It is the distance between that number and 0 on a number line. The symbol for absolute value is |number|. For instance, the absolute value of −5 is written as |−5|.

Example 1: The absolute value of 5 is 5:



Example 2: The absolute value of −5 is also 5:



When you face an expression like |4 − 7|, treat the absolute value symbol like parentheses. Solve the arithmetic problem inside first, and then find the absolute value of the answer. In this case, 4 − 7 = −3, and −3 is three units from zero, so |4 − 7| = |−3| = 3.

Check Your Skills

Mark the following expressions as True or False.

13. |3| = 3

14. |−3| = −3

15. |3| = −3

16. |−3| = 3

17. |3 – 6| = 3

18. |6 – 3|= −3

SOLVING ABSOLUTE VALUE EQUATIONS

On the GRE, some absolute value equations place a variable inside the absolute value signs:

Example: $|y| = 3$

What's the trap here? The trap is that there are two numbers, 3 and −3, that are 3 units away from 0. That means both of these numbers could be possible values for y. So how do you figure that out? Here, you can't. All you can say is that y could be either the positive value or the negative value; y is either 3 or −3.

When there is a variable inside an absolute value, you should look for the variable to have two possible values. Although you will not always be able to determine which of the two is the correct value, it is important to be able to find both values. Following is a step-by-step process for finding all solutions to an equation that contains a variable inside an absolute value:

$\|y\| = 3$	Step 1: Isolate the absolute value expression on one side of the equation. In this case, the absolute value expression is already isolated.
$+(y) = 3$ or $-(y) = 3$	Step 2: Take what's inside the absolute value sign and set up two equations. The first sets the positive value equal to the other side of the equation, and the second sets the negative value equal to the other side.
$y = 3$ or $-y = 3$	Step 3: Solve both equations.
$y = 3$ or $y = -3$	Note: There are two possible values for y.

Sometimes people take a shortcut and go right to “*y* equals plus or minus 3.” This shortcut works as long as the absolute value expression is by itself on one side of the equation.

Here’s a slightly more difficult problem, using the same technique:

Example: $6 \times |2x + 4| = 30$

To solve this, you can use the same approach:

$6 \times |2x + 4| = 30$ $|2x + 4| = 5$	Step 1: Isolate the absolute value expression on one side of the equation or inequality.
$(2x + 4) = 5$ or $-(2x + 4) = 5$ $2x + 4 = 5$ or $-2x - 4 = 5$	Step 2: Set up two equations—the positive and the negative values are set equal to the other side.
$2x = 1$ or $-2x = 9$	Step 3: Solve both equations/inequalities.
$x = \frac{1}{2}$ or $x = -\frac{9}{2}$	Note: There are two possible values for *x*.

Check Your Skills

Solve the following equations with absolute values in them.

19. $|a| = 6$

20. $|x + 2| = 5$

21. $|3y - 4| = 17$

22. $4|x + \frac{1}{2}| = 18$

Putting Them Together: Inequalities and Absolute Values

Some problems on the GRE include both inequalities and absolute values. You can solve these problems by combining what you have learned about solving inequalities with what you have learned about solving absolute values:

Example 1: $|x| \geq 4$

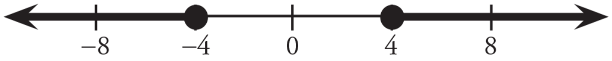
Even though you're now dealing with an inequality, and not an equals sign, the basic process is the same. The absolute value is already isolated on one side, so now you need to set up your two equations or, in this case, inequalities. The first inequality replaces the absolute value with the *positive* of what's inside, and the second replaces the absolute value with the *negative* of what's inside:

$+(x) \geq 4$ *or* $-(x) \geq 4$

Now that you have your two equations, isolate the variable in each equation:

$$\begin{array}{l|ll} +(x) \geq 4 & -(x) \geq 4 & \\ x \geq 4 & -x \geq 4 & \text{Divide by } -1. \\ & x \leq -4 & \text{Remember to flip the sign when dividing by a negative.} \end{array}$$

So the two solutions to the original equation are $x \geq 4$ and $x \leq -4$. Here it is represented on a number line:



As before, any number that is covered by the black arrow will make the inequality true. Because of the absolute value, there are now two arrows instead of one, but nothing else has changed. Any number to the left of −4 will make the inequality true, as will any number to the right of 4 (and don't forget that 4 and −4 themselves work as well here).

Looking back at the inequality $|x| \geq 4$, you can now interpret it in terms of distance. $|x| \geq 4$ means "x is at least 4 units away from zero, in either direction." The black arrows indicate all numbers for which that statement is true.

Example 2: $|x + 3| < 5$

Once again, the absolute value is already isolated on one side, so now you need to set up the two inequalities. The first inequality replaces the absolute value with the *positive* of what's inside, and the second replaces the absolute value with the *negative* of what's inside:

$+(x + 3) < 5$ *and* $-(x + 3) < 5$

Next, isolate the variable in each equation:

$$\begin{array}{l|l} x+3<5 & -x-3<5 \\ x<2 & -x<8 \\ & x>-8 \end{array}$$

So the two equations are $x < 2$ and $x > -8$. But now something curious happens if those two equations are plotted on the number line:

It seems like every number should be a solution to the equation However, if you start testing numbers, that isn't the case. Test $x = 5$, for example. Is $|5 + 3| < 5$? No, it isn't. As it turns out, the only numbers that make the original inequality true are those that are true for *both* inequalities. Your number line should look like this:

In the first example, it was the case that x could be greater than or equal to 4 *or* less than or equal to −4. For this example, however, it seems to make more sense to say that x is greater than −8 *and* less than 2.

The inequality you just graphed means "($x + 3$) is less than five units away from from 0, in either direction." The shaded segment indicates all numbers x for which this is true. As the inequalities become more complicated, don’t worry about interpreting their meaning—simply solve them algebraically.

To summarize, when representing inequalities on the number line, absolute value expressions where variables are *greater than some quantity* will show up *as two ranges in opposite directions* (or “double arrows”); however, absolute value expressions where variables are *less than some quantity* will show up *as a single range* (or “line segment”).

Check Your Skills

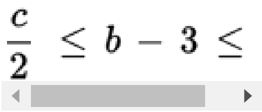
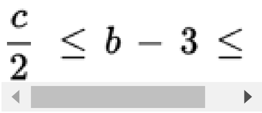
23. $|x + 1| > 2$

24. $|-x - 4| \geq 8$

25. $|x - 7| < 9$

Manipulating Compound Inequalities

Sometimes a problem with compound inequalities will require you to manipulate the inequalities to solve the problem. You can perform operations on a compound inequality as long as you remember to perform those operations on **every term** in the inequality, not just the outside terms. For example:

$x + 3 < y < x + 5$	$\xrightarrow{\times}$	$x < y < x + 2$	**Wrong**: You must subtract 3 from every term in the inequality.
$x + 3 < y < x + 5$	$\xrightarrow{\checkmark}$	$x < y - 3 < x + 2$	CORRECT
 $\frac{c}{2} \leq b - 3 \leq$	$\xrightarrow{\times}$	$c \leq b - 3 \leq d$	**Wrong:** You must multiply by 2 in every term in the inequality.
 $\frac{c}{2} \leq b - 3 \leq$	$\xrightarrow{\checkmark}$	$c \leq 2b - 6 \leq d$	Correct

> If 1 > 1 – ab > 0, which of the following must be true?

> Indicate <u>all</u> that apply.
>
> A $\quad \frac{a}{b} > 0$
>
> B $\quad \frac{a}{b} > 0$
>
> C $\quad ab < 1$

You can manipulate the original compound inequality as follows, making sure to perform each manipulation on every term:

$1 > 1 - ab > 0$	
$0 > \; -ab > -1$	Subtract 1 from all three terms.
$0 < \; ab < 1$	Multiply all three terms by −1 and flip the inequality signs.

Therefore, you know that $0 < ab < 1$. This tells you that *ab* is positive, so $\frac{a}{b}$ must be positive (*a* and *b* have the same sign). Therefore, (A) must be true. However, you do not know whether $\frac{a}{b} < 1$, so (B) is not necessarily true. But you do know that *ab* must be less than 1, so (C) must be true. Therefore, the correct answers are (A) and (C).

Check Your Skills

> 26. Find the range of values for *x* if $-7 < 3 - 2x < 9$.

Using Extreme Values

One effective technique for solving GRE inequality problems is to focus on the **Extreme Values** of a given inequality. This is particularly helpful when solving the following types of inequality problems:

- Problems with multiple inequalities where the question involves the potential range of values for variables in the problem
- Problems involving both equations and inequalities

INEQUALITIES WITH RANGES

Whenever a question asks about the possible range of values for a problem, consider using Extreme Values:

> If $0 \le x \le 3$ and $y < 8$, which of the following could NOT be the value of xy?
>
>)
>
> 0
>
>)
>
> 8
>
>)
>
> 12
>
>)
>
> 16

)

24

To solve this problem, consider the **Extreme Values** of each variable:

Extreme Values for *x*	**Extreme Values for *y***
The lowest value for *x* is **0**.	The lowest value for *y* is negative infinity.
The highest value for *x* is **3**.	The highest value for *y* is **less than 8**.

(*y* cannot be 8, therefore, this upper limit is termed “less than 8” or “LT8” for shorthand.)

What is the lowest value for *xy*? Plug in the lowest values for both *x* and *y*. In this problem, *y* has no lower limit, so there is no lower limit to *xy*.

What is the highest value for *xy*? Plug in the highest values for both *x* and *y*. In this problem, the highest value for *x* is **3**, and the highest value for *y* is **LT8**.

Multiplying these two extremes together yields: 3 × LT8 = LT24. Notice that you can multiply LT8 by another number (if that other number is positive) just as though it were 8. You just have to remember to include the “LT” tag on the result.

Because the upper extreme for *xy* is less than 24, *xy* CANNOT be 24, and the answer is **(E)**.

Notice that you would run into trouble if x did not have to be non-negative. Consider this slight variation:

> If $-1 \leq x \leq 3$ and $y < 8$, what is the possible range of values for xy?

Because x could be negative and because y could be a very negative number, there is no longer an upper extreme on xy. For example, if $x = -1$ and $y = -1{,}000$, then $xy = 1{,}000$. Obviously, even greater positive results are possible for xy if both x and y are very negative. Likewise, because x can be positive and y can be infinitely negative, xy can be infinitely negative. Therefore, xy can equal any number.

Check Your Skills

27. If $-4 < a < 4$ and $-2 < b < -1$, which of the following could NOT be the value of ab?

(A) -3
(B) 0
(C) 4
(D) 6
(E) 9

Optimization Problems

Related to extreme values are problems involving optimization: specifically, minimization or maximization problems. In these problems, you need to **focus on the largest and smallest possible values for each of the variables**, as some combination of them will usually lead to the largest or smallest possible result:

> If $-7 \le a \le 6$ and $-7 \le b \le 8$, what is the maximum possible value for ab?

Once again, you are looking for a maximum possible value, this time for ab. You need to test the extreme values for a and for b to determine which combinations of extreme values will maximize ab:

Extreme Values for *a*	**Extreme Values for *b***
The lowest value for a is -7.	The lowest value for b is -7.
The highest value for a is 6.	The highest value for b is 8.

Now consider the different extreme value scenarios for a, b, and ab:

a	***b***	***ab***
Min **−7**	Min **−7**	**$(-7) \times (-7) = 49$**

a	***b***	***ab***
Min −7	Max 8	(−7) × 8 = −56
Max 6	Min −7	6 × (−7) = −42
Max 6	Max 8	6 × 8 = 48

This time, *ab* is maximized when you take the *negative* extreme values for both *a* and *b*, resulting in *ab* = 49. Notice that you could have focused right away on the first and fourth scenarios, because they are the only scenarios that produce positive products.

> If $-4 \leq m \leq 7$ and $-3 < n < 10$, what is the maximum possible integer value for $m - n$?

Again, you are looking for a maximum possible value, this time for $m - n$. You need to test the extreme values for *m* and for *n* to determine which combinations of extreme values will maximize $m - n$:

Extreme Values for *m*	**Extreme Values for *n***
The lowest value for *m* is −4. The highest value for *m* is 7.	The lowest value for *n* is greater than −3. The highest value for *n* is less than 10.

Now consider the different extreme value scenarios for *m*, *n*, and $m - n$:

m	***n***	***m* − *n***

m	***n***	***m − n***
Min −4	Min GT(−3)	(−4) − GT(−3) = LT(−1)
Min −4	Max LT10	(−4) − LT10 = GT(−14)
Max 7	Min **GT(−3)**	**7** − **GT(−3)** = **LT10**
Max 7	Max LT10	7 − LT10 = GT(−3)

Thus, $m - n$ is maximized when you take the *maximum* extreme for m and the *minimum* extreme for n, resulting in $m - n$ = less than 10. The largest integer less than 10 is 9, so the correct answer is $\boldsymbol{m - n = 9}$. Look at another, slightly different, problem:

> If $x \geq 4 + (z + 1)^2$, what is the minimum possible value for x?

The key to this type of problem—where you need to maximize or minimize when one of the variables has an even exponent—is to recognize that the squared term will be minimized when it is set equal to 0. Therefore, you need to set $(z + 1)^2$ equal to 0:

$$x \geq 4 + (0)$$

Therefore, 4 is the minimum possible value for x.

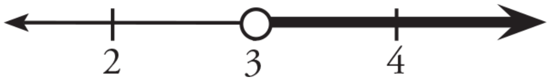
Check Your Skills

28. If $-1 \leq a \leq 4$ and $-6 \leq b \leq -2$, what is the minimum value for $b - a$?

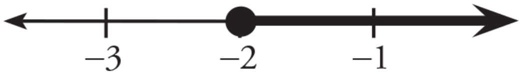
29. If $(x + 2)^2 \leq 2 - y$, what is the maximum possible value for y?

Check Your Skills Answer Key

1.



2.



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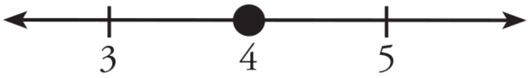


4. ***z* > *v***

5. Let *a* = total amount.

***a* > $2,000**

6. **(A), (B), (C), (D)**

All of these numbers are to the left of 10 on the number line.

7. $x - 6 < 13$

$\boldsymbol{x < 19}$

8. $y + 11 \geq -13$

$\boldsymbol{y \geq -24}$

9. $x + 7 > 7$

$\boldsymbol{x > 0}$

10. $x + 3 \geq -2$

$\boldsymbol{x \geq -5}$

11. $-2y < 8$

$\boldsymbol{y > -4}$

12. $a + 4 \geq 2a$

$\boldsymbol{4 \geq a}$

13. **True**

14. **False** *(Note that absolute value is always positive!)*

15. **False**

16. **True**

17. **True** ($|3 - 6| = |-3| = 3$)

18. **False**

19. $|a| = 6$

$\boldsymbol{a = 6}$ *or* $\boldsymbol{a = -6}$

20. **$\boldsymbol{x}$ = 3 or −7**

$|x + 2| = 5$

$+(x + 2) = 5$ *or* $-(x + 2) = 5$

$x + 2 = 5$ *or* $-x - 2 = 5$

$x = 3$ *or* $-x = 7$

$x = -7$

21. **$\boldsymbol{y}$ = 7 or $\boldsymbol{-\frac{25}{2}}$**

$|3y - 4| = 17$

$+(3y - 4) = 17$ *or* $-(3y - 4) = 17$

$3y - 4 = 17$ *or* $-3y + 4 = 17$

$3y = 21$ *or* $-3y = 13$

$y = 7$ *or* $y = -\frac{25}{2}$

22. **$\boldsymbol{x}$ = 4 or −5**

or $4|x + \frac{1}{2}| = 18$

$+ (x + \frac{1}{2}) = 4\frac{1}{2}$ *or* $-(x + \frac{1}{2}) = 4\frac{1}{2}$

$x + \frac{1}{2} = 4\frac{1}{2}$ *or* $-x - \frac{1}{2} = 4\frac{1}{2}$

$x = 4$ *or* $-x = 5$

$x = -5$

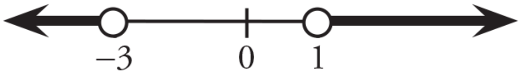
23. $|x + 1| > 2$

$+ (x + 1) > 2$ *or* $-(x + 1) > 2$

$x + 1 > 2$ *or* $-x - 1 > 2$

$x > 1$ *or* $-x > 3$

$x < -3$



$\boldsymbol{x < -3}$ **or** $\boldsymbol{x > 1}$

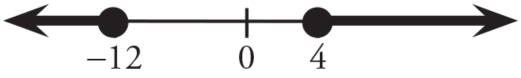
24. $|-x - 4| \geq 8$

$+ (-x - 4) \geq 8$ *or* $-(-x - 4) \geq 8$

$-x - 4 \geq 8$ *or* $x + 4 \geq 8$

$-x \geq 12$ *or* $x \geq 4$

$x \leq -12$



$\boldsymbol{x \le -12}$ **or** $\boldsymbol{x \ge 4}$

25. $|x - 7| < 9$

$+(x - 7) < 9$ *and* $-(x - 7) < 9$

$x - 7 < 9$ *and* $-x + 7 < 9$

$x < 16$ *and* $-x < 2$

$x > -2$

$x > -2$ *and* $x < 16$, $\boldsymbol{-2 < x < 16}$

26. $\boldsymbol{-3 < x < 5}$

$-7 < 3 - 2x < 9$

$-10 < -2x < 6$ Subtract 3 from all three terms.

$5 > x > -3$ Divide all three terms by −2, and flip the inequality signs.

or $\boldsymbol{-3} < x < \boldsymbol{5}$

27. **(E)**

Extreme values for *a* are greater than −4 and less than 4. Extreme values for *b* are greater than −2 and less than −1.

Note that *a* can be positive, zero, or negative, while *b* can only be negative, so *ab* can be positive, zero, and negative.

The most negative *ab* can be is (less positive than 4) × (less negative than −2) = less negative than −8.

The most positive *ab* can be is (less negative than −4) × (less negative than −2) = less positive than 8.

28. **−10**

a	***b***	***b* − *a***
−1	−6	−6 − (−1) = −5
−1	−2	−2 − (−1) = −1
4	−6	−6 − 4 = −10
4	−2	−2 − 4 = −6

29. **2**

$(x + 2)^2 \le 2 - y$	
$y + (x + 2)^2 \le 2$	Add y to both sides.
$y \le 2 - (x + 2)^2$	Subtract $(x + 2)^2$ from both sides.

Note that y is maximized when $(x + 2)^2$ is minimized. The smallest possible value for $(x + 2)^2$ is 0, when $x = -2$. When $(x + 2)^2 = 0$, $y = 2$.

Problem Set

$$4x - 12 \geq x + 9$$

1.

Quantity A	**Quantity B**
x	6

2. Which of the following is equivalent to $-3x + 7 \leq 2x + 32$?

(A) $x \geq -5$
(B) $x \geq 5$
(C) $x \leq 5$
(D) $x \leq -5$

3. If $G^2 < G$, which of the following could be G?

(A) 1

(B) $\frac{23}{7}$

(C) $\frac{23}{7}$

(D) −4

(E) −2

4. If $|A| > 19$, which of the following *cannot* be equal to A?

(A) 26
(B) 22
(C) 18
(D) −20
(E) −24

5. If $B^3A < 0$ and $A > 0$, which of the following must be negative?

(A) AB

(B) B^2A

(C) B^4

(D) $\frac{A}{B^2}$

(E) $-\frac{B}{A}$

6. $|2x - 5| \le 7$

Quantity A	**Quantity B**
x	3

7. $1 \le x \le 5$ and $1 \ge y \ge -2$

Quantity A	**Quantity B**
xy	-10

8. $x = 4$

Quantity A	**Quantity B**
$\|2 - x\|$	2

Solutions

1. **(A)**

$$\begin{aligned} 4x - 12 &\geq x + 9 \\ 3x &\geq 21 \\ x &\geq 7 \end{aligned}$$

$$\text{If } x \geq 7\text{, then } x > 6.$$

2. **(A)**

$$\begin{aligned} -3x + 7 &\leq 2x + 32 \\ -5x &\leq 25 \\ x &>= -5 \end{aligned}$$

When you divide by a negative number, you must reverse the direction of the inequality symbol.

3. **(C)**

If $G^2 < G$, then G must be positive (because G^2 will never be negative), and G must be less than 1, because otherwise, $G^2 > G$. Thus, $0 < G < 1$. You can eliminate choices (D) and (E) because they violate the condition that G be positive. Then test choice (A): 1 is not less than 1, so you can eliminate (A). Choice (B) is greater than 1, so only choice (C) satisfies the inequality.

4. **(C)**

If $|A| > 19$, then $A > 19$ OR $A < -19$. The only answer choice that does not satisfy either of these inequalities is choice (C), 18.

5. **(A)**
If *A* is positive, B^3 must be negative. Therefore, *B* must be negative. If *A* is positive and *B* is negative, the product *AB* must be negative.

6. **(D)**
To evaluate the absolute value, set up two equations and isolate *x*:

$+(2x-5) \le 7$	*and*	$-(2x-5) \le 7$
$2x-5 \le 7$		$-2x+5 \le 7$
$2x \le 12$		$-2x \le 2$
$x \le 6$		$x \ge -1$

Combine the information from the two equations:

$$|2x-5| \le 7$$

Quantity A	**Quantity B**
$-1 \le x \le 6$	3

There are possible values of *x* greater than *and* less than 3. **The relationship cannot be determined.**

7. **(D)**
To find the minimum and maximum values of *xy*, test the boundaries of *x* and *y*:

x	y	xy
Min 1	Min −2	$(1) \times (-2) = -2$

Min 1	Max 1	$(1) \times (1) = 1$
Max 5	Min −2	$(5) \times (-2) = -10$
Max 5	Max 1	$(5) \times (1) = 5$

Combine the information from the chart to show the range of *xy*:

$$1 \leq x \leq 5 \text{ and } 1 \geq y \geq -2$$

Quantity A	**Quantity B**
$-10 \leq xy \leq 5$	-10

Quantity A can be either greater than or equal to −10. **The relationship cannot be determined.**

8. **(C)**

Plug in 4 for *x* in Quantity A.

$$x = 4$$

Quantity A	**Quantity B**
$\lvert 2 - x \rvert =$ $\lvert 2 - (4) \rvert = \lvert -2 \rvert = 2$	2

The two quantities are equal.

Chapter 4

FORMULAS & FUNCTIONS

In This Chapter...

Plug in Formulas
Strange Symbol Formulas
Formulas with Unspecified Amounts
Sequence Formulas: Direct and Recursive
Sequence Problems
Sequences and Patterns
Functions

Chapter 4

Formulas & Functions

Plug In Formulas

The most basic GRE formula problems provide you with a formula and ask you to solve for one of the variables in the formula by plugging in given values for the other variables. For example:

> The formula for determining an individual's comedic aptitude, C, on a given day is defined as $\frac{QL}{J}$, where J represents the number of jokes told, Q represents the overall joke quality on a scale of 1 to 10, and L represents the number of individual laughs generated. If Nicole told 12 jokes, generated 18 laughs, and earned a comedic aptitude of 10.5, what was the overall quality of her jokes?

Solving this problem involves plugging the given values into the formula to solve for the unknown variable Q:

$$C = \frac{QL}{J} \rightarrow 10.5 = \frac{18Q}{12} \rightarrow 10.5(12) = 18Q \rightarrow Q = \frac{10.5(12)}{18} \rightarrow Q = 7$$

The quality of Nicole's jokes was rated a 7.

Notice that you will typically have to do some rearrangement after plugging in the numbers, to isolate the desired unknown. The actual computations are not complex. What makes formula problems tricky is the unfamiliarity of the given formula, which may seem to come from "out of the blue." Do not be intimidated. Simply write the equation down, plug in the numbers carefully, and solve for the required unknown. Alternatively, you can rearrange the original equation to solve for the unknown *before* plugging in the numbers. Either order works fine.

Be sure to write the formula as a part of an equation. For instance, do not just write "$\frac{QL}{J}$" on your paper. Rather, write "$C = \frac{QL}{J}$." Look for language such as "is defined as" to identify what equals what.

Check Your Skills

1. The baking time in minutes for a certain cake is defined as $\frac{Vk}{T}$, where V is the volume of the cake in inches cubed, T is the oven temperature in degrees Fahrenheit, and k is a constant. If the baking time was 30 minutes at 350 degrees Fahrenheit for a cake with a volume of 150 cubic inches, what is the value of constant k?

Strange Symbol Formulas

Another type of GRE formula problem involves the use of strange symbols. In these problems, the GRE introduces an arbitrary symbol, which defines a certain procedure. These problems may look confusing because of the unfamiliar symbols. However, the symbol is *irrelevant*. All that is important is that you carefully follow each step in the procedure that the symbol indicates.

A technique that can be helpful is to break the operations down one-by-one and say them aloud (or in your head)—to “hear” them explicitly. Here are some examples:

Formula Definition	**Step-by-Step Breakdown**
$x \heartsuit y = x^2 + y^2 - xy$	“The first number squared, plus the second number squared, minus the product of the two …”
$s \circ t = (s - 2)(t + 2)$	“Two less than the first number times two more than the second number …”
$\boxed{x}$ is defined as the product of all integers smaller than x but greater than 0 …	“… x minus 1, times x minus 2, times x minus 3 … Aha! So this is $(x - 1)$ factorial!”

Notice that it can be helpful to refer to the variables as “the first number,” “the second number,” and so on. In this way, you use the physical position of the numbers to keep them straight in relation to the strange symbol.

Now that you have interpreted the formula step-by-step and can understand what it means, you can calculate a solution for the formula with actual numbers. Consider the following example:

> $\boldsymbol{W \,\psi\, F} = \left(\sqrt{W}\right)^{F}$ for all integers W and F. What is $4\ \psi\ 3$?

The symbol ψ between two numbers indicates the following procedure: Take the square root of the *first* number and then raise that value to the power of the *second* number:

$$4 \,\psi\, 3 = \left(\sqrt{4}\right)^{3} = 2^{3} = 8$$

Watch for symbols that *invert* the order of an operation. It is easy to automatically translate the function in a “left to right” manner even when that is *not* what the function specifies:

> $\boldsymbol{W \,\Phi\, F} = \left(\sqrt{F}\right)^{W}$ for all integers W and F. What is $4\ \Phi\ 9$?

It would be easy in this example to mistakenly calculate the formula in the same way as the first example. However, notice that the order of the operation is *reversed*—you need to take the square root of the *second* number, raised to the power of the *first* number:

$$\mathbf{4 \,\Phi\, 9} = \left(\sqrt{\mathbf{9}}\right)^{4} = 3^{4} = 81$$

More challenging strange-symbol problems require you to use the given procedure more than once. For example:

> $\boldsymbol{A \,\Phi\, B} = \left(\sqrt{B}\right)^{A}$ for all integers A and B. What is 2 Φ(3 Φ 16) ?

Always perform the procedure inside the parentheses FIRST:

$$\mathbf{3 \,\Phi\, 16} = \left(\sqrt{16}\right)^{3} = 4^{3} = 64$$

Now you can rewrite the original formula as follows: 2 Φ (3 Φ 16) = 2 Φ 64.

Performing the procedure a second time yields the answer:

$$\mathbf{3 \,\Phi\, 16} = \left(\sqrt{16}\right)^{3} = 4^{3} = 64$$

Check Your Skills

2. $A \,\Delta\, B = A^{B} + B$ for all integers A and B. What is the value of $-2\Delta(3 \,\Delta\, 1)$?

3. $\boldsymbol{s \,\lambda\, t} = \frac{t}{s} + \frac{s}{t}$ for all integers s and t. What is the value of $2\lambda 16$?

Formulas with Unspecified Amounts

Some of the most challenging formula problems on the GRE are those that involve unspecified amounts. Typically, these questions focus on the increase or decrease in the value of a certain formula, given a change in the value of the variables. Just as with other GRE problems with unspecified amounts, solve these problems by picking numbers.

> If the length of the side of a cube decreases by two-thirds its original value, by what percent will the volume of the cube decrease?

First consider the formula involved here. The volume of a cube is defined by the formula $V = s^3$, where s represents the length of a side. Then pick a number for the length of the side of the cube.

Say the cube has a side of 3 units. Note that this is a “smart” number to pick because it is divisible by 3 (the denominator of two-thirds).

Then its volume equals $s^3 = 3 \times 3 \times 3 = 27$.

If the cube’s side decreases by two-thirds, its new length is $\mathbf{3 - \frac{2}{3}(3) = 1}$ unit.

Its new volume equals $s^3 = 1 \times 1 \times 1 = 1$.

You determine percent decrease as follows:

$$\frac{\text{change}}{\text{original}} = \frac{27-1}{27} = \frac{26}{27} \approx 0.963 = 96.3\%\,\text{decrease}$$

Check Your Skills

4. When Tom moved to a new home, his distance to work decreased by $\frac{1}{2}$ the original distance and the constant rate at which he travels to work increased by $\frac{1}{2}$ the original rate. By what percent has the time it takes Tom to travel to work decreased?

Sequence Formulas: Direct and Recursive

The final type of GRE formula problem involves sequences. A sequence is a collection of numbers in a set order. The order of a given sequence is determined by a **rule**. Here are examples of sequence rules:

For all integers $n \geq 1$…

$A_n = 9n + 3$	The *n*th term of this sequence is defined by the rule $9n + 3$, for integers $n \geq 1$. For example, the fourth term in this sequence is $9n + 3 = 9(4) + 3 = 39$. The first 10 terms of the sequence are as follows: 12, 21, 30, 39, 48, 57, 66, 75, 84, 93 (notice that successive terms differ by 9)
$Q_n = n^2 + 4$	The *n*th term of this sequence is defined by the rule $n^2 + 4$, for integers $n \geq 1$. For example, the first term in this sequence is $1^2 + 4 = 5$. The first 10 terms of the sequence are as follows: 5, 8, 13, 20, 29, 40, 53, 68, 85, 104

In these cases, each item of the sequence is defined as a function of *n*, the place in which the term occurs in the sequence. For example, the value of A_5 is a function of its being the fifth item in the sequence. This is a **direct** definition of a sequence formula.

The GRE also uses **recursive** formulas to define sequences. With **direct** formulas, the value of each item in a sequence is defined in terms of its item number in the sequence. With **recursive** formulas, each item of a sequence is defined in terms of the value of previous items in the sequence.

A recursive formula looks like this:

$$A_n = A_{n-1} + 9$$

This formula simply means “This term (A_n) equals the previous term (A_{n-1}) plus 9.” It is shorthand for a series of specific relationships between successive terms:

$$\begin{aligned} A_2 &= A_1 + 9 \\ A_3 &= A_2 + 9 \\ A_4 &= A_3 + 9, \text{etc.} \end{aligned}$$

Whenever you look at a recursive formula, *articulate its meaning in words in your mind*. If necessary, also write out one or two specific relationships that the recursive formula stands for. Think of a recursive formula as a “domino” relationship: if you know A_1, then you can find A_2, and then you can find A_3, then A_4, and so on for all the terms. You can also work backward: If you know A_4, then you can find A_3, A_2, and A_1. However, if you do not know the value of any one term, then you cannot calculate the value of any other. You need one domino to fall, so to speak, to knock down all the others.

Thus, to solve for the values of a recursive sequence, you need to be given the recursive rule and *also* the value of one of the items in the sequence. For example:

$A_n = A_{n-1} + 9$ $A_1 = 12$	In this example, A_n is defined in terms of the previous item, A_{n-1}. Recall the meaning

Because the first term is 12, this sequence is identical to the sequence defined by the direct definition, $A_n = 9n + 3$, given at the beginning of this section. Here is another example:

$F_n = F_{n-1} + F_{n-2}$ $F_1 = 1, F_2 = 1$ For all integers $n \geq 3$	In this example, F_n is defined in terms of both the previous item, F_{n-1}, and the iten

Check Your Skills

5. $S_n = 2n - 5$ for all integers $n \geq 1$. What is the 11th term of the sequence?

6. $B_n = (-1)^n \times n + 3$ for all integers $n \geq 1$. What is the 9th term of the sequence?

7. If $A_n = 2A_{n-1} + 3$ for all $n \geq 1$, and $A_4 = 45$, what is A_1?

Sequence Problems

For sequence problems on the GRE, you may be asked to do any of the following:

- Determine which answer choice corresponds to the correct *definition* (or *rule*) for a sequence (direct or recursive).
- Determine the value of a particular *item* in a sequence.
- Determine the sum or difference of a *set of items* in a sequence.

For simple linear sequences, in which the same number is added to any term to yield the next term, you can use the following alternative method—rather than find the rule or definition for the sequence, you can sometimes logically derive one item in the sequence based on the information given:

> If each number in a sequence is three more than the previous number, and the sixth number is 32, what is the 100th number?

Instead of finding the rule for this sequence, consider the following reasoning:

From the sixth term to the one hundredth term, there are 94 “jumps” of 3. Because 94 × 3 = 282, there is an increase of 282 from the sixth term to the one hundredth term:

$32 + 282 = 314$

Check Your Skills

8. If each number of a sequence is 4 more than the previous number, and the 3rd number in the sequence is 13, what is the 114th number in the sequence?

Sequences and Patterns

As has been discussed, sequence problems generally involve finding patterns among the *items in a sequence*, or the *defining rule* for the sequence. Generally, for questions involving the sequence items themselves, the best approach involves writing down information (often in the form of an equation) for specific items in the sequence, and trying to find a pattern among these items.

> If $S_n = 3^n$, what is the units digit of S_{65}?

Clearly, you cannot be expected to multiply out 3^{65} on the GRE, even with a calculator. Therefore, you must look for a pattern in the powers of three.

$$\begin{aligned}
3^1 &= 3\\
3^2 &= 9\\
3^3 &= 27\\
3^4 &= 81\\
3^5 &= 243\\
3^6 &= 729\\
3^7 &= 2{,}187\\
3^8 &= 6{,}561
\end{aligned}$$

You can see that the units digits of powers of 3 follow the pattern “3, 9, 7, 1” before repeating. The units digit of 3^{65} will thus be 3, because the 64th term will be “1” as 64 is divisible by 4 (and the pattern repeats every four terms).

As a side note, most sequences on the GRE are defined for integers $n \geq 1$. That is, sequence S_n almost always starts at S_1. Occasionally, a sequence might start at S_0, but in that case, you will be told that n could equal 0.

Check Your Skills

9. If $A_n = 7^n - 1$, what is the units digit of A_{33}?

Functions

Functions are very much like the “magic boxes” you may have learned about in elementary school. For example:

> You put a 2 into the magic box, and a 7 comes out. You put a 3 into the magic box, and a 9 comes out. You put a 4 into the magic box, and an 11 comes out. What is the magic box doing to your number?

There are many possible ways to describe what the magic box is doing to your number. One possibility is as follows: The magic box is doubling your number and adding 3:

$2(2) + 3 = 7$ $\qquad$ $2(3) + 3 = 9$ $\qquad$ $2(4) + 3 = 11$

It’s possible that the magic box is actually doing something different to your number. But assuming that the box is really following the "double and add 3" rule, you can express this result this way: $2x + 3$. This whole process can be written in function form as:

$$f(x) = 2x + 3$$

In words, you say “f of x equals $2x$ plus 3.” Here, x is the number you put in the box, f is the magic box itself, and the value of $2x + 3$ is the result that comes out of the box. Functions always work the same way every time, so you can think of the function *f* as actually representing the rule that the magic box is using to transform your number.

The magic box analogy is a helpful way to conceptualize a function as a rule built on an independent variable. The value of a function changes as the value of the independent variable changes. In other words, the value of a function is dependent on the value of the independent variable. Examples of functions include:

$f(x) = 4x^2 - 11$	The value of the function *f* is *dependent* on the *independent* variable *x*.
$= \frac{3a + 12 - 8a + 8}{12}$	The value of the function *g* is *dependent* on the *independent* variable *t*.

Think of functions as consisting of an “input” variable (the number you put into the magic box), and a corresponding “output” value (the number that comes out of the box). The function is simply the rule that turns the “input” variable into the “output” variable.

By the way, the expression $f(x)$ is pronounced “*f* of *x*”, not “*fx*.” It does not mean “*f* times *x*.”The letter *f* does not stand for a variable; rather, it stands for the rule that dictates how the input *x* changes into the output $f(x)$.

The “domain” of a function indicates the set of possible inputs. The “range” of a function indicates the set of possible outputs. For instance, the function $f(x) = x^2$ can take any input but never produces a negative number. So the domain is all numbers, but the range is $f(x) \geq 0$.

NUMERICAL SUBSTITUTION

This is the most basic type of function problem. Input the numerical value (say, 5) in place of the independent variable *x* to determine the value of the function:

> If $f(x) = x^2 - 2$, what is the value of $f(5)$?

In this problem, you are given a rule for *f*(*x*): square *x* and subtract 2. Then, you are asked to apply this rule to the number 5. Square 5 and subtract 2 from the result:

$$f(5) = (5)^2 - 2 = 25 - 2 = 23$$

VARIABLE SUBSTITUTION

This type of problem is slightly more complicated. Instead of finding the output value for a numerical input, you must find the output when the input is an algebraic expression:

> If $f(z) = z^2 - \frac{z}{3}$, what is the value of $f(w + 6)$?

Input the variable expression (*w* + 6) in place of the independent variable (*z*) to determine the value of the function:

$$f(w + 6) = (w + 6)^2 - \frac{w + 6}{3}$$

Compare this equation to the equation for *f*(*z*). The expression (*w* + 6) has taken the place of every *z* in the original equation. In a sense, you are treating the expression (*w* + 6) as one thing, as if it were a single letter or variable.

The rest is algebraic simplification:

$$\begin{aligned} f(w+6) &= (w+6)(w+6) - \left(\frac{w}{3} + \frac{6}{3}\right) \\ &= w^2 + 12w + 36 - \frac{w}{3} - 2 \\ &= w^2 + 11\frac{2}{3}w + 34 \end{aligned}$$

COMPOUND FUNCTIONS

Imagine putting a number into one magic box, and then putting the output directly into another magic box. This is the situation you have with compound functions:

> $$\text{If } f(x) = x^3 + \sqrt{x} \text{ and } g(x) = 4x - 3, \text{ what is } f(g(3))?$$

The expression *f*(*g*(3)), pronounced "*f* of *g* of 3," looks ugly, but the key to solving compound function problems is to work from the inside out. In this case, start with *g*(3). Notice that you put the number into *g*, not into *f*, which may seem backward at first:

$g(3) = 4(3) - 3 = 12 - 3 = 9$

Use the result from the *inner* function *g* as the new input variable for the *outer* function *f*:

The final result is 732.

$$f\left(g(3)\right) = f(9) = (9)^3 + \sqrt{9} = 729 + 3 = 732$$

Note that changing the order of the compound functions changes the answer:

> $\mathbf{If}\, f(x) = x^3 + \sqrt{x}\, \mathbf{and}\, g(x) = 4x - 3, \mathbf{what\,is}\, g(f(3))?$

Again, work from the inside out. This time, start with *f*(3), which is now the inner function:

$$f(3) = (3)^3 + \sqrt{3} = 27 + \sqrt{3}$$

Use the result from the *inner* function *f* as the new input variable for the *outer* function *g*:

$$g\left(f(3)\right) = g\left(27 + \sqrt{3}\right) = 4\left(27 + \sqrt{3}\right) - 3 = 108 + 4\sqrt{3} - 3 = 105 + 4\sqrt{3}$$

Thus, $g(f(3)) = 105 + 4\sqrt{3}.$

In general, *f*(*g*(*x*)) and *g*(*f*(*x*)) are **not the same rule overall** and will often lead to different outcomes. As an analogy, think of “putting on socks” and “putting on shoes” as two functions: the order in which you perform these steps obviously matters!

You may be asked to find a value of *x* for which *f*(*g*(*x*)) = *g*(*f*(*x*)). In that case, use variable substitution, working as always from the inside out:

> If $f(x) = x^3 + 1$, and $g(x) = 2x$, for what value of x does $f(g(x)) = g(f(x))$?

Simply evaluate as you did in the problems above, using *x* instead of an input value:

$$f(g(x)) = g(f(x))$$
$$f(2x) = g(x^3 + 1)$$
$$(2x)^3 + 1 = 2(x^3 + 1)$$
$$8x^3 + 1 = 2x^3 + 2$$
$$6x^3 = 1$$
$$x = \sqrt[3]{\frac{1}{6}}$$

FUNCTIONS WITH UNKNOWN CONSTANTS

On the GRE, you may be given a function with an unknown constant. You will also be given the value of the function for a specific number. You can combine these pieces of information to find the complete function rule:

> If $f(x) = ax^2 - x$, and $f(4) = 28$, what is $f(-2)$?

Solve these problems in three steps. First, use the value of the input variable and the corresponding output value of the function to solve for the unknown constant:

$$\begin{aligned} f(4) = a(4)^2 - 4 &= 28 \\ 16a - 4 &= 28 \\ 16a &= 32 \\ a &= 2 \end{aligned}$$

Then, rewrite the function, replacing the constant with its numerical value:

$$f(x) = ax^2 - x = 2x^2 - x$$

Finally, solve the function for the new input variable:

$$f(-2) = 2(-2)^2 -(-2) = 8 + 2 = 10$$

FUNCTION GRAPHS

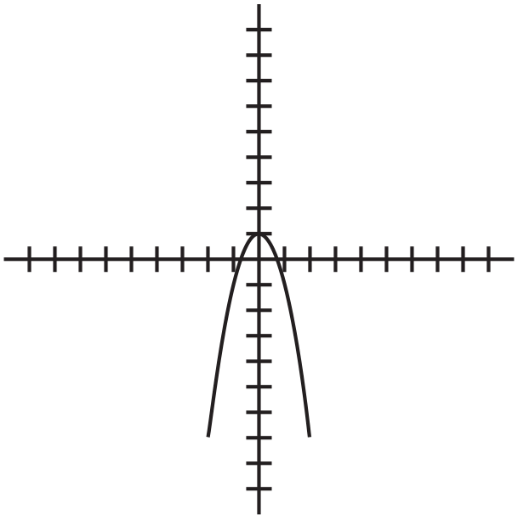
A function can be visualized by graphing it in the coordinate plane. The input variable is considered the **domain** of the function, or the *x*-coordinate. The corresponding output is considered the **range** of the function, or the *y*-coordinate.

What is the graph of the function $f(x) = -2x^2 + 1$?

INPUT	**OUTPUT**	**(x, y)**
-3	$-2(-3)^2 + 1 = -17$	$(-3, -17)$
-2	$-2(-2)^2 + 1 = -7$	$(-2, -7)$
-1	$-2(-1)^2 + 1 = -1$	$(-1, -1)$
0	$-2(0)^2 + 1 = 1$	$(0, 1)$
1	$-2(1)^2 + 1 = -1$	$(1, -1)$
2	$-2(2)^2 + 1 = -7$	$(2, -7)$
3	$-2(3)^2 + 1 = -17$	$(3, -17)$

Create an **input-output** table by evaluating the function for several input values.

Then, plot points to see the shape of the graph:



Check Your Skills

10. If $f(x) = \dfrac{1}{x+2} + (x-1)^2$, what is $f(-1)$?

11. If $t(u) = au^2 - 3u + 1$ and $t(3) = 37$, what is a?

12. If $f(x) = 3x - \sqrt{x}$ and $g(x) = x^2$, what is $g(f(4))$?

13. If $g(y) = y^2 - \dfrac{1}{y+1}$, what is $g\left(\dfrac{1}{x}\right)$?

Check Your Skills Answer Key

1. ***k* = 70**

   Baking time in minutes $= \dfrac{Vk}{T}$

   $$30 = \frac{150 \times k}{350}$$

   $$k = \frac{30 \times 350}{150} = 70$$

2. **20**

   Deal with the formula in the parentheses first:

   $3 \Delta 1 = 3^1 + 1 = 3 + 1 = 4$

   $-2 \Delta (3 \Delta 1) = -2 \Delta 4$

   $-2 \Delta 4 = (-2)^4 + 4 = 16 + 4 = 20$

3. $\dfrac{Vk}{T}$

   $$2\lambda16 = \frac{16}{2} + \frac{2}{16} = 8 + \frac{1}{8} = 8\,\frac{1}{8}$$

4. **62.5% decrease**

   No numbers are specified, so you should choose values for the original distance and the original rate. Good numbers to pick for the distance are multiples of 2, because the rate is decreased by $\frac{1}{2}$. Good numbers to pick for the rate are multiples of 3, because the rate is increased by $\frac{1}{2}$:

	Old	**New**
Distance	12	6
Rate	3	4
Time = Distance/Rate	12/3 = 4	6/4 = 1.5

$$\text{Percent decrease in time} = \frac{\text{change in time}}{\text{original time}} = \frac{4 - 1.5}{4} = \frac{2.5}{4} = 62.5\%\ \text{decrease.}$$

5. **17**

   $S_{11} = 2 \times (11) - 5 = 22 - 5 = 17$

6. **–6**

   $B_9 = (-1)^{(9)} \times 9 + 3 = -9 + 3 = -6$

7. **3**

   You know the value of A_4, therefore, you can write the definition for A_4 to solve for A_3, write the definition for A_3 to solve for A_2, and so on:

$$\begin{array}{lllllllll} A_4 & = & 2A_3+3 & \rightarrow & 45 & = & 2A_3+3 & \rightarrow & A_3 & = & 21 \\ A_3 & = & 2A_2+3 & \rightarrow & 21 & = & 2A_2+3 & \rightarrow & A_2 & = & 9 \\ A_2 & = & 2A_1+3 & \rightarrow & 9 & = & 2A_1+3 & \rightarrow & A_1 & = & 3 \end{array}$$

8. **457**

   There are 114 − 3 = 111 “jumps” of 4 between the 3rd and the 114th terms. Because 111 × 4 = 444, there is an increase of 444 from the 3rd term to the 114th term: 13 + 444 = 457.

9. **6**

   The units digits of the powers of 7 follow a repeating pattern: **7**, 4**9**, 34**3**, 240**1**, 1680**7**, etc. Pattern = {7, 9, 3, 1}. There are 8 repeats of the pattern from A_1 to A_{32}, inclusive. The pattern begins again on A_{33}, so A_{33} has the same units digit as A_1, which is 7. The units digit of 7^{33} is 7, and 7 − 1 = 6.

10. **5**

    Simply plug in (−1) for each occurrence of *x* in the function definition and evaluate:

$$f(x) = \frac{1}{x+2} + (x-1)^2 \qquad f(-1) = \frac{1}{(-1)+2} + ((-1)-1)^2$$

$$f(1) = \frac{1}{1} + (-2)^2 = 1 + 4 = 5$$

11. **5**

    Plug in 3 for *u* in the definition of *t*(*u*), set it equal to 37, and solve for *a*:

$$t(u) = au^2 - 3u + 1 \rightarrow t(3) = a(3)^2 - 3(3) + 1 = 37$$

$$9a - 9 + 1 = 37$$

$$9a = 45$$

$$a = 5$$

12. **100**

    First, find the output value of the inner function:

    $$f(4) = 3(4) - \sqrt{4} = 12 - 2 = 10.$$

    Then, find $g(10)$: $10^2 = 100$.

13. $\dfrac{-x^3 + x + 1}{x^3 + x^2}$

    Simply plug in $\left(\dfrac{1}{x}\right)$ for y in $g(y)$, and simplify the expression:

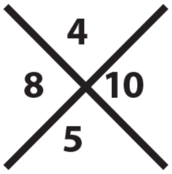
$$g(y) = y^2 - \frac{1}{y+1} \rightarrow g\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 - \frac{1}{\left(\frac{1}{x}\right) + 1}$$

$$g\left(\frac{1}{x}\right) = \frac{1}{x^2} - \frac{1}{\frac{x+1}{x}} = \frac{1}{x^2} - \frac{x}{x+1}$$

$$g\left(\frac{1}{x}\right) = \frac{x + 1 - x^3}{x^2(x+1)} = \frac{-x^3 + x + 1}{x^3 + x^2}$$

Problem Set

1. If $A \diamond B = 4A - B$, what is the value of $(3 \diamond 2) \diamond 3$?

2. If  $= \dfrac{u + y}{x + z}$, what is ?

3. Life expectancy is defined by the formula $\dfrac{2SB}{G}$, where S = shoe size, B = average monthly electric bill in dollars, and G = GRE score. If Melvin's GRE score is twice his monthly electric bill, and his life expectancy is 50, what is his shoe size?

4. The formula for spring factor in a shoe insole is $\dfrac{w^2 + x}{3}$, where w is the width of the insole in centimeters and x is the grade of rubber on a scale of 1 to 9. What is the maximum spring factor for an insole that is 3 centimeters wide?

5. Cost is expressed by the formula tb^4. If b is doubled, by what factor has the cost increased?

(A)	2
(B)	6
(C)	8
(D)	16
(E)	$\frac{1}{2}$

6. If the scale model of a cube sculpture is 0.5 cm per every 1 m of the real sculpture, what is the volume of the model, if the volume of the real sculpture is 64 m^3?

7. The "competitive edge" of a baseball team is defined by the formula $\sqrt{\frac{W}{L}}$, where W represents the number of wins, and L represents the number of losses. This year, the GRE All-Stars had three times as many wins and one-half as many losses as they had last year. By what factor did their "competitive edge" increase?

8. If the radius of a circle is tripled, what is the ratio of the area of half the original circle to the area of the whole new circle? (Area of a circle = πr^2, where r = radius)

For problems #9–10, use the following sequence definition: $A_n = 3 - 8n$.

9. What is A_1?

10. What is $A_{11} - A_9$?

11. A sequence S is defined as follows: $S_n = \dfrac{S_{n+1} + S_{n-1}}{2}$. If $S_1 = 15$ and $S_4 = 10.5$, what is S_2?

12. If $f(x) = 2x^4 - x^2$, what is the value of $f\left(2\sqrt{3}\right)$?

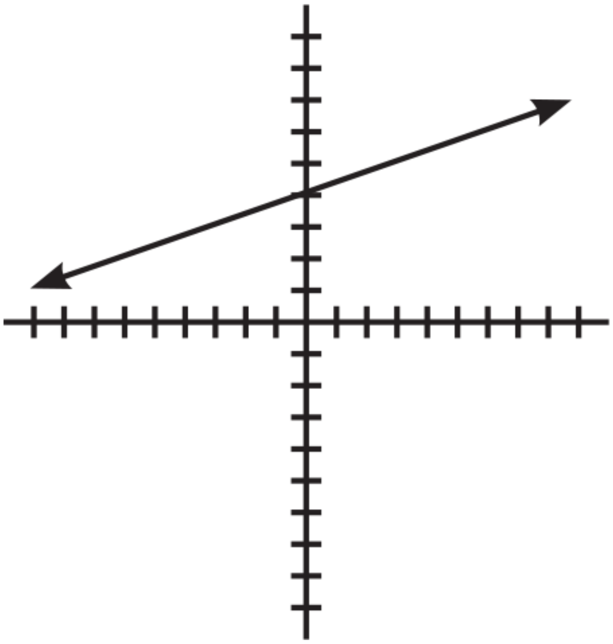
13. If $k(\mathrm{x}) = 4x^3a$, and $k(3) = 27$, what is $k(2)$?

14. If $f(x) = 3x - \sqrt{x}$ and $g(x) = x^2$, what is $f(g(4))$?

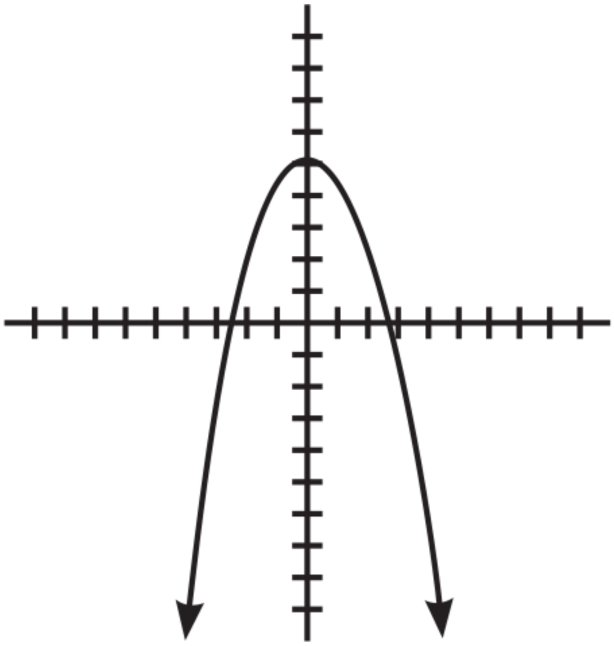
15. If $f(x) = 2x^2 - 4$ and $g(x) = 2x$, for what values of x will $f(x) = g(x)$?

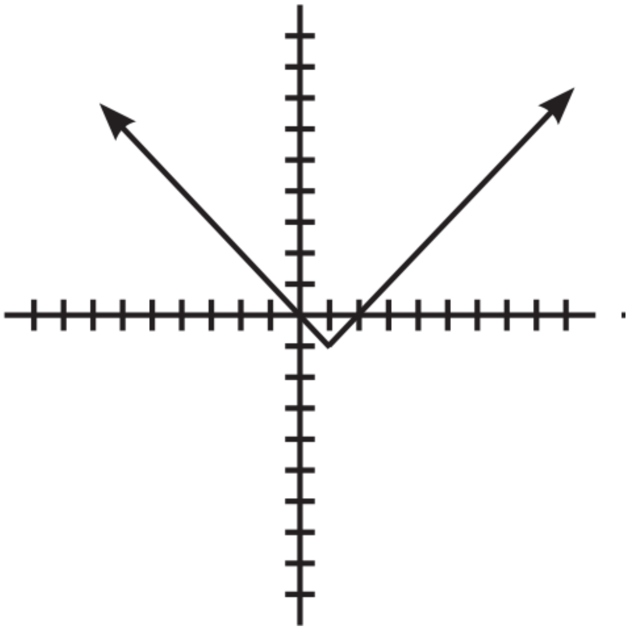
16. Which of the following graphs is the graph of function $g(x) = |x - 1| - 1$?

(A)

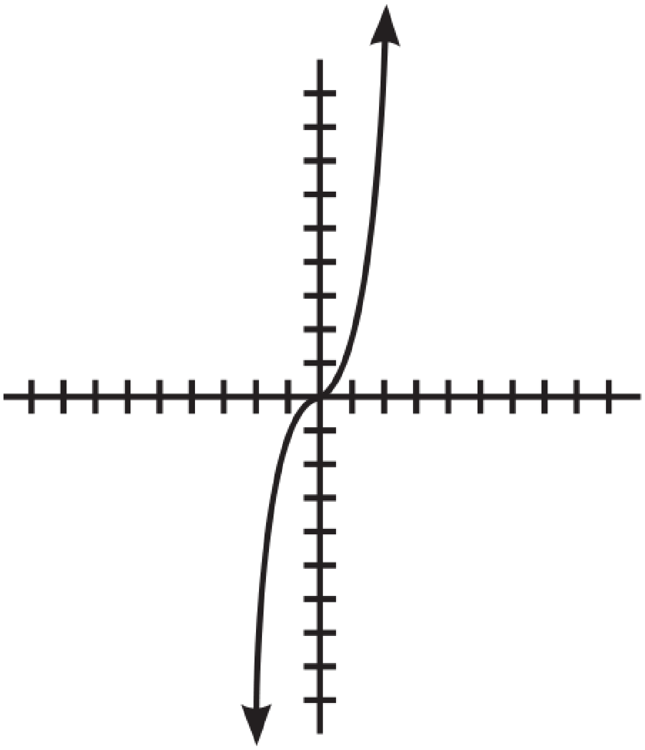


(B)





(C)



(D)

$A_n = 2^n - 1$ for all integers $n \geq 1$

17.

Quantity A

Quantity B

The units digit of A_{26}	The units digit of A_{34}

18. $P \blacksquare Q = P + 2Q$ for all integers P and Q

Quantity A	**Quantity B**
$11 \blacksquare 5$	$5 \blacksquare 11$

19. The length of a rectangle increased by a factor of 2, and at the same time its area increased by a factor of 6.

Quantity A	**Quantity B**
The factor by which the width of the rectangle increased	3

Solutions

1. **37**

   First, simplify 3 ◊ 2: 4(3) − 2 = 12 − 2 = 10. Then, solve 10 ◊ 3: 4(10) − 3 = 40 − 3 = 37.

2. **2**

   Plug the numbers in the grid into the formula, matching up the number in each section with the corresponding variable in the formula:

   $$\frac{u+y}{x+z} = \frac{8+10}{4+5} = \frac{18}{9} = 2.$$

3. **Size 50**

   $$\frac{2SB}{2B} = 50$$
   $$S = 50$$

   Substitute $2B$ for G in the formula. Note that the term $2B$ appears in both the numerator and denominator, so they cancel out.

4. **6**

   Determine the maximum spring factor by setting $x = 9$.

   Let s = spring factor:

   $$s = \frac{w^2 + x}{3}$$

   $$s = \frac{(3)^2 + 9}{3} = \frac{18}{3} = 6$$

5. **D**

Pick numbers to see what happens to the cost when *b* is doubled. If the original value of *b* is 2, the cost is 16*t*. When *b* is doubled to 4, the new cost value is 256*t*. The cost has increased by a factor of $\frac{256}{16}$, or 16.

6. **8cm^3**

$$V = s^3 \rightarrow 64 = s^3 \rightarrow s = 4$$

The length of a side on the real sculpture is 4 m.

$$\frac{0.5\,\text{cm}}{1\,\text{m}} = \frac{x\,\text{cm}}{4\,\text{m}} \rightarrow x = 2$$

The length of a side on the model is 2 cm.

$V = s^3 = (2)^3 = 8$

The volume of the model is 8.

7. $\sqrt{9}$

Let *c* = competitive edge:

$$c = \sqrt{\frac{W}{L}}$$

Pick numbers to see what happens to the competitive edge when *W* is tripled and *L* is halved. If the original value of *W* is 2 and the original value of L is 2, the original value of c is $\sqrt{\frac{2}{2}} = 1$. If W triples to 6 and L is halved to 1, the new value of c is $\sqrt{\frac{6}{1}} = \sqrt{6}$. The competitive edge has increased from 1 to $\sqrt{9}$.

The competitive edge has increased by a factor of $\sqrt{9}$.

8. $\frac{23}{7}$

Pick real numbers to solve this problem. Set the radius of the original circle equal to 2. Therefore, the radius of the new circle is equal to 6. Once you compute the areas of both circles, you can find the ratio:

$$\frac{\text{Area of half the original circle}}{\text{Area of the new circle}} = \frac{2\pi}{36\pi} = \frac{1}{18}$$

9. **−5**

$A_n = 3 - 8n$

$A_1 = 3 - 8(1) = 3 - 8 = -5$

10. **–16**

$A_n = 3 - 8n$

$A_{11} = 3 - 8(11) = 3 - 88 = -85$

$A_9 = 3 - 8(9) = 3 - 72 = -69$

$A_{11} - A_9 = -85 - (-69) = -16$

11. **13.5**

The easiest way to solve this problem is to write equations for S_2 and S_3 in terms of the other items in the sequence and solve for S_2:

$$S_2 = \frac{S_3 + S_1}{2} \rightarrow S_2 = \frac{S_3 + 15}{2}$$
$$S_3 = \frac{S_4 + S_2}{2} \rightarrow S_3 = \frac{10.5 + S_2}{2}$$

Now substitute the expression for S_3 into the first equation and solve:

$$S_2 = \frac{\left(\frac{10.5 + S_2}{2}\right) + 15}{2} \rightarrow 2S_2 = \left(\frac{10.5 + S_2}{2}\right) + 15 \rightarrow 4S_2 = 10.5 + S_2 + 30 \rightarrow 3S_2 = 40.5 \rightarrow S_2 = 13.5$$

By this logic, $S_3 = \frac{10.5 + 13.5}{2} = 12$.

12. **276**

$$\begin{aligned} f(x) &= 2\left(2\sqrt{3}\right)^4 - \left(2\sqrt{3}\right)^2 = 2(2)^4\left(\sqrt{3}\right)^4 - (2)^2\left(\sqrt{3}\right)^2 \\ &= (2 \cdot 16 \cdot 9) - (4 \cdot 3) \\ &= 288 - 12 = 276 \end{aligned}$$

13. **8**

k(3) = 27. Therefore:

$$\begin{aligned} &4(3)^3 a = 27 \\ &108a = 27 \\ &a = \frac{1}{4} \rightarrow k(x) = 4x^3\left(\frac{1}{4}\right) = x^3 \rightarrow k(2) = (2)^3 = 8 \end{aligned}$$

14. **44**

First, find the output value of the inner function: $g(4) = 16$. Then, find $f(16)$: $3(16) - \sqrt{16} = 48 - 4 = 44$.

15. ***x* = {−1, 2}**

To find the values for which $f(x) = g(x)$, set the functions equal to each other:

$$\begin{aligned} 2x^2 - 4 &= 2x \\ 2x^2 - 2x - 4 &= 0 \\ 2\left(x^2 - x - 2\right) &= 0 \\ 2(x-2)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x - 2 &= 0 & x + 1 &= 0 \\ x &= 2 \text{ or} & x &= -1 \end{aligned}$$

16. **(C)**

$g(x) = |x - 1| - 1$. This function is an absolute value, which typically has a V-shape. You can identify the correct graph by trying $x = 0$, which yields $g(0) = 0$, the origin. Then, try $x = 1$, which yields $g(1) = -1$ and the point $(1, -1)$. Next, try $x = 2$: $g(2) = |2 - 1| - 1 = 1 - 1 = 0$. These three points fall on the V-shape.

17. **(C)**

The powers of 2 have a repeating pattern of four terms for their units digits: {2, 4, 8, 6}. That means that every fourth term, the pattern repeats. For instance, the fifth term has the same units as the first term, because 5 −

1 = 4. So terms that are four terms apart, or a multiple of four terms apart, will have the same units digit.

The 34th term and the 26th term are 34 − 26 = 8 terms apart. Because 8 is a multiple of 4, the terms will have the same units digit. **The two quantities are equal.** Incidentally, the units digit of A_{26} and A_{34} is 3.

18. **(B)**

$$P \blacksquare Q = P + 2Q \text{ for all integers } P \text{ and } Q$$

Quantity A	**Quantity B**
$11 \blacksquare 5 =$	$5 \blacksquare 11 =$
$(11) + 2 \times (5) =$	$(5) + 2 \times (11) =$
$11 + 10 = \mathbf{21}$	$5 + 22 = \mathbf{27}$

Quantity B is greater.

19. **(C)**

Plug in numbers to answer this question. Use a table to organize the information:

	Old	**New**
Length	2	$2 \times 2 = 4$
Width	1	W
Area	2	$2 \times 6 = 12$

$4 \times W = 12$
$W = 3$

Compare the new width to the original: $\frac{\text{New}}{\text{Old}} = \frac{3}{1} = 3.$ The width increased by a factor of 3.

The two quantities are equal.

Unit Two: Fractions, Decimals, & Percents

This section provides an in-depth look at the array of GRE questions that test knowledge of Fractions, Decimals, and Percents. Learn to see the connections among these part–whole relationships and practice implementing strategic shortcuts.

In This Unit...

Chapter 5: Fractions

Chapter 6: Digits and Decimals

Chapter 7: Percents

Chapter 8: FDP Connections

Chapter 5

FRACTIONS

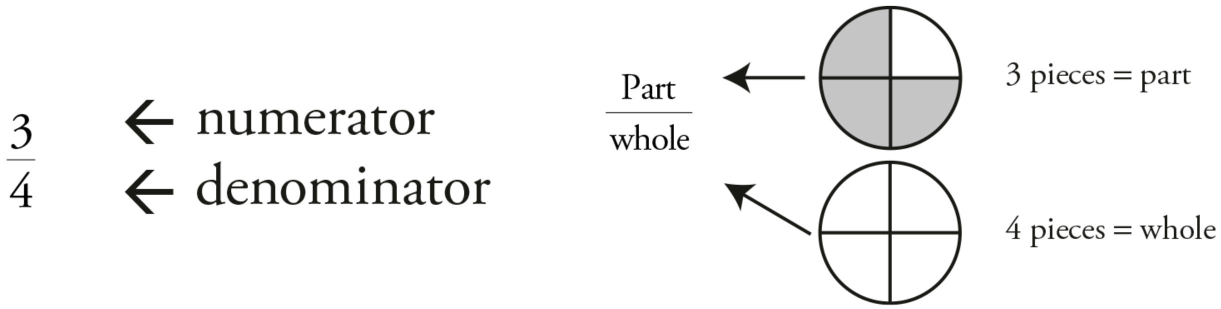
In This Chapter...

Manipulating Fractions
Switching Between Improper Fractions and Mixed Numbers
Division in Disguise
Fraction Operations: Know What to Expect
Comparing Fractions: Cross-Multiply
NEVER Split the Denominator!
Benchmark Values
Picking Smart Numbers: Multiples of the Denominators
When NOT to Use Smart Numbers

Chapter 5

Fractions

This chapter is devoted entirely to understanding what fractions are and how they work from the ground up. Begin by reviewing the two parts of a fraction: the **numerator** and the **denominator**.

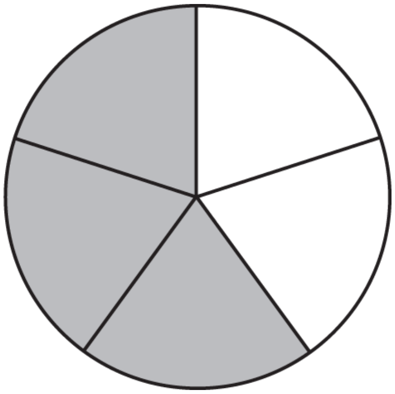


In the picture above, each circle represents a whole unit. One full circle means the number 1, two full circles represent the number 2, and so on. Fractions essentially divide units into parts. The units shown here have been divided into four equal parts, because the denominator of our fraction is 4. In any fraction, the denominator tells you how many equal pieces a unit has been broken into.

The circle at the top has three of the pieces shaded in, and one piece unshaded. That’s because the top of the fraction is 3. For any fraction, the numerator tells you how many of the equal pieces you have.

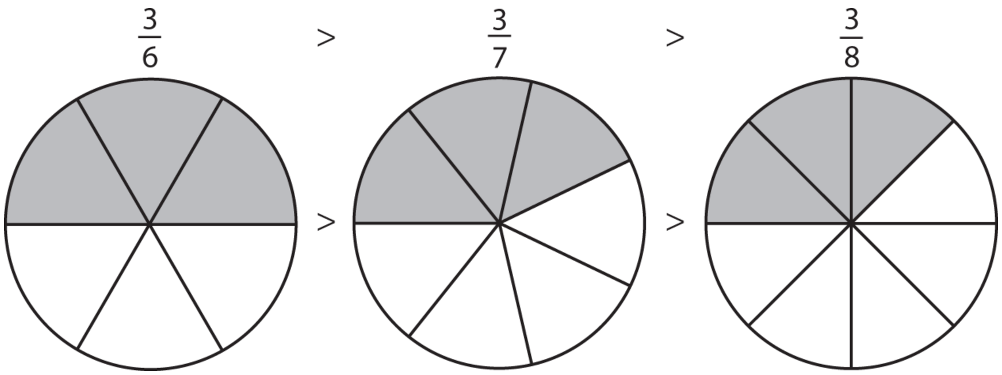
Take a look at how changes to the numerator and denominator change a fraction. First, consider how changes affect the denominator. You've already seen what $\frac{1}{2}$ looks like; here is what $\frac{1}{2}$ looks like.

$\frac{3}{5} =$



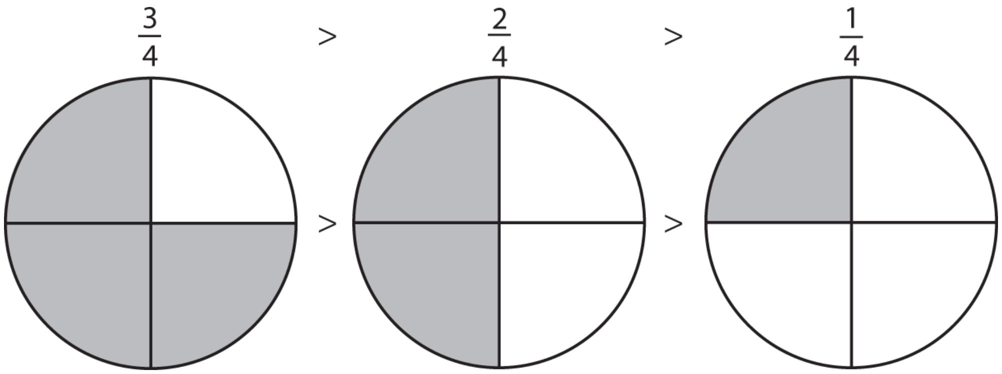
The numerator hasn't changed (it's still 3), so you still have three shaded pieces, but now the circle has been divided into five pieces instead of four. One effect is that each piece is now smaller. The fraction $\frac{1}{2}$ is smaller than $\frac{1}{2}$. Rule: as the denominator of a number gets bigger, the value of the fraction gets smaller. The fraction $\frac{1}{2}$ is smaller than $\frac{1}{2}$, because each fraction has three pieces, but when the circle (or number) is divided into five equal portions, each portion is smaller, so three $\frac{1}{2}$ portions are less than three $\frac{1}{2}$ portions.

As you split the circle into more and more pieces, each piece gets smaller and smaller:



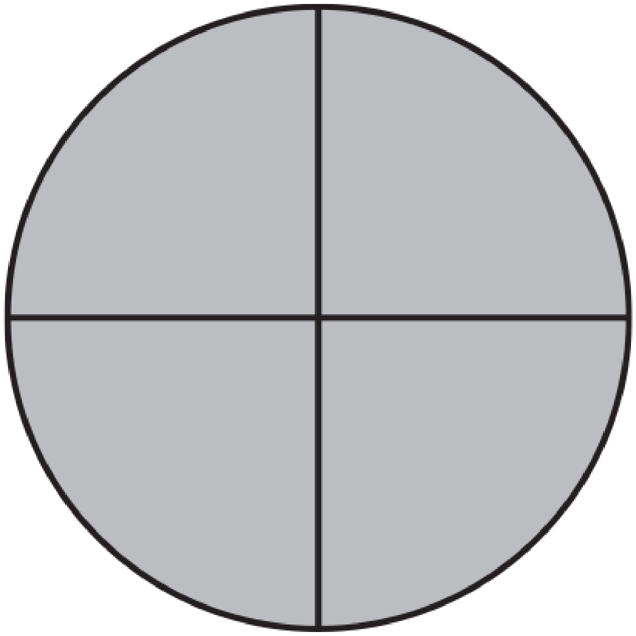
Conversely, as the denominator gets smaller, each piece becomes bigger and bigger.

Now look at what happens as you change the numerator. The numerator tells you how many pieces you have, so if you make the numerator smaller, we get fewer pieces:



Conversely, if you make the numerator larger, you get more pieces. Look more closely at what happens as you get more pieces. In particular, you want to know what happens when the numerator becomes equal to or greater than the denominator. First, notice what happens when you have the same numerator and denominator. If you have $\frac{1}{2}$ pieces, this is what the circle looks like:

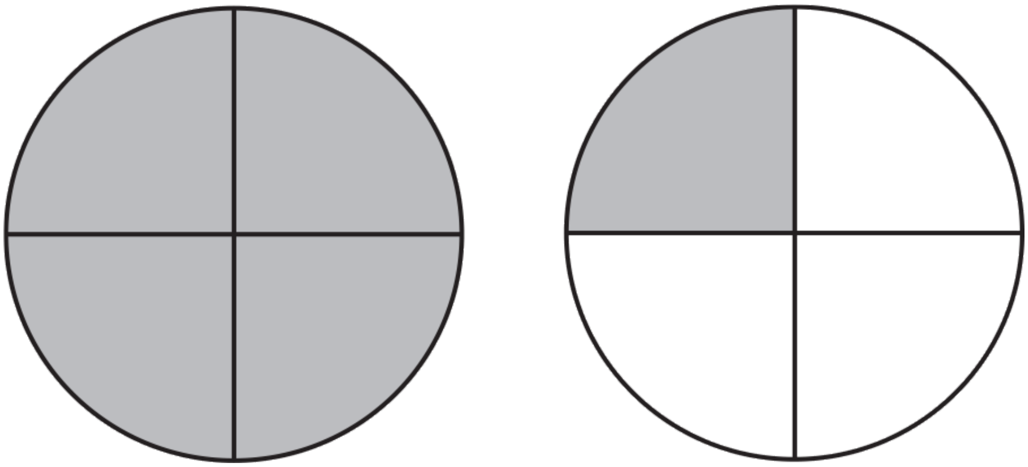
$$\frac{4}{4} =$$



Remember, the circle represents one whole unit. So when all four parts are filled, you have one full unit, or 1. So $\frac{1}{2}$ is equal to 1. Rule: If the numerator and denominator of a fraction are the same, that fraction equals 1.

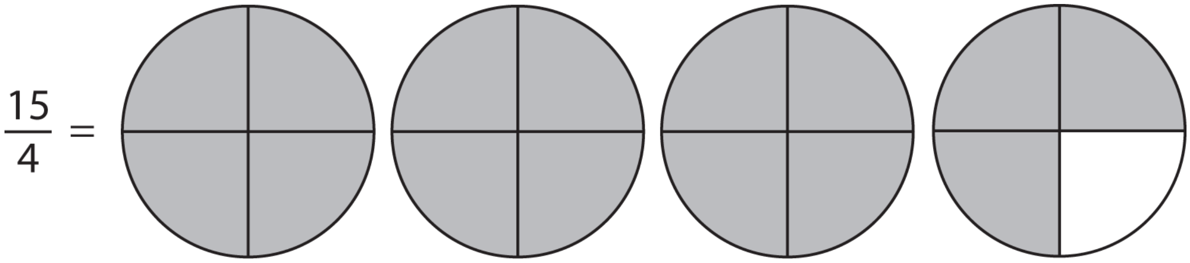
Following is what happens as the numerator becomes larger than the denominator. What does $\frac{1}{2}$ look like?

$$\frac{5}{4} =$$



Each circle is only capable of holding four pieces, so when you fill up one circle, you have to move on to a second circle and begin filling it up, too. So one way of looking at $\frac{1}{2}$ is that you have one complete circle, which you know is equivalent to 1, and you have an additional $\frac{1}{2}$. So another way to write $\frac{1}{2}$ is 1 + $\frac{1}{2}$. This can be shortened to $1\frac{1}{4}$ ("one and one-fourth").

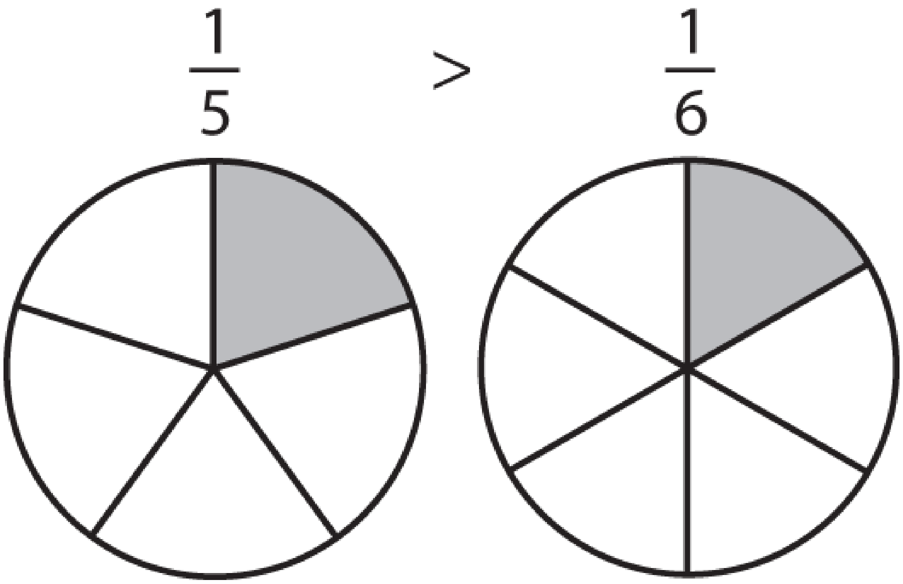
In the last example, the numerator was only a little larger than the denominator. However, that will not always be the case. The same logic applies to any situation. Look at the fraction $\frac{23}{7}$. Once again, this means that each circle (i.e., each whole number) is divided into four pieces, and you have 15 pieces.

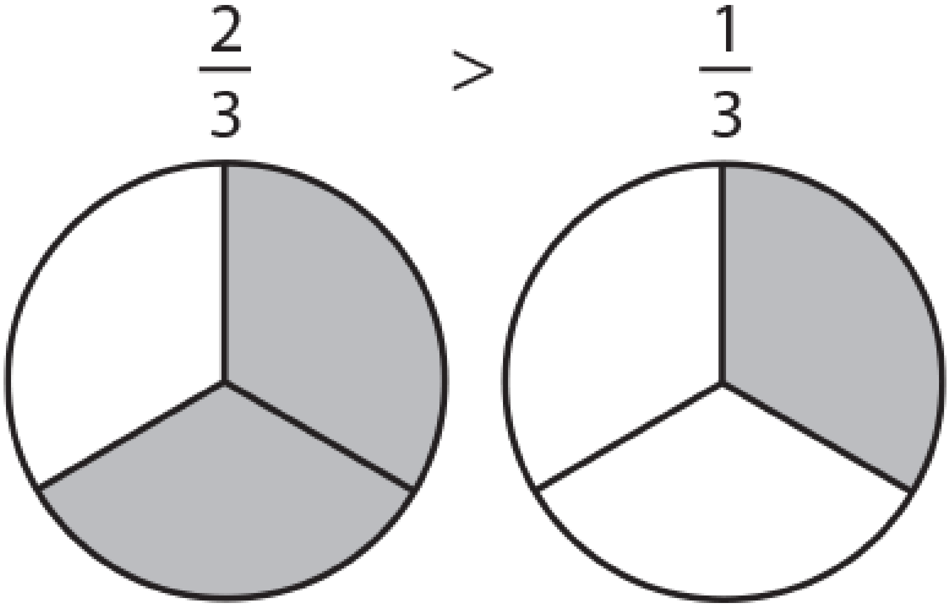


In this case, you have three circles completely filled. To fill three circles, you needed 12 pieces. (Note: 3 circles × 4 pieces per circle = 12 pieces.) In addition to the three full circles, you have three additional pieces. So you have: $\frac{15}{4} = 3 + \frac{3}{4} = 3\,\frac{3}{4}$.

Whenever you have both an integer and a fraction in the same number, you have a **mixed number**. Any fraction in which the numerator is larger than the denominator (e.g., $\frac{1}{2}$) is known as an **improper fraction**.

Improper fractions and mixed numbers express the same thing. How to convert from improper fractions to mixed numbers and vice-versa will be discussed later in the chapter.



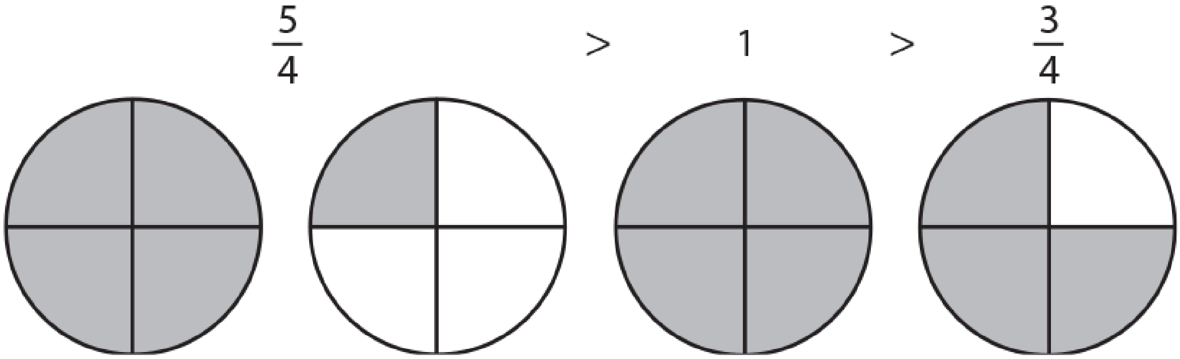


Take a moment to review what you've learned about fractions so far. Every fraction has two components: the numerator and the denominator.

The denominator tells you how many equal pieces each unit circle has. Assuming that the numerator stays the same, as the denominator gets bigger, each piece gets smaller, so the fraction gets smaller as well.

The numerator tells you how many equal pieces you have. Assuming that the denominator stays the same, as the numerator gets bigger, you have more pieces, so the fraction gets bigger.

When the numerator is smaller than the denominator, the fraction will be less than 1. When the numerator equals the denominator, the fraction equals 1. When the numerator is larger than the denominator, the fraction is greater than 1.



Check Your Skills

For each of the following sets of fractions, decide which fraction is larger:

1. $\frac{5}{7}$ vs. $\frac{3}{7}$

2. $\frac{3}{10}$ vs. $\frac{3}{13}$

Manipulating Fractions

The next two sections discuss how to add, subtract, multiply, and divide fractions. You should already be familiar with these four basic manipulations of arithmetic, but when fractions enter the picture, things can become more complicated.

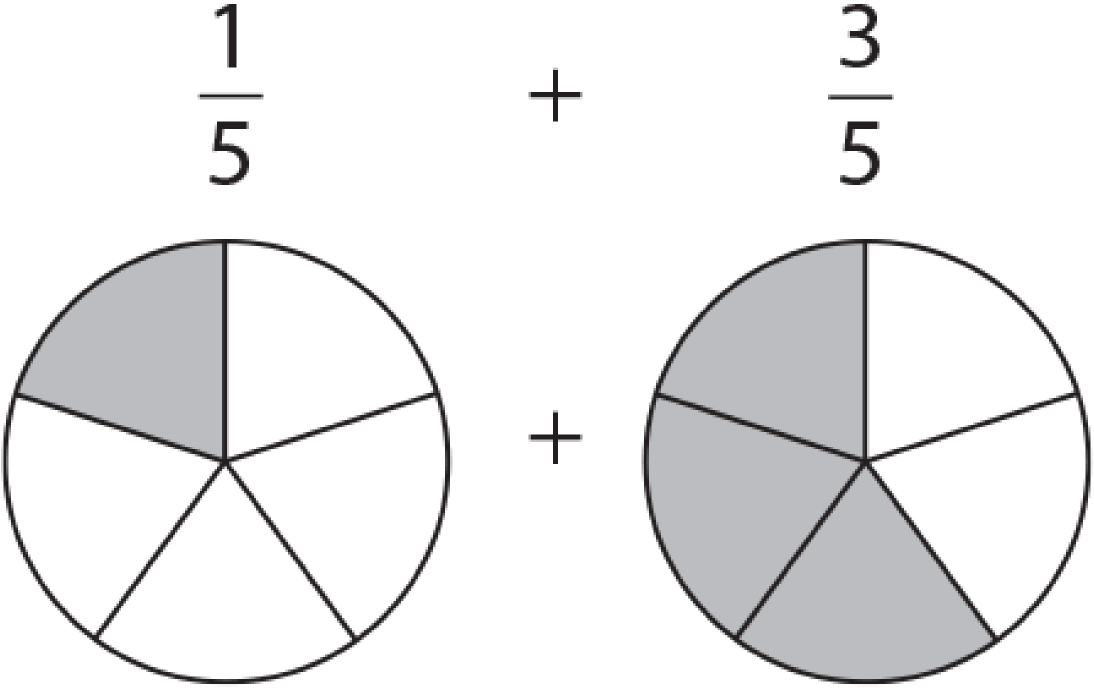
Each manipulation is discussed in turn in the following sections. Each discussion talks conceptually about what changes are being made with each manipulation, then goes through the actual mechanics of performing the manipulation.

Up first is how to add and subtract fractions.

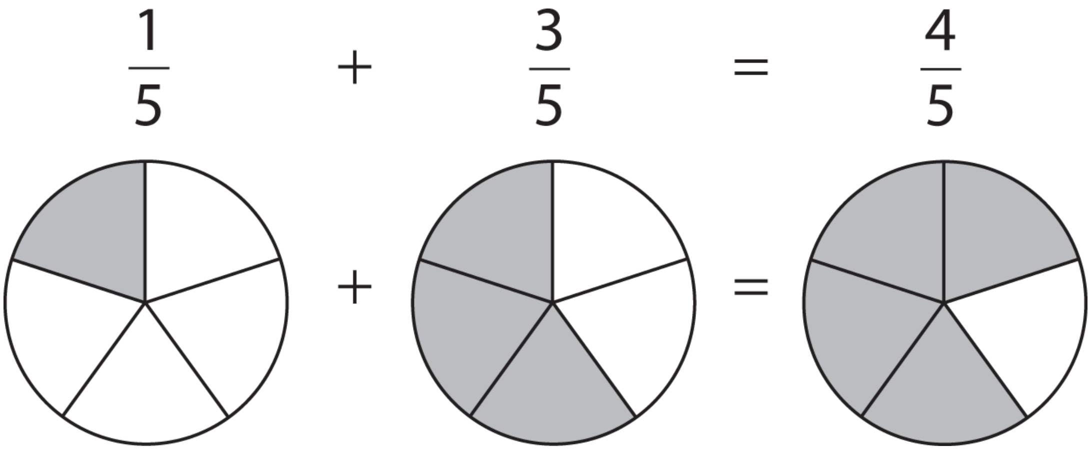
FRACTION ADDITION AND SUBTRACTION

The first thing to recall about addition and subtraction in general is that they affect how many things you have. If you have three things, and you add six more things, you have 3 + 6 or 9 things. If you have seven things and you subtract two of those things, you now have 7 − 2 or 5 things. That same basic principle holds true with fractions as well. What this means is that addition and subtraction affect the numerator of a fraction, because the numerator tells you how many things, or pieces, you have.

For example, say you want to add the two fractions $\frac{1}{2}$ and $\frac{1}{2}$. What you are doing is adding 3 fifths to 1 fifth. (A “fifth” is the very specific pie slice, as seen here.)



If you were dealing with integers, and added 3 to 1, you would get 4. The idea is the same with fractions. Now, instead of adding three complete units to one complete unit, you’re adding 3 fifths to 1 fifth: 1 fifth plus 3 fifths equals 4 fifths.



Notice that when you added the two fractions, the denominator stayed the same. Remember, the denominator tells you how many pieces each unit has been broken into. In other words, it determines the size of the slice. Adding three pieces to one piece did nothing to change the size of the pieces. Each unit is still broken into five pieces; hence there is no change to the denominator. The only effect of the addition was to end up with more pieces, which means that you ended up with a larger numerator.

Be able to conceptualize this process both ways: adding $\frac{1}{2}$ and $\frac{1}{2}$ to get $\frac{1}{2}$, *and* regarding $\frac{1}{2}$ as the sum of $\frac{1}{2}$ and $\frac{1}{2}$.

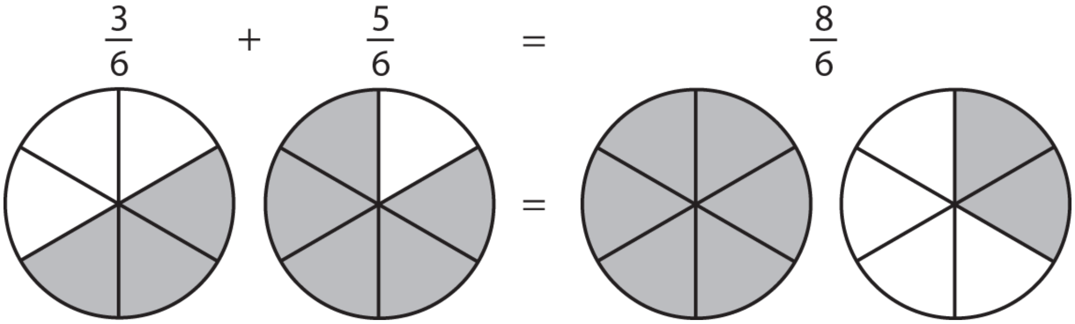
$$\frac{1}{5}+\frac{3}{5}=\frac{1+3}{5}=\frac{4}{5}$$
$$\frac{4}{5}=\frac{1+3}{5}=\frac{1}{5}+\frac{3}{5}$$

Also, you should be able to handle an *x* (or any variable) in place of one of the numerators:

$$\frac{1}{5}+\frac{x}{5}=\frac{4}{5} \quad \text{becomes} \quad 1+x=4$$

$$x=3$$

You can apply the same thinking no matter what the denominator is. Say you want to add $\frac{1}{2}$ and $\frac{1}{2}$. This is how it looks:



Notice that once again, the only thing that changes during the operation is the numerator. Adding 5 sixths to 3 sixths gives you 8 sixths. The principle is still the same even though it results in an improper fraction.

Again, see the operation both ways:

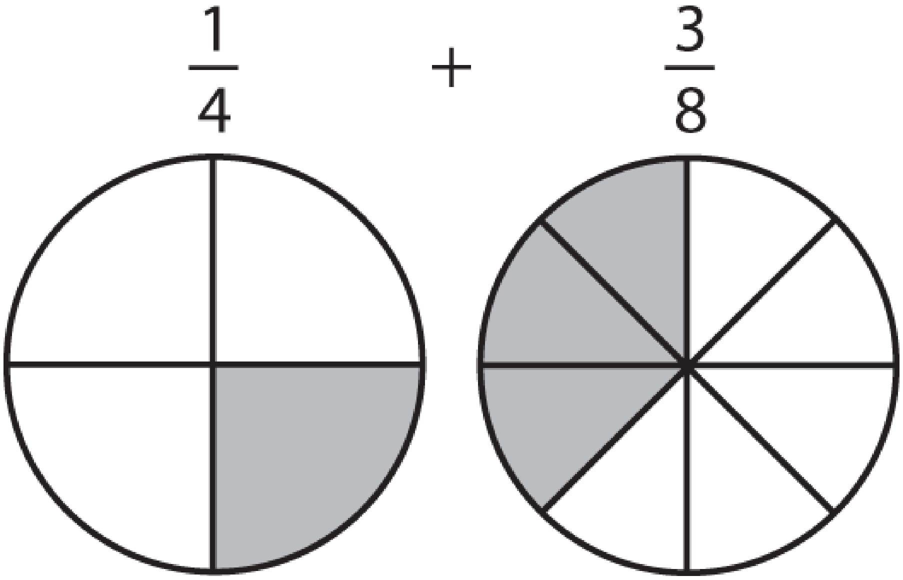
$$\frac{3}{6}+\frac{5}{6}=\frac{3+5}{6}=\frac{8}{6} \qquad \frac{3}{6}+\frac{5}{6}=\frac{3+5}{6}=\frac{8}{6}$$

Be ready for a variable as well:

$$\frac{1}{5}+\frac{x}{5}=\frac{4}{5} \quad \text{becomes} \quad 3+x=8$$

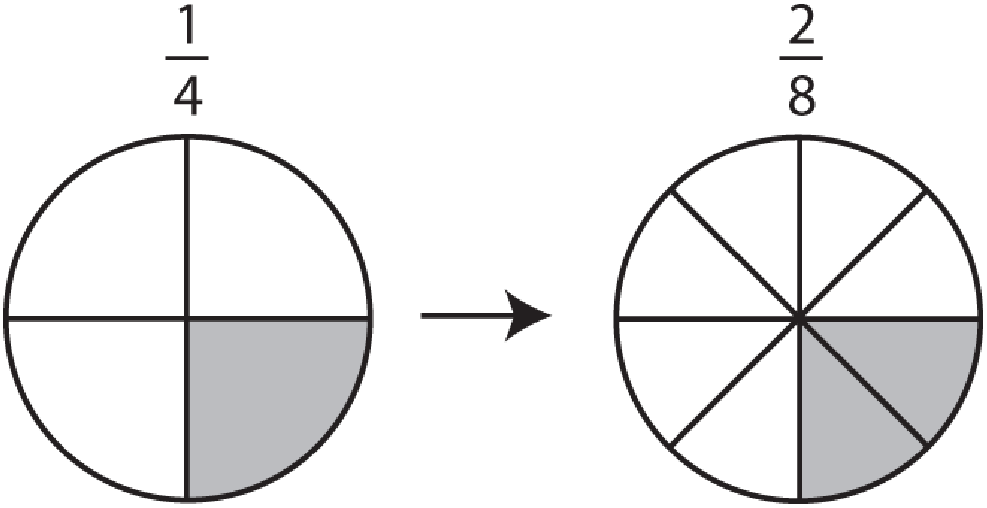
$x = 5$

Now look at a slightly different problem. This time you want to add $\frac{1}{2}$ and $\frac{1}{2}$:



Do you see the problem here? You have one slice on the left and three slices on the right, but the denominators are different, so the sizes of the slices are different. It doesn't make sense in this case simply to add the numerators and get 4 of anything. Fraction addition only works if you can add slices that are all the same size. So now the question becomes, how can you make all the slices the same size?

What you need to do is find a *new* way to express both of the fractions so that the slices are the same size. For this particular addition problem, take advantage of the fact that one-fourth is twice as big as one-eighth. Look what happens if you take all the fourths in the first circle and divide them in two:

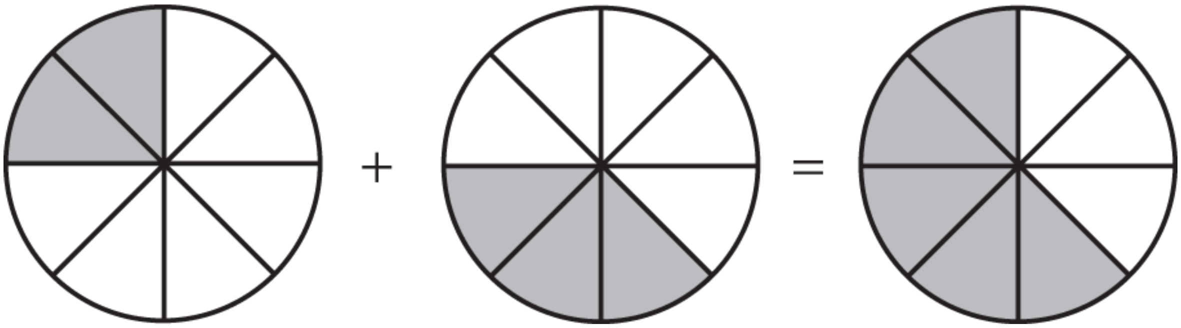


What happened to the fraction? The first thing to note is that the value of the fraction hasn't changed. Originally, you had 1 piece out of 4. Once you divided every part into 2, you ended up with 2 pieces out of 8. So you ended up with twice as many pieces, but each piece was half as big. So you actually ended up with the same amount of "stuff" overall.

What did you change? You ended up with twice as many pieces, which means you multiplied the numerator by 2, and broke the circle into twice as many pieces, which means you also multiplied the denominator by 2. So you ended up with $\frac{1 \times 2}{4 \times 2} = \frac{2}{8}$. This concept will be reviewed later, but for now, simply make sure that you understand that $\frac{1}{4} = \frac{2}{8}$.

So without changing the value of $\frac{1}{2}$, you have now found a way to *rename* $\frac{1}{2}$ as $\frac{1}{2}$, so you can add it to $\frac{1}{2}$. Now the problem looks like this:

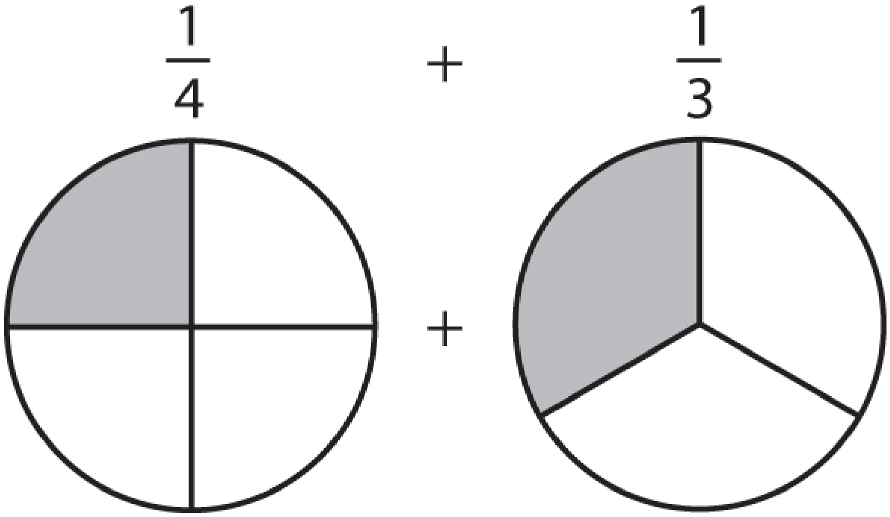
$$\cancel{\frac{1}{4}}\ \frac{2}{8} \quad + \quad \frac{3}{8} \quad = \quad \frac{5}{8}$$



The key to this addition problem was to find what's called a **common denominator**. Finding a common denominator simply means renaming the fractions so they have the same denominator. Then, and *only* then, can you add the renamed fractions.

All the details of fraction multiplication shouldn't concern you just yet (don't worry—it's coming), but you need to take a closer look at what you did to the fraction $\frac{1}{2}$ to rename it. Essentially what you did was multiply this fraction by $\frac{2}{2}$: $\frac{1}{4} = \frac{1}{4} \times \frac{2}{2} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$. As was discussed earlier, any fraction in which the numerator equals the denominator is 1. So $\frac{1}{2}$ = 1. That means that all the process did was multiply $\frac{1}{2}$ by 1. And anything times 1 equals itself. So the *appearance* of $\frac{1}{2}$ was changed by multiplying the top and bottom by 2, but its *value* was not.

Finding common denominators is a critical skill when dealing with fractions. Here's another example to consider (pay close attention to how the process works). This time, add $\frac{1}{2}$ and $\frac{1}{2}$:

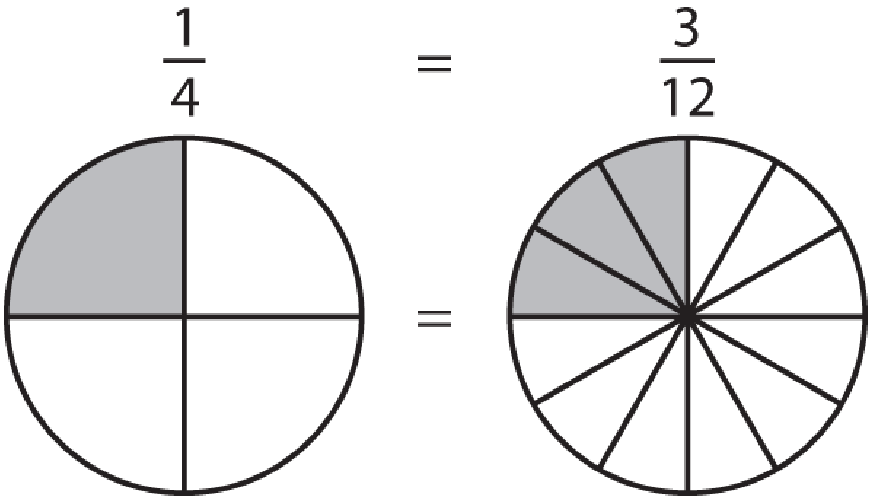


Once again, you are adding two fractions with different-sized pieces. There's no way to complete the addition without finding a common denominator. But remember, the only way to find common denominators is by multiplying one or both of the fractions by some version of 1 (such as $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, etc.). Because you can only multiply by 1 (the number that won't change the value of the fraction), the only way you can change the denominators is through multiplication. In the last example, the two denominators were 4 and 8. You could make them equal because 4 × 2 = 8.

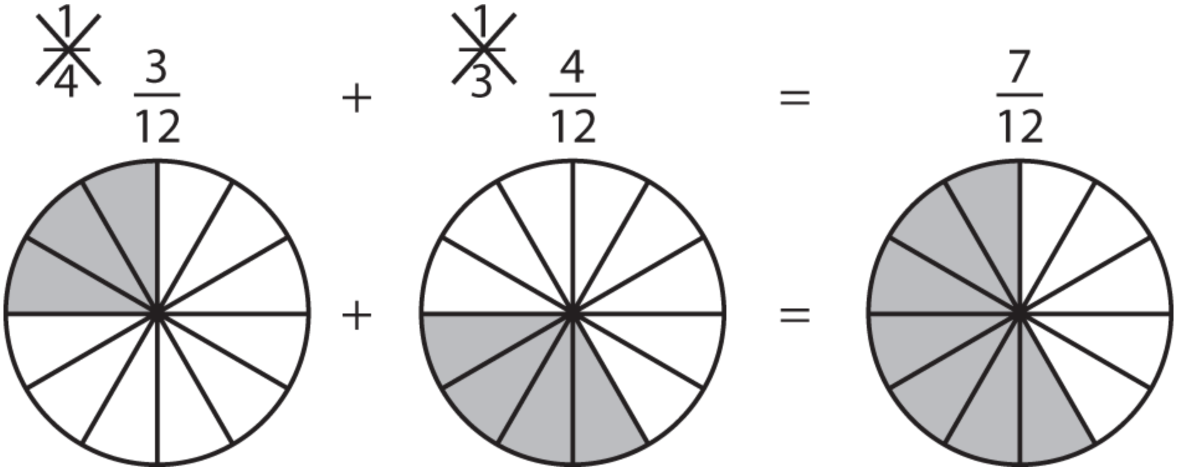
Because all you can do is multiply, what you really need when you look for a common denominator is a common *multiple* of both denominators. In the last example, 8 was a multiple of both 4 and 8.

In this problem, find a number that is a multiple of both 4 *and* 3. List a few multiples of 4: 4, 8, 12, 16. Also list a few multiples of 3: 3, 6, 9, 12. Stop; the number 12 is on both lists, so 12 is a multiple of both 3 and 4. Now change both fractions so that they have a denominator of 12.

Begin by changing $\frac{1}{2}$. You have to ask the question, what times 4 equals 12? The answer is 3. That means that you want to multiply $\frac{1}{2}$ by $\frac{1}{2}$: $\frac{1}{4} \times \frac{3}{3} = \frac{3}{12}$. So $\frac{1}{2}$ is the same as $\frac{23}{7}$. Once again, look at the circles to verify these fractions are the same:



Now you need to change $\frac{1}{2}$. What times 3 equals 12? The answer is 4 × 3 = 12, so you need to multiply $\frac{1}{2}$ by $\frac{1}{2}$: $\frac{c}{2} \leq b - 3 \leq \frac{d}{2}$. Now both of your fractions have a common denominator, so you're ready to add:



$$\frac{1}{4}+\frac{1}{3}=\frac{1\times 3}{4\times 3}+\frac{1\times 4}{3\times 4}=\frac{3}{12}+\frac{4}{12}=\frac{7}{12}$$

And now you know everything you need to add any two fractions together.

Here's a recap what you've done so far:

- When adding fractions, you have to add equal-sized pieces. That means you need the denominators to be the same for any fractions you want to add. **If the denominators are the same, then you add the numerators and keep the denominator the same.**

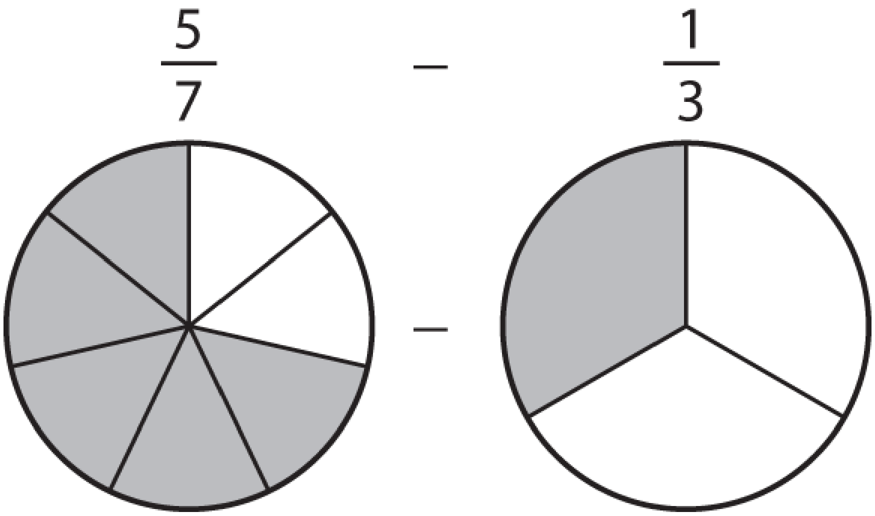
$$\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$$

- If the two fractions have different denominators,

you need to find a common multiple for the two denominators first.	$\frac{1}{4}+\frac{2}{5}=?$
• Once you know the common multiple, figure out what number for each fraction multiplies the denominator to reach the common multiple.	Common multiple of 4 and 5 = 20
• Using the number you found in the last step, multiply each fraction that needs to be changed by the appropriate fractional version of 1 (such as $\frac{1}{2}$).	$\frac{1}{4}\times\frac{5}{5}=\frac{5}{20}$ $\frac{2}{5}\times\frac{4}{4}=\frac{8}{20}$
• Now that the denominators are the same, you can add the fractions.	$\frac{5}{20}+\frac{8}{20}=\frac{13}{20}$

This section would not be complete without a discussion of subtraction. The good news is that subtraction works exactly the same way as addition! The only difference is that when you subtract, you end up with fewer pieces instead of more pieces, so you end up with a smaller numerator.

Consider the following problem. What is $\frac{1}{2}-\frac{1}{2}$?



Just like addition, subtraction of fractions requires a common denominator. So you need to figure out a common multiple of the two denominators: 7 and 3. The number 21 is a common multiple, so use that.

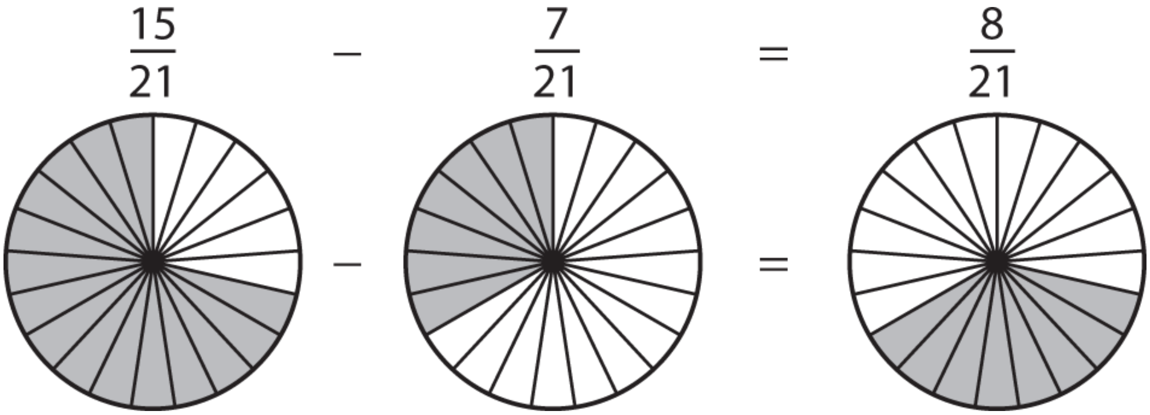
Change $\frac{1}{2}$ so that its denominator is 21. Because 3 times 7 equals 21, you multiply $\frac{1}{2}$ by $\frac{1}{2}$:

$$\frac{c}{2} \leq b - 3 \leq \frac{d}{2}.$$

Now do the same for $\frac{1}{2}$. Because 7 times 3 equals 21, you multiply $\frac{1}{2}$ by $\frac{1}{2}$:

$$\frac{c}{2} \leq b - 3 \leq \frac{d}{2}.$$

The subtraction problem can be rewritten as $\frac{15}{21} - \frac{7}{21}$, which you can easily solve:



Finally, if you have a variable in the subtraction problem, nothing really changes. One way or another, you still have to find a common denominator.

Here’s another problem:

$$\text{Solve: } \frac{1}{4} + \frac{x}{5} = \frac{13}{20}$$

First, subtract 1/4 from each side:

$$\frac{x}{5} = \frac{13}{20} - \frac{1}{4}$$

Perform the subtraction by finding the common denominator, which is 20:

$$\frac{13}{20}-\frac{1\times 5}{4\times 5}=\frac{13}{20}-\frac{5}{20}=\frac{8}{20}$$

So you have $\dfrac{20-5a}{12}$.

There are several options at this point. The one you should use right now is to convert to the common denominator again (which is still 20):

$$\frac{15}{4}=3+\frac{3}{4}=3\,\frac{3}{4}$$

Now set the numerators equal: $4x = 8$

Divide by 4: $x = 2$

If you spot the common denominator of all three fractions at the start, you can save work:

$$\frac{x}{5}=\frac{13}{20}-\frac{1}{4}$$

Convert to a common denominator of 20: $\dfrac{1\ \times\ 5}{4\ \times\ 5}+\dfrac{x\ \times\ 4}{5\ \times\ 4}=\dfrac{13}{20}$

Clean up: $\dfrac{5}{20}+\dfrac{4x}{20}=\dfrac{13}{20}$

Set numerators equal: $5 + 4x = 13$

Subtract 5: $4x = 8$

Divide by 4:

$x = 2$

Check Your Skills

Evaluate the following expressions:

3. $\frac{1}{2} + \frac{3}{4} =$

4. $\frac{1}{2} + \frac{3}{4} =$

5. Solve for x. $\frac{\mathrm{x}}{5} + \frac{2}{5} = \frac{13}{5}$

6. Solve for x. $\frac{x}{3} - \frac{4}{9} = \frac{8}{9}$

SIMPLIFYING FRACTIONS

Suppose you were presented with this question on the GRE:

$$\frac{1}{4}+\frac{2}{5}=?$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

This question involves fraction addition, which you now know how to do. So begin by adding the two fractions: $\frac{5}{9}+\frac{1}{9}=\frac{5\,+\,1}{9}=\frac{6}{9}$. But $\frac{1}{2}$ isn't one of the answer choices. Did something go wrong? No, it didn't, but there is an important step missing.

The fraction $\frac{1}{2}$ doesn't appear as an answer choice because it isn't simplified (in other words, **reduced**). To understand what that means, recall a topic that should be familiar: prime factors. Break down the numerator and denominator into prime factors: $\frac{6}{9}\rightarrow\frac{2\,\times\,3}{3\,\times\,3}$.

Notice that both the numerator and the denominator have a 3 as one of their prime factors. Because neither multiplying nor dividing by 1 changes the value of a number, you can effectively cancel the $\frac{1}{2}$, leaving behind only $\frac{1}{2}$. That is, $\frac{6}{9} = \frac{2 \times 3}{3 \times 3} = \frac{2}{3} \times \cancel{\frac{3}{3}} = \frac{2}{3}$.

Look at another example of a fraction that can be reduced: $\frac{23}{7}$. Once again, begin by breaking down the numerator and denominator into their respective prime factors: $\frac{18}{60} = \frac{2 \times 3 \times 3}{2 \times 2 \times 3 \times 5}$. This time, the numerator and the denominator have two factors in common: a 2 and a 3. Once again, split this fraction into two pieces:

$$\frac{1}{4} + \frac{1}{3} = \frac{1 \times 3}{4 \times 3} + \frac{1 \times 4}{3 \times 4} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$$

The fraction $\frac{1}{2}$ is the same as 1, so really you have $\frac{23}{7}$, which leaves you with $\frac{23}{7}$.

As you practice, you should be able to simplify fractions by recognizing the largest common factor in the numerator and denominator and canceling it out. For example, you should recognize that in the fraction $\frac{Vk}{T}$ both the numerator and the denominator are divisible by 6. That means you could think of the fraction as $\frac{20-5a}{12}$. You can then cancel out the common factors on top and bottom and simplify the fraction:

$$\frac{18}{60} = \frac{3 \times \cancel{6}}{10 \times \cancel{6}} = \frac{3}{10}.$$

Check Your Skills

Simplify the following fractions.

7. $\frac{23}{7}$

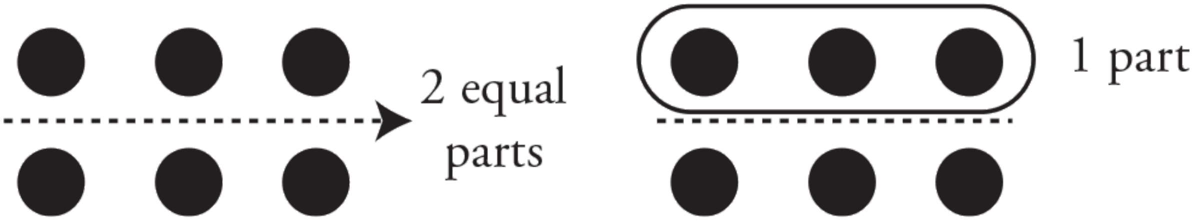
8. $\frac{23}{7}$

FRACTION MULTIPLICATION

Now that you know how to add and subtract fractions, you're ready to multiply and divide them. First up is multiplication. Consider what happens when you multiply a fraction by an integer.

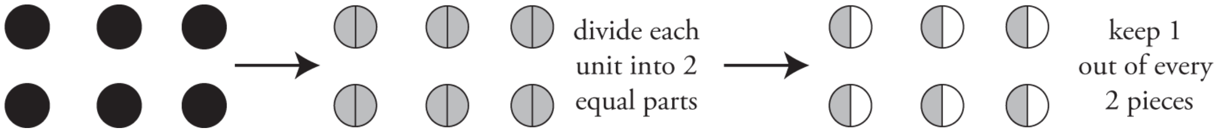
Start by considering the question, what is $\frac{1}{2}$ × 6? When you added and subtracted fractions, you were really adding and subtracting pieces of numbers. With multiplication, conceptually it is different: you start with an amount, and leave a fraction of it behind. For instance, in this example, what it's really asking is, what is $\frac{1}{2}$ *of* 6? There are a few ways to conceptualize what that means.

You want to find one-half of six. One way to do that is to split 6 into two equal parts and keep one of those parts.

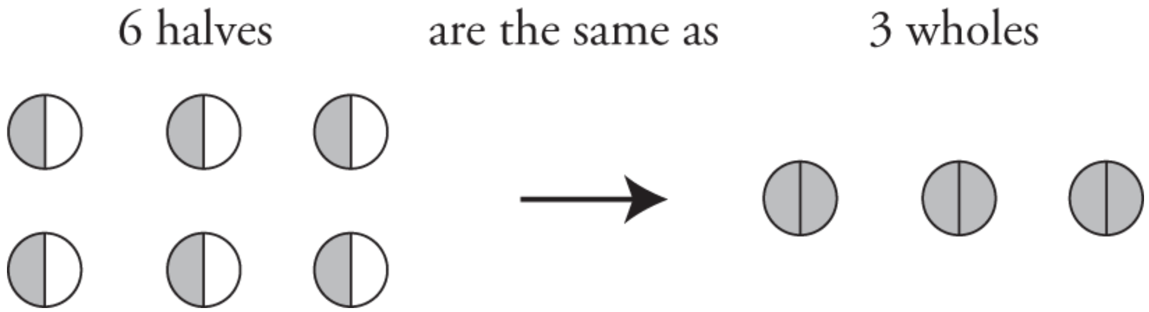


Because the denominator of the fraction is 2, divide 6 into two equal parts of 3. Then, because the numerator is 1, keep one of those parts. So $\frac{1}{2}$ × 6 = 3.

You can also think of this multiplication problem a slightly different way. Consider each unit circle of the 6. What happens if you break each of those circles into two parts, and keep one part?



Divide every circle into two parts, and keep one out of every two parts. You end up with six halves, or $\frac{1}{2}$, written as a fraction. But $\frac{1}{2}$ is the same as 3, so really, $\frac{1}{2}$ of 6 is 3:

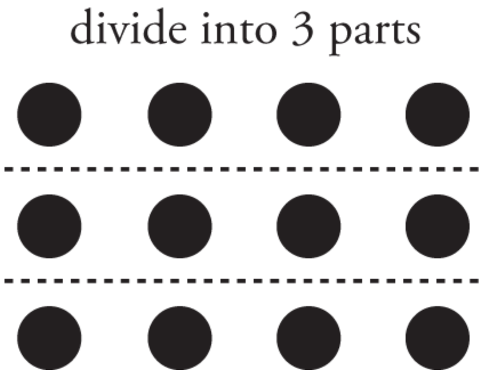


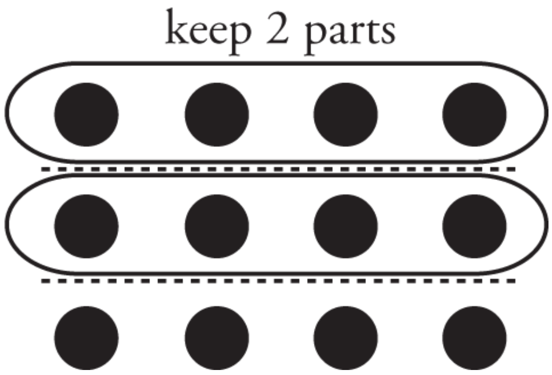
Either way you conceptualize this multiplication, you end up with the same answer. Try another example:

What is 2/3 × 12?

Once again, it's really asking, what is $\frac{1}{2}$ *of* 12? In the previous example, when you multiplied a number by $\frac{1}{2}$, you divided the number into two parts (as indicated by the denominator). Then you kept one of those parts (as indicated by the numerator).

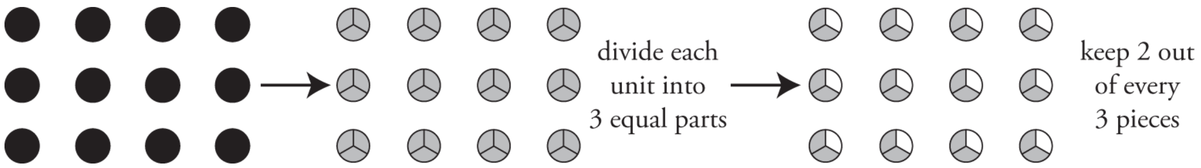
By the same logic, if you want to get $\frac{1}{2}$ of 12, you need to divide 12 into three equal parts, because the denominator is 3. Then keep two of those parts, because the numerator is 2. As with the first example, there are several ways of conceptualizing this. One way is to divide 12 into three equal parts, and keep two of those parts:



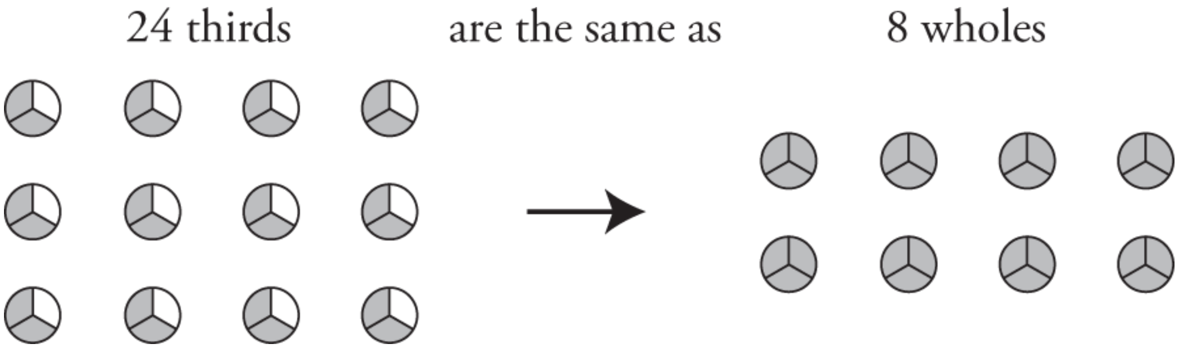


The number 12 is divided into three equal parts of 4, and two of those parts are kept. Because two groups of 4 is 8, then $\frac{1}{2}$ × 12 = 8.

Another way to conceptualize $\frac{1}{2}$ × 12 is to once again look at each unit of 12. If you break each unit into three pieces (because the denominator of the fraction is 3) and keep two out of every three pieces (because the numerator is 2) you end up with this:



You ended up with 24 thirds, or $\frac{23}{7}$. But $\frac{23}{7}$ is the same as 8, so $\frac{1}{2}$ of 12 is 8:

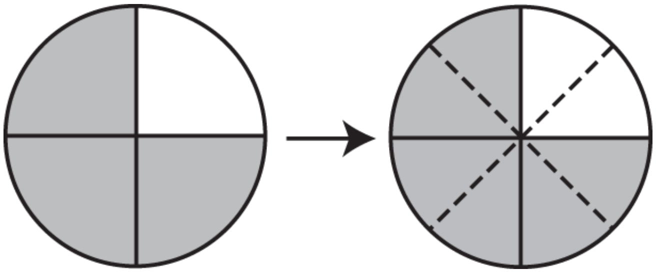


Once again, either way you think about this multiplication problem, you arrive at the same conclusion: $\frac{1}{2}$ × 12 = 8.

Now that you’ve seen what happens when you multiply an integer by a fraction, it’s time to multiply a fraction by a fraction. It’s important to remember that the basic logic is the same. When you multiply any number by a fraction, the denominator of the fraction tells you how many parts to divide the number into, and the numerator tells you how many of those parts to keep. Now consider how that logic applies to fractions:

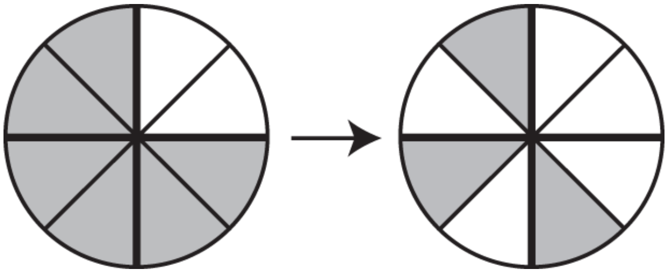
$$\frac{3x-7}{2}+20=6$$

This question is asking, what is $\frac{1}{2}$ *of* $\frac{1}{2}$? So once again, divide $\frac{1}{2}$ into two equal parts. This time, though, because you’re splitting a fraction, you’re going to do things a little differently. Because $\frac{1}{2}$ is a fraction, the unit circle has already broken a number into four equal pieces. So, break each of those pieces into two smaller pieces:



Cut each piece in half.

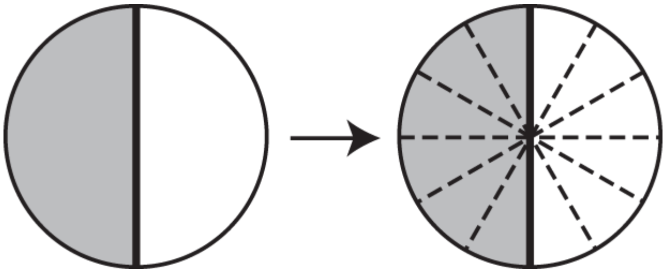
Now that you’ve divided each piece into two smaller pieces, keep one from each pair of those smaller pieces:



Keep one out of each of the two resulting pieces.

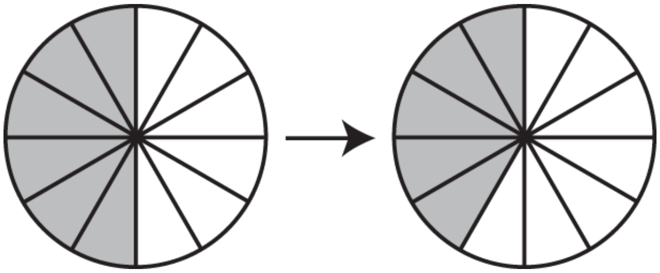
So what did you end up with? First of all, the result is going to remain a fraction. The original number was $\frac{1}{2}$. In other words, a number was broken into four parts, and you kept three of those parts. Now the number has been broken into eight pieces, not four, so the denominator is now 8. However, you still have 1 × 3 = 3 of those parts, so the numerator is still 3. So $\frac{1}{2}$ of $\frac{1}{2}$ is $\frac{1}{2}$.

Try one more. What is $\frac{1}{2} \times \frac{1}{2}$? Once again, start by dividing the fraction into six equal pieces:



Cut each piece into six smaller pieces.

Now keep five out of every six parts:



Keep five of the six.

So what did you end up with? Now you have a number divided into 12 parts, so the denominator is 12, and you keep 1 × 5 = 5 parts, so the numerator is 5. Thus, $\frac{1}{2}$ of $\frac{1}{2}$ is $\frac{23}{7}$.

Multiplying fractions would get very cumbersome if you always resorted to slicing circles up into increasingly tiny pieces. So now consider, in a *general* way, the mechanics of multiplying a number by a fraction.

First, note the following *crucial* difference between two types of arithmetic operations on fractions:

<u>Addition & Subtraction:</u>
Only the numerator changes (once you've found a *common* denominator).

<u>Multiplication & Division:</u>
Both the numerator and the denominator typically change.

The way to generalize **fraction multiplication** is this: *Multiply the numerators together to get the new numerator, and multiply the denominators together to get the new denominator*. Then, simplify (or reduce):

$$\frac{1}{2}\times\frac{6}{1}=\frac{1\times 6}{2\times 1}=\frac{6}{2}=3$$
$$\frac{2}{3}\times\frac{12}{1}=\frac{2\times 12}{3\times 1}=\frac{24}{3}=8$$
$$\frac{1}{2}\times\frac{3}{4}=\frac{1\times 3}{2\times 4}=\frac{3}{8}$$
$$\frac{5}{6}\times\frac{1}{2}=\frac{5\times 1}{6\times 2}=\frac{5}{12}$$

In practice, when you multiply fractions, *don't worry about the conceptual foundation* once you understand the mechanics:

$$1 \quad 3 \longrightarrow \text{Mechanical: } \frac{1}{2}\times\frac{3}{4}=\frac{1\times 3}{2\times 4}=\frac{3}{8} \quad \textit{easier}$$

$\overline{2}$ ^ $\overline{4}$ → Conceptual: "one-half of 3/8"…cut up the circles further… *harder*

Finally, whenever you multiply fractions, always look to cancel common factors to reduce your answer without doing unnecessary work:

$$\frac{33}{7} \times \frac{14}{3} = ?$$

The long way to do this is:

$$33 \times 14 = \qquad \begin{array}{r} {}^{1}33 \\ \times 14 \\ \hline 132 \\ 330 \\ \hline 462 \end{array}$$

$$7 \times 3 = 21$$

You wind up with $\frac{256}{16}$:

$$\begin{array}{r} 22 \\ 21\overline{)462} \\ -42 \\ \hline 42 \\ -42 \\ \hline \end{array}$$

This work can be simplified greatly by *canceling* parts of each fraction *before* multiplying. Always look for common factors in the numerator and denominator:

$$\frac{33}{7} \times \frac{14}{3} = \frac{3 \times 11}{7} \times \frac{2 \times 7}{3}$$

It's now clear that the numerator of the first fraction has a 3 as a factor, which can be canceled out with the 3 in the denominator of the second fraction. (This is because multiplication and division operate at the same level of priority in the PEMDAS operations.) Similarly, the 7 in the denominator of the first fraction can be canceled out by the 7 in the numerator of the second fraction. By cross-canceling these factors, you can save yourself a lot of work:

$$\frac{\cancel{3} \times 11}{\cancel{7}} \times \frac{2 \times \cancel{7}}{\cancel{3}} = \frac{11}{1} \times \frac{2}{1} = \frac{22}{1} = 22$$

Check Your Skills

Evaluate the following expressions. Simplify all fractions:

9. $\frac{3}{10} \times \frac{6}{7} =$

10. $\frac{5}{14} \times \frac{7}{20} =$

FRACTION DIVISION

Next up is the last of the four basic arithmetic operations on fractions (addition, subtraction, multiplication, and division). This section will be a little different than the other three—it will be different because you're actually going to do fraction division by avoiding division altogether.

You can avoid division entirely because of the relationship between multiplication and division. Multiplication and division are opposite sides of the same coin. Any multiplication problem can be expressed as a division problem, and vice-versa. This is useful because, although the mechanics for multiplication are straightforward, the mechanics for division are more *work* and therefore more *difficult*. Thus, you should express every fraction division problem as a fraction multiplication problem.

Now the question becomes: how do you rephrase a division problem so that it becomes a multiplication problem? The key is **reciprocals**.

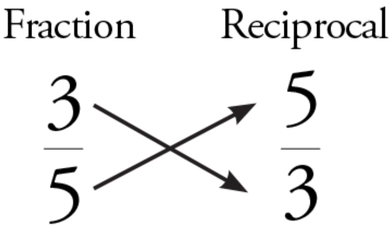
Reciprocals are numbers that, when multiplied together, equal 1. For instance, $\frac{1}{2}$ and $\frac{1}{2}$ are reciprocals, because

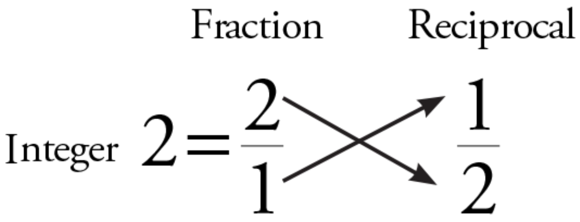
$$\frac{3}{5} \times \frac{5}{3} = \frac{3 \times 5}{5 \times 3} = \frac{15}{15} = 1.$$

Another pair of reciprocals is 2 and $\frac{1}{2}$, because

$2 \times \frac{1}{2} = \frac{2}{1} \times \frac{1}{2} = \frac{2 \times 1}{1 \times 2} = \frac{2}{2} = 1.$ (Once again, it is important to remember that *every integer can be thought of as a fraction*.)

The way to find the reciprocal of a number turns out to generally be very easy—take the numerator and denominator of a number, and switch them around:

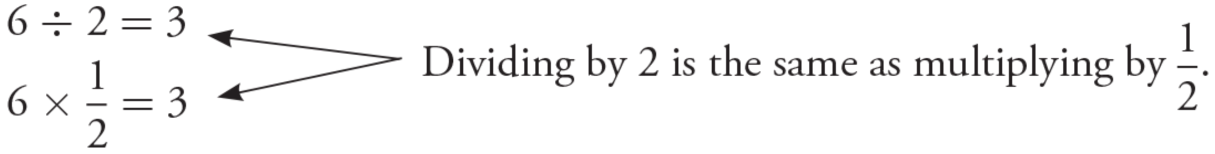




Reciprocals are important because *dividing by a number is the exact same thing as multiplying by its reciprocal*. Look at an example to clarify:

What is 6 ÷ 2?

This problem shouldn't give you any trouble: 6 divided by 2 is 3. But it should also seem familiar because it's the exact same problem you dealt with in the discussion on fraction multiplication: 6 ÷ 2 is the *exact same thing* as 6 × $\frac{1}{2}$.



To change from division to multiplication, you need to do two things. First, take the divisor (the number to the right of the division sign—in other words, what you are dividing *by*) and replace it with its reciprocal. In this problem, 2 is the divisor, and $\frac{1}{2}$ is the reciprocal of 2. Then, change the division sign to a multiplication sign. So 6 ÷ 2 becomes 6 × $\frac{1}{2}$. Then, proceed to do the multiplication:

$$\frac{1}{4}+\frac{1}{3}=\frac{1\times 3}{4\times 3}+\frac{1\times 4}{3\times 4}=\frac{3}{12}+\frac{4}{12}=\frac{7}{12}$$

This is obviously overkill for 6 ÷ 2, but try another one. What is $\frac{1}{2} \div \frac{1}{2}$?

Once again, start by taking the divisor ($\frac{1}{2}$) and replacing it with its reciprocal ($\frac{1}{2}$). Then change the division sign to a multiplication sign. So $\frac{1}{2} \div \frac{1}{2}$ is the same as $\frac{1}{2} \times \frac{1}{2}$. Now do fraction multiplication:

$$\frac{5}{6}\div\frac{4}{7}=\frac{5}{6}\times\frac{7}{4}=\frac{5\times 7}{6\times 4}=\frac{35}{24}$$

Note that the fraction bar (sometimes indicated with a slash) is another way to express division. After all, $6\div 2=6/2=\frac{6}{2}=3.$ In fact, the division sign, ÷, looks like a little fraction. So if you see a “double-decker” fraction, don’t worry. It’s just one fraction divided by another fraction.

$$\frac{\frac{5}{6}}{\frac{4}{7}} = \frac{5}{6} \div \frac{4}{7} = \frac{5}{6} \times \frac{7}{4} = \frac{35}{24}$$

To recap:

- When you are confronted with a division problem involving fractions, it is *always* easier to perform multiplication than division. For that reason, every fraction division problem should be rewritten as a multiplication problem.

- To do so, replace the divisor with its reciprocal. To find the reciprocal of a number, you simply need to switch the numerator and denominator (e.g., 2/9 becomes 9/2).

$$\begin{array}{ccc} \textbf{Fraction} & & \textbf{Reciprocal} \\ \frac{2}{9} & \rightarrow & \frac{9}{2} \end{array}$$

- Remember that a number multiplied by its reciprocal equals 1.

$$\frac{1}{4} + \frac{2}{5} = ?$$

- After that, switch the division symbol to a multiplication symbol, and perform fraction multiplication.

$$\frac{3}{4} \div \frac{2}{9} \rightarrow \frac{3}{4} \times \frac{9}{2} = \frac{27}{8}$$

Check Your Skills

Evaluate the following expressions. Simplify all fractions.

11. $\frac{3}{10} \times \frac{6}{7} =$

12. $\frac{3}{10} \times \frac{6}{7} =$

FRACTIONS IN EQUATIONS

When an *x* appears in a fraction multiplication or division problem, you'll use essentially the same concepts and techniques to solve:

$$\frac{3}{10} \text{ vs. } \frac{3}{13}$$

Divide both sides by $\frac{4}{3}$:

$$x = \frac{15}{8} \div \frac{4}{3}$$

$$x = \frac{15}{8} \times \frac{3}{4} = \frac{45}{32}$$

An important tool to add to your arsenal at this point is *cross-multiplication*. This tool comes from the principle of making common denominators.

$$\frac{x}{7} = \frac{5}{8}$$

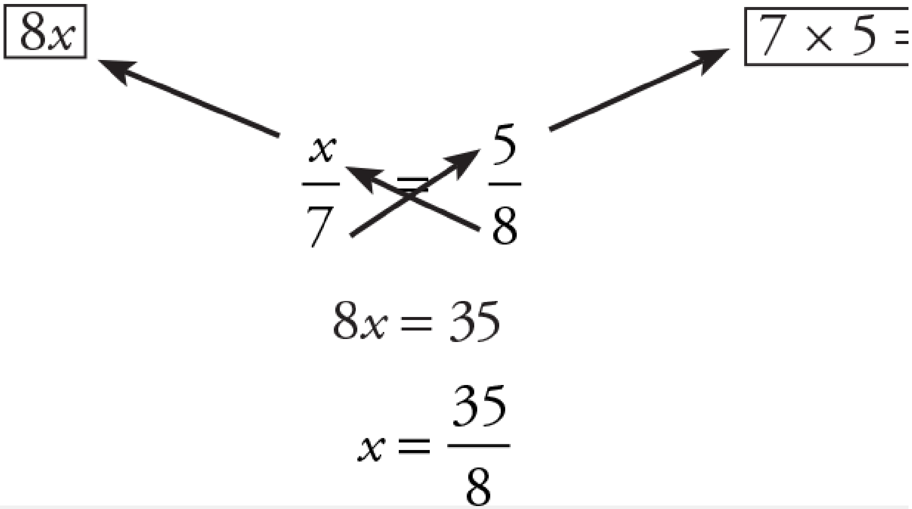
The common denominator of 7 and 8 is 7 × 8 = 56. So you have to multiply the left fraction by $\frac{1}{2}$ and the right fraction by $\frac{1}{2}$:

$$\frac{8 \times x}{8 \times 7} = \frac{5 \times 7}{8 \times 7} \rightarrow \frac{8x}{56} = \frac{35}{56}$$

Now you can set the numerators equal:

$$\begin{aligned} 8x &= 5 \times 7 = 35 \\ x &= 35/8 \end{aligned}$$

However, in this situation you can avoid having to determine the common denominator explicitly by cross-multiplying each numerator by the other denominator and setting the products equal to each other:



When you cross-multiply, you are in fact multiplying each side by the common denominator. You're just doing so really efficiently, without having to determine the actual value of that common denominator. Consider what happens when you multiply each side of the original equation by 56 = 7 × 8. When you multiply *x*/7 on the left by 56 = 7 × 8, the 7's cancel and you're left with 8*x*. Likewise, when you multiply 5/8 on the right by 56 = 7 × 8, the 8's cancel and you're left with = 7 × 5. By cross-multiplying, you're already doing the right canceling, and you don't have to figure out that 56 in the first place.

Check Your Skills

Solve for *x* in the following equations:

13. $\frac{3}{4}x = \frac{3}{2}$

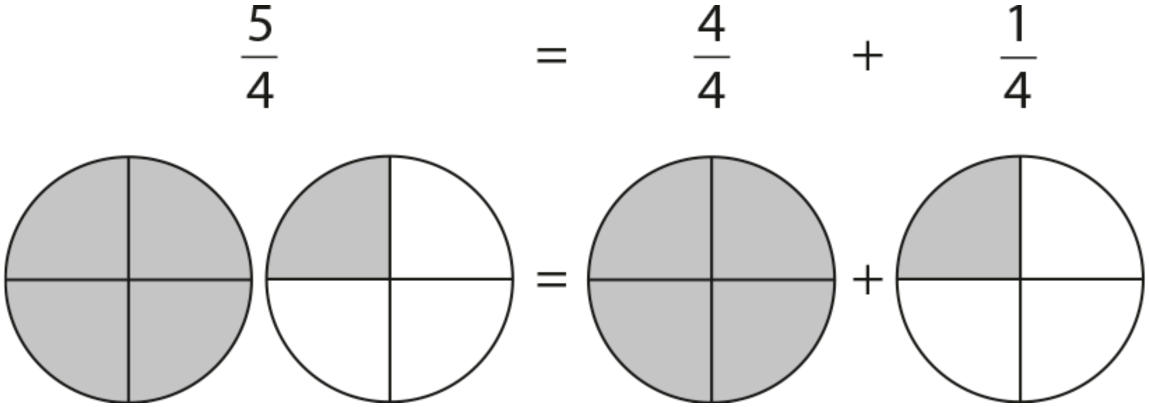
14. $\frac{x}{7} = \frac{5}{8}$

Switching Between Improper Fractions and Mixed Numbers

Recall the earlier discussion of why $\frac{1}{2}$ equals $1\frac{1}{4}$ and how to switch between improper fractions and mixed numbers.

To do this, the numerator needs to be discussed in more detail. The numerator is a description of how many parts you have. The fraction $\frac{1}{2}$ tells you that you have five parts. But you have some flexibility in how you arrange those five parts. For instance, you already expressed it as $\frac{1}{2}$ + $\frac{1}{2}$, or 1 + $\frac{1}{2}$. Essentially, what you did was to split the numerator into two pieces: 4 and 1. If you wanted to express this as a fraction, you could say that $\frac{1}{2}$ becomes $\frac{4+1}{4}$. This hasn't changed anything, because 4 + 1 equals 5, so you still have the same number of parts.

Then, as noted above, you can split the fraction into two separate fractions. For instance $\frac{4+1}{4}$ becomes $\frac{4}{4}+\frac{1}{4}$. This is the same as saying that 5 fourths equals 4 fourths plus 1 fourth. So there are several different ways of representing the same fraction. For example, $\frac{3}{6}+\frac{5}{6}=\frac{3+5}{6}=\frac{8}{6}$. Here is a visual representation:



As a general rule, *you can always split the numerator of a fraction into different parts and thus split a fraction into multiple fractions*. This is just *reversing* the process of adding fractions. When you add fractions, take two fractions with the same denominator and combine them into one fraction. Here do the exact opposite—turn one fraction into two separate fractions, each with the same denominator. Now that the fraction $\frac{1}{2}$ is split into two fractions, you can take advantage of the fact that fractions, at their essence, represent *division*. As was discussed earlier, $\frac{1}{2}$ = 1, and another way to think of $\frac{1}{2}$ is 4 ÷ 4.

To switch from an improper fraction to a mixed number, figure out how many complete units there are. To do that, figure out the *largest multiple of the denominator that is less than or equal to the numerator*. For the fraction $\frac{1}{2}$, 4 is the largest multiple of 4 that is less than 5. So split the fraction into $\frac{1}{2}$ and 1/4. Note that $\frac{1}{2}$ equals 1, so the mixed number is $1\frac{1}{4}$.

Try it again with the fraction $\frac{23}{7}$. This time, the largest multiple of 4 that is less than 15 is 12. So you can split the fraction $\frac{23}{7}$ into $\frac{23}{7}$ + $\frac{1}{2}$. In other words, $\frac{15}{4} = \frac{12+3}{4} = \frac{12}{4} + \frac{3}{4}$. And $\frac{23}{7}$ = 3, so the fraction $\frac{23}{7}$ becomes the mixed number $1\frac{1}{4}$.

Try one with a different denominator. How do you turn the fraction $\frac{23}{7}$ into a mixed number? This time you need the largest multiple of 7 that is less than or equal to 16. The number 14 is the largest multiple of 7 that is less than 16, so once again split the fraction $\frac{23}{7}$ into $\frac{23}{7}$ and $\frac{1}{2}$. In other words, $\frac{15}{4} = \frac{12+3}{4} = \frac{12}{4} + \frac{3}{4}$. Because 14 divided by 7 equals 2, your mixed number is $1\frac{1}{4}$.

Check Your Skills

Change the following improper fractions to mixed numbers:

15. $\frac{23}{7}$

16. $\frac{256}{16}$

CHANGING MIXED NUMBERS TO IMPROPER FRACTIONS

Now that you know how to change a number from an improper fraction to a mixed number, you also need to be able to do the reverse. Suppose you have the mixed number $1\frac{1}{4}$. How do you turn this number into a fraction?

Remember that you can think of any integer as a fraction. The number 1, for instance, can be thought of many different ways. It can be thought of as $\frac{1}{2}$. It can also be thought of as $\frac{1}{2}$. In other words, a unit circle can be split into 2 equal pieces, with 2 of those pieces forming a whole unit circle again. 1 can also be written as $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, and so on.

In fact, you can think of the process of turning mixed numbers into improper fractions as simple fraction addition. The fraction $1\,\frac{1}{4}$ is the same thing as 5 + $\frac{1}{2}$, so you can think of it as $\frac{4}{4} + \frac{1}{4}$. Now you know what to do—change $\frac{1}{2}$ so that it has a denominator of 3. The way to do that is to multiply $\frac{1}{2}$ by $\frac{1}{2}$, which equals $\frac{5}{9} + \frac{1}{9} = \frac{5\,+\,1}{9} = \frac{6}{9}$. So the mixed number is really $\frac{15}{4} = \frac{12+3}{4} = \frac{12}{4} + \frac{3}{4}$.

Check Your Skills

Change the following mixed numbers to improper fractions.

17. $1\,\frac{1}{4}$

18. $1\,\frac{1}{4}$

Division in Disguise

Sometimes, dividing fractions can be written in a confusing way. Consider one of the previous examples:

$\frac{4}{4} + \frac{1}{4}$ can also be written as a “double-decker,” or **complex**, fraction like this: $\dfrac{\frac{1}{2}}{\frac{3}{4}}$

Do not be confused. You can rewrite this as the top fraction divided by the bottom fraction, and solve it normally (by using the reciprocal of the bottom fraction and then multiplying):

$$\dfrac{\frac{1}{2}}{\frac{3}{4}} = \frac{1}{2} \div \frac{3}{4} = \frac{1}{2} \times \frac{4}{3} = \frac{4}{6} = \frac{2}{3}$$

Here’s a **speed tip** for problems like this: notice that, quite often, you can quickly simplify by multiplying both the top fraction and the bottom fraction by a common denominator:

$$\frac{\frac{1}{2}}{\frac{3}{4}} = \frac{\frac{1}{2} \times 4}{\frac{3}{4} \times 4} = \frac{2}{3}$$

Check Your Skills

Evaluate the following complex fractions by multiplying the top and bottom fractions by a common denominator:

19. $\frac{\frac{3}{5}}{\frac{2}{3}} = ?$

20. $\frac{\frac{3}{5}}{\frac{2}{3}} = ?$

Fraction Operations: Know What to Expect

Adding and subtracting fractions leads to expected results: when you add two positive fractions, you get a larger number; when you subtract a positive fraction from something else, you get a smaller number.

However, multiplying by fractions between 0 and 1 yields **unexpected** results:

$$9 \times \frac{1}{3} = 3 \qquad\qquad 3 < 9$$

Multiplying a number by a fraction between 0 and 1 creates a product **smaller** than the original number. Note that this is also true when the original number is a fraction:

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8} \qquad\qquad \frac{1}{8} < \frac{1}{2}$$

Conversely, dividing by a fraction between 0 and 1 yields a quotient, or result, that is **larger** than the original number:

$$\frac{6}{\frac{3}{4}} = 6 \div \frac{3}{4} = 6 \times \frac{4}{3} = \frac{24}{3} = 8 \qquad 8 > 6$$

This is also true when the original number is a fraction:

$$\frac{\frac{1}{4}}{\frac{5}{6}} = \frac{1}{4} \div \frac{5}{6} = \frac{1}{4} \times \frac{6}{5} = \frac{6}{20} = \frac{3}{10} \qquad \frac{3}{10} > \frac{1}{4}$$

Check Your Skills

21. $\frac{1}{4} + \frac{2}{5} = ?$

22. $\frac{1}{4} + \frac{2}{5} = ?$

Comparing Fractions: Cross–Multiply

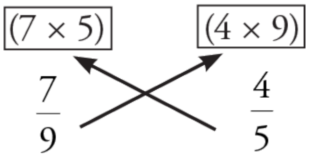
Earlier you were introduced to the technique of *cross-multiplying* in the context of solving for a variable in an equation that involved fractions. Now look at another use of cross-multiplication:

Which fraction is greater, $\frac{4}{4} + \frac{1}{4}$?

The traditional technique used to compare fractions involves finding a common denominator, multiplying, and comparing the two fractions. The common denominator of 9 and 5 is 45.

Thus, $\frac{7}{9} = \frac{35}{45} \text{ and } \frac{4}{5} = \frac{36}{45}$. Because 35 < 36, you can see that $\frac{1}{2}$ is slightly bigger than $\frac{1}{2}$.

However, there is a shortcut to comparing fractions called (you guessed it): **cross-multiplication**. This process involves multiplying the numerator of one fraction with the denominator of the other fraction, and vice-versa. Again, this maneuver is legal because you’re just secretly and efficiently multiplying each side by the common denominator. Here’s how the procedure works:

$\frac{1}{2}$ $\frac{1}{2}$	Set up the fractions next to each other.
	Cross-multiply the fractions and put each answer by the corresponding *numerator.* (**Not** the denominator!)
35 < 36	Because 35 is less than 36, the first fraction must be less than the second one.

Check Your Skills

23. Which fraction is greater? $\frac{4}{13}$ or $\frac{1}{3}$

24. Which fraction is smaller? $\frac{4}{13}$ or $\frac{1}{3}$

NEVER Split the Denominator!

One final rule—perhaps the most important—is one that you must *always* remember when working with fractions that have an expression (more than one term) in the numerator or denominator. Three examples are:

$$\frac{15+10}{5}$$

$$\frac{5}{15+10}$$

$$\frac{5}{15+10}$$

In example (a), the numerator is expressed as a sum.

In example (b), the denominator is expressed as a sum.

In example (c), both the numerator and the denominator are expressed as sums.

When simplifying fractions that incorporate sums (or differences), remember this rule: You may split up the terms of the numerator, *but you may **never** split the terms of the **denominator***.

Thus, the terms in example (a) may be split:

$$\frac{15+10}{5} = \frac{15}{5} + \frac{10}{5} = 3 + 2 = 5$$

But the terms in example (b) *may not* be split:

$$\frac{5}{15+10} \neq \frac{5}{15} + \frac{5}{10} \quad \textbf{No!}$$

Instead, simplify the denominator first:

$$\frac{5}{15+10} = \frac{5}{25} = \frac{1}{5}$$

The terms in example (c) may not be split either:

$$\frac{5}{15+10} \neq \frac{5}{15} + \frac{5}{10} \quad \textbf{No!}$$

Instead, simplify both parts of the fraction:

$$\frac{15+10}{5+2} = \frac{25}{7} = 3\,\frac{4}{7}$$

Often, GRE problems will involve complex fractions with variables. On these problems, it is tempting to split the denominator. *Do not fall for it!*

It is tempting to perform the following simplification:

$$\frac{5x-2y}{x-y}=\frac{5x}{x}-\frac{2y}{y}=5-2=3 \quad \textbf{No!}$$

This is **wrong** because you cannot split terms in the denominator.

The reality is that $\frac{5x-2y}{x-y}$ *cannot be simplified further*.

On the other hand, the expression $\frac{6x-15y}{10}$ can be simplified by splitting the difference, *because this difference appears in the numerator*. Thus:

$$\frac{6x-15y}{10}=\frac{6x}{10}-\frac{15y}{10}=\frac{3x}{5}-\frac{3y}{2}$$

Check Your Skills

Simplify the following fractions:

25. $\frac{13+7}{5}$

26. $\frac{21+6}{7+6}$

27. $\frac{48a+12b}{a+b}$

28. $\frac{9g-6h}{6g-4h}$

Benchmark Values

You will use a variety of estimating strategies on the GRE. One important strategy for estimating with fractions is to use **Benchmark Values**. These are simple fractions with which you are already familiar:

$$\frac{1}{10}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \text{and } \frac{3}{4}$$

You can use Benchmark Values to compare fractions:

Which is greater: $\frac{127}{255}$ or $\frac{162}{320}$?

If you recognize that 127 is *less* than half of 255, and 162 is *more* than half of 320, you will save yourself a lot of cumbersome computation.

You can also use Benchmark Values to estimate computations involving fractions:

Approximately what is $\frac{10}{22}$ of $\frac{5}{18}$ of 2,000?

If you recognize that these fractions are very close to the Benchmark Values $\frac{1}{2}$ and $\frac{1}{2}$, you can estimate:

$$\frac{1}{2} \text{ of } \frac{1}{4} \text{ of } 2{,}000 = \frac{1}{2} \times \frac{1}{4} \times 2{,}000 = 250\text{; therefore, } \frac{10}{22} \text{ of } \frac{5}{18} \text{ of } 2{,}000 \approx 250$$

Notice that the rounding errors compensated for each other:

$\frac{10}{22} \approx \frac{10}{20} = \frac{1}{2}$ You decreased the denominator, so you rounded up: $\frac{15+10}{5}$.

$\frac{10}{22} \approx \frac{10}{20} = \frac{1}{2}$ You increased the denominator, so you rounded down: $\frac{15+10}{5}$.

Note also that $\frac{10}{22} \times \frac{5}{18} \times 2{,}000 = \frac{100,000}{396} = \frac{25,000}{99} = 252.5\overline{25}$, so your estimation was very close.

If instead you had rounded $\frac{5}{18}$ to $\frac{6}{18} = \frac{1}{3}$ instead, then you would have rounded *both* fractions up. This would lead to a *slight* but *systematic* overestimation:

$$\frac{1}{2} \times \frac{1}{3} \times 2000 \approx 333$$

Try to make your rounding errors partially cancel each other out by rounding some numbers up and others down.

Check Your Skills

29. Which is greater: $\frac{127}{255}$ or $\frac{162}{320}$?

30. Approximate $\left(\frac{15}{58}\right)\left(\frac{9}{19}\right)403$

Picking Smart Numbers: Multiples of the Denominators

Sometimes, fraction problems on the GRE include **unspecified** numerical amounts; sometimes these unspecified amounts are described by variables, other times they are not. In these cases, you often can pick **real numbers** to stand in for the variables. To make the computation easier, choose **Smart Numbers** equal to *common multiples of the denominators of the fractions in the problem*.

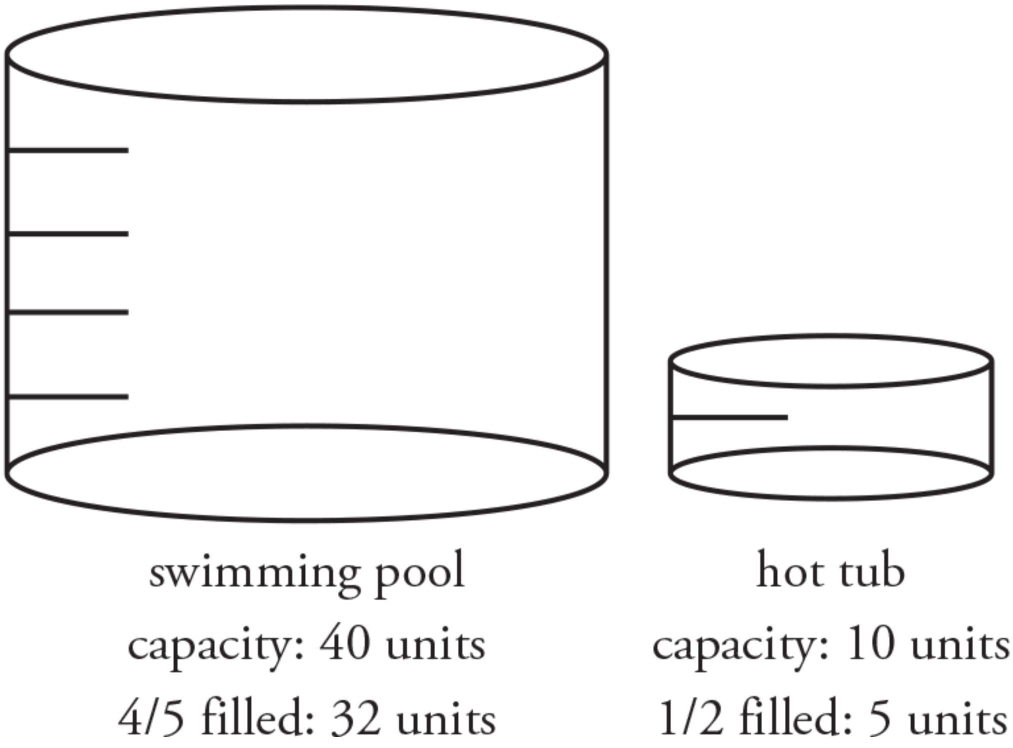
For example, consider this problem:

> The Crandalls's hot tub is halfway filled. Their swimming pool, which has a capacity four times that of the hot tub, is filled to four-fifths of its capacity. If the hot tub is drained into the swimming pool, to what fraction of its capacity will the swimming pool be filled?

The denominators in this problem are 2 (from $\frac{1}{2}$ of the hot tub) and 5 (from $\frac{1}{2}$ of the swimming pool). The Smart Number in this case is the least common denominator, which is 10. Therefore, assign the hot tub, *the smaller quantity*, a capacity of 10 units. Because the swimming pool has a capacity four times that of the hot tub, the swimming pool has a capacity of 40 units. The hot tub is only halfway filled; therefore, it has 5 units of water in it. The swimming pool is four-fifths of the way filled, so it has 32 units of water in it.

Add the 5 units of water from the hot tub to the 32 units of water that are already in the swimming pool: 32 + 5 = 37.

With 37 units of water and a total capacity of 40, the pool will be filled to $\frac{23}{7}$ of its total capacity:



Check Your Skills

Choose Smart Numbers to solve the following problem:

31. Mili's first-generation uHear is filled to $\frac{1}{2}$ capacity with songs. Her second-generation uHear, which has 3 times the capacity of her first-generation uHear, is filled to $\frac{1}{2}$ capacity. Will Mili be able to transfer all of her music from her first-generation uHear to her second-generation uHear?

When NOT to Use Smart Numbers

In some problems, even though an amount might be unknown to you, it is actually specified in the problem in another way—specifically, because some other related quantity *is given*. In these cases, you *cannot use* Smart Numbers to assign real numbers to the variables. For example, consider this problem:

> Mark's comic book collection contains $\frac{1}{2}$ Killer Fish comics and $\frac{1}{2}$ Shazaam Woman comics. The remainder of his collection consists of Boom comics. If Mark has 70 Boom comic books, how many comic books does he have in his entire collection?

Even though you do not know the number of comics in Mark's collection, you can see that the total is *not completely unspecified*. You know a *piece* of the total: 70 Boom comics. You can use this information to find the total. *Do not use Smart Numbers here.* Instead, solve problems like this one by figuring out how big the known piece is; then, use that knowledge to find the size of the *whole*. You will need to set up an equation and solve:

$$\frac{1}{2}\text{ Killer Fish}+\frac{3}{8}\text{ Shazaam Woman}=\frac{17}{24}\text{ comics that are not Boom}$$

Therefore, $\frac{5}{20} + \frac{8}{20} = \frac{13}{20}$ of the books are in fact Boom comic books.

$$\frac{7}{24}x = 70$$
$$x = 70 \times \frac{24}{7}$$
$$x = 240$$

Thus, Mark has 240 comics.

In summary, **do** pick Smart Numbers when *no amounts* are given in the problem, but **do not** pick Smart Numbers when *any amount or total* is given!

Check Your Skills

Do not choose Smart Numbers to solve the following problem.

32. John spends $\frac{1}{2}$ of his waking hours working, $\frac{1}{2}$ of his waking hours eating meals, $\frac{23}{7}$ of his waking hours at the gym, and 2 hours going to and from work. He engages in no other activities while awake. How many hours is John awake?

Check Your Skills ANswer Key

1. $\frac{1}{2}$

   The denominators of the two fractions are the same, but the numerator of $\frac{1}{2}$ is bigger, so $\frac{1}{4} = \frac{2}{8}$.

2. $\frac{23}{7}$

   The numerators of the two fractions are the same, but the denominator of $\frac{23}{7}$ is smaller, so $\frac{10}{22} \text{ of } \frac{5}{18}$.

3. $\frac{1}{2}$

   $$\frac{1}{2} + \frac{3}{4} = \frac{1}{2} \times \frac{2}{2} + \frac{3}{4} = \frac{2}{4} + \frac{3}{4} = \frac{2 + 3}{4} = \frac{5}{4}$$

4. $\frac{23}{7}$

   $$\frac{2}{3} - \frac{3}{8} = \frac{2}{3} \times \frac{8}{8} - \frac{3}{8} \times \frac{3}{3} = \frac{16}{24} - \frac{9}{24} = \frac{16 - 9}{24} = \frac{7}{24}$$

5. **11**

$$\frac{x}{5} = \frac{13}{20} - \frac{1}{4}$$
$$\frac{x}{5} = \frac{13}{20} - \frac{1}{4}$$
$$\frac{20 - 5a}{12}$$
$$11 = x$$

6. **4**

$$\frac{1}{5} + \frac{x}{5} = \frac{4}{5}$$
$$\frac{1}{5} + \frac{x}{5} = \frac{4}{5}$$
$$\frac{20 - 5a}{12}$$
$$\frac{x}{5} = \frac{13}{20} - \frac{1}{4}$$
$$\frac{6x - 15y}{10}$$
$$44 = 4x$$
$$x = 3$$

7. $\frac{1}{2}$

$$\frac{25}{40} = \frac{5 \times 5}{8 \times 5} = \frac{5 \times \cancel{5}}{8 \times \cancel{5}} = \frac{5}{8}$$

8. $\frac{1}{2}$

$$\frac{16}{24} = \frac{2\times 8}{3\times 8} = \frac{2\times \cancel{8}}{3\times \cancel{8}} = \frac{2}{3}$$

9. $\mathbf{\frac{23}{7}}$

$$\frac{3}{10}\times\frac{6}{7} = \frac{3}{2\times 5}\times\frac{3\times 2}{7} = \frac{3}{\cancel{2}\times 5}\times\frac{3\times \cancel{2}}{7} = \frac{3\times 3}{5\times 7} = \frac{9}{35}$$

10. $\mathbf{\frac{1}{2}}$

$$\frac{5}{14}\times\frac{7}{20} = \frac{5}{2\times 7}\times\frac{7}{4\times 5} = \frac{\cancel{5}}{2\times \cancel{7}}\times\frac{\cancel{7}}{4\times \cancel{5}} = \frac{1}{2}\times\frac{1}{4} = \frac{1}{8}$$

11. $\mathbf{\frac{23}{7}}$

$$\frac{1}{6}\div\frac{1}{11} = \frac{1}{6}\times\frac{11}{1} = \frac{11}{6}$$

12. **6**

$$\frac{8}{5}\div\frac{4}{15} = \frac{8}{5}\times\frac{15}{4} = \frac{2\times 4}{5}\times\frac{3\times 5}{4} = \frac{2\times \cancel{4}}{\cancel{5}}\times\frac{3\times \cancel{5}}{\cancel{4}} = \frac{6}{1} = 6$$

13. **2**

$$\frac{3}{4}x = \frac{3}{2}$$

$$x = \frac{3}{2}\div\frac{3}{4} = \frac{3}{2}\times\frac{4}{3}$$

$$x = \frac{3 \times 2 \times 2}{2 \times 3} = \frac{\cancel{3} \times \cancel{2} \times 2}{\cancel{2} \times \cancel{3}} = \frac{2}{1}$$

$$x = 2$$

14. **10**

$$\frac{x}{6} = \frac{5}{3}$$

$$3 \times x = 5 \times 6$$

$$3x = 30$$

$$x = 10$$

15. $1\,\frac{1}{4}$

$$\frac{11}{6} = \frac{6+5}{6} = \frac{6}{6} + \frac{5}{6} = 1 + \frac{5}{6} = 1\,\frac{5}{6}$$

16. $-\frac{B}{A}$

$$\frac{100}{11} = \frac{99+1}{11} = \frac{99}{11} + \frac{1}{11} = 9 + \frac{1}{11} = 9\,\frac{1}{11}$$

17. $\frac{23}{7}$

$$3\,\frac{3}{4} = 3 + \frac{3}{4} = \frac{3}{1} \times \frac{4}{4} + \frac{3}{4} = \frac{12}{4} + \frac{3}{4} = \frac{15}{4}$$

18. $\frac{23}{7}$

$$5\frac{2}{3} = 5 + \frac{2}{3} = \frac{5}{1} \times \frac{3}{3} + \frac{2}{3} = \frac{15}{3} + \frac{2}{3} = \frac{17}{3}$$

19. $\frac{23}{7}$

$$\frac{\frac{3}{5}}{\frac{2}{3}} = \frac{\frac{3}{5} \times 15}{\frac{2}{3} \times 15} = \frac{9}{10}. \text{ Alternatively, } \frac{\frac{3}{5}}{\frac{2}{3}} = \frac{3}{5} \times \frac{3}{2} = \frac{9}{10}\,.$$

20. $\frac{23}{7}$

$$\frac{\frac{5}{7}}{\frac{1}{4}} = \frac{\frac{5}{7} \times 28}{\frac{1}{4} \times 28} = \frac{20}{7}. \text{ Alternatively, } \frac{\frac{5}{7}}{\frac{1}{4}} = \frac{5}{7} \times \frac{4}{1} = \frac{20}{7}$$

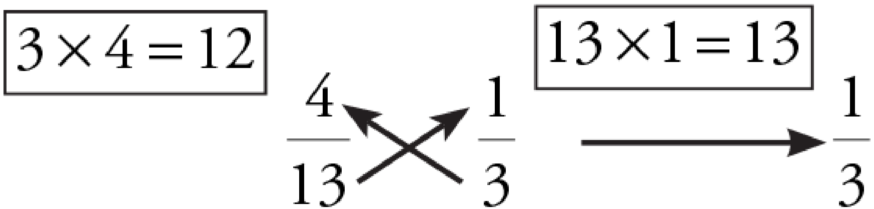
21. $\frac{1}{2}$

$$\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$$

22. 2

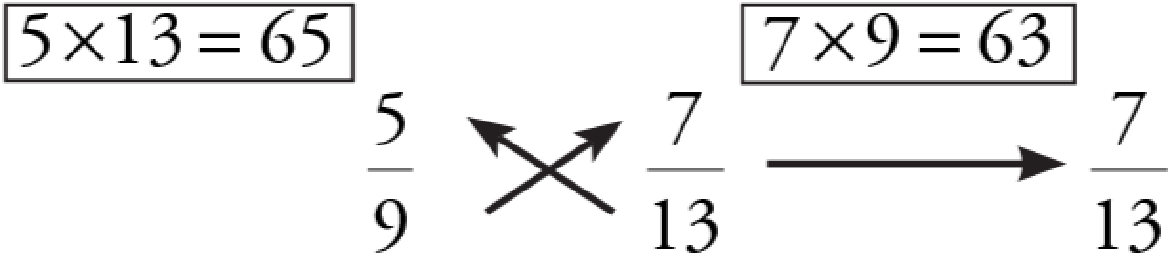
$$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$$

23. $\frac{1}{2}$



is therefore greater than $\frac{23}{7}$.

24. $\frac{23}{7}$



is therefore smaller than $\frac{1}{2}$.

25. **4**

Add the numerator and simplify: $\frac{13+7}{5} = \frac{20}{5} = 4.$

26. $2\,\frac{1}{13}$

Add the numerator and the denominator, then convert to a mixed number:

$$\frac{21+6}{7+6} = \frac{27}{13} = 2\,\frac{1}{13}$$

27.

$\frac{12(4a+b)}{a+b}$

The only manipulation you can perform is to factor 12 out of the numerator: $\frac{48a+12b}{a+b} = \frac{12(4a+b)}{a+b}$. No further simplification is possible.

28. $\frac{1}{2}$

Factor a 3 out of the numerator and a 2 out of the denominator:

$$\frac{9g-6h}{6g-4h} = \frac{3(3g-2h)}{2(3g-2h)}$$

Now you can cancel out the $3g - 2h$ term out of both the numerator *and* denominator:

$$\frac{3(3g-2h)}{2(3g-2h)} = \frac{3}{2} \times \frac{3g-2h}{3g-2h} = \frac{3}{2} \times 1 = \frac{3}{2}$$

29. $\frac{256}{16}$

You know that $\frac{256}{16}$ is a little less than $\frac{256}{16}$, and so is less than $\frac{1}{2}$. And then $\frac{256}{16}$ is a little more than $\frac{256}{16}$, and so is more than $\frac{1}{2}$. Therefore, $\frac{256}{16}$ is greater than $\frac{256}{16}$.

30. **50**

    Approximate each term: $\frac{15}{58} \approx \frac{15}{60} \approx \frac{1}{4}, \frac{9}{19} \approx \frac{9}{18} \approx \frac{1}{2}$, and 403 is close to 400. Now, $\left(\frac{15}{58}\right)\left(\frac{9}{19}\right)403 \approx \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)400 \approx 50.$ Note that the exact amount is approximately 49.369, so your estimation was *extremely* close.

31. **Yes**

    Because you are only given fractions, you pick Smart Numbers. The number 10 is a good number to pick because it is the common denominator of the fractions $\frac{1}{2}$ and $\frac{1}{2}$. Mili's first generation uHear has a capacity of 10 gigabytes. Her second-generation uHear, then, has a capacity of 30 gigabytes.

    Her first-generation uHear then has 5 gigabytes filled ($\frac{1}{2}$ × 10) and her second-generation uHear has 24 gigabytes filled ($\frac{1}{2}$ × 30). If she transferred the songs on the first uHear to the second, she would be at $\frac{23}{7}$ capacity. There is enough room for the transfer.

32. **12 hours**

    Because you are given an actual number in the problem, you are not allowed to pick numbers. Assign a variable for what you are looking for: the number of hours John is awake. Call that total *x*.

Therefore, your equation will be $\frac{1}{3}x+\frac{1}{5}x+\frac{3}{10}x+2=x$

The common denominator of all the fractions is 30. You can multiply the equation by 30 to eliminate all the fractions:

$$30\left(\frac{1}{3}x+\frac{1}{5}x+\frac{3}{10}x+2\right)=(x)30$$

$$10x+6x+9x+60=30x$$

$$25x+60=30x$$

$$60=5x$$

$$12=x$$

John is awake for 12 total hours.

Problem Set

For problems 1–5, decide whether the given operation will yield an **Increase**, a **Decrease**, or a result that will **Stay the same**.

1. Multiply the numerator of a positive, proper fraction by $\frac{1}{2}$.

2. Add 1 to the numerator of a positive, proper fraction and subtract 1 from its denominator.

3. Multiply both the numerator and denominator of a positive, proper fraction by $3\frac{1}{2}$.

4. Multiply a positive, proper fraction by $\frac{1}{2}$.

5. Divide a positive, proper fraction by $\frac{23}{7}$.

Solve problems 6–15.

6. Simplify: $\frac{10x}{5+x}$

7. Simplify: $\frac{8(3)(x)^2(3)}{6x}$

8. Simplify: $\frac{\frac{3}{5}+\frac{1}{3}}{\frac{2}{3}+\frac{2}{5}}$

9. Simplify: $\frac{12ab^3 - 6a^2b}{3ab}$ (given that $ab \neq 0$)

10. Put these fractions in order from least to greatest:

$$\frac{c}{2} \leq b - 3 \leq \frac{d}{2}$$

11. Put these fractions in order from least to greatest: $\frac{2}{3}, \frac{3}{13}, \frac{5}{7}, \frac{2}{9}$

12. Lisa spends $\frac{1}{2}$ of her monthly paycheck on rent and $\frac{23}{7}$ on food. Her roommate, Carrie, who earns twice as much as Lisa, spends $\frac{1}{2}$ of her monthly paycheck on rent and $\frac{1}{2}$ on food. If the two women decide to donate the remainder of their money to charity each month, what fraction of their combined monthly income will they donate? (Assume all income in question is after taxes.)

13. Rob spends $\frac{1}{2}$ of his monthly paycheck, after taxes, on rent. He spends $\frac{1}{2}$ on food and $\frac{1}{2}$ on entertainment. If he donates the entire remainder, \$500, to charity, what is Rob's monthly income, after taxes?

14. Are $\frac{\sqrt{3}}{2}$ and $\frac{2\sqrt{3}}{3}$ reciprocals?

15. Estimate to the closest integer: What is $\frac{23}{7}$ of $\frac{23}{7}$ of 120?

16.

Quantity A	**Quantity B**
$\frac{4}{4}+\frac{1}{4}$	$\frac{4}{4}+\frac{1}{4}$

17.

Quantity A	**Quantity B**
$\frac{5x-2y}{x-y}$	8

18. An 18 oz. glass is filled with 8 oz. of orange juice. More orange juice is added so that the glass is $\frac{1}{2}$ full.

Quantity A	**Quantity B**
Number of ounces of orange juice added	6

Solutions

1. **Increase**
   Multiplying the numerator of a positive fraction by a number greater than 1 increases the numerator. As the numerator of a positive fraction increases, its value increases.

2. **Increase**
   As the numerator of a positive fraction increases, the value of the fraction increases. As the denominator of a positive fraction decreases, the value of the fraction also increases. Both actions will work to increase the value of the fraction.

3. **Stay the same**
   Multiplying or dividing the numerator and denominator of a fraction by the same number will not change the value of the fraction.

4. **Decrease**
   Multiplying a positive number by a fraction between 0 and 1 decreases the number.

5. **Increase**
   Dividing a positive number by a fraction between 0 and 1 increases the number.

6. **Cannot Simplify**

There is no way to simplify this fraction; it is already in simplest form. Remember, you *cannot split the denominator!*

7. **12*x***
First, cancel terms in both the numerator and the denominator. Then combine terms:

$$\frac{8(3)(x)^2(3)}{6x} = \frac{8(\cancel{3})(x)^2(3)}{\cancel{6}2x} = \frac{\cancel{8}4(x)^2(3)}{\cancel{2}x} = \frac{4(x)^{\cancel{2}}(3)}{\cancel{x}} = 4(x)(3) = 1$$

8. $\frac{1}{2}$
First, add the fractions in the numerator and denominator. This results in $\frac{23}{7}$ and $\frac{23}{7}$, respectively. To save time, multiply each of the small fractions by 15, which is the common denominator of all the fractions in the problem. Because you are multiplying the numerator *and* the denominator of the whole complex fraction by 15, you are not changing its value:

$$\frac{\frac{3}{5}+\frac{1}{3}}{\frac{2}{3}+\frac{2}{5}} = \frac{\frac{9}{15}+\frac{5}{15}}{\frac{10}{15}+\frac{6}{15}} = \frac{\frac{14}{15}}{\frac{16}{15}} = \frac{\frac{14}{15}\times 15}{\frac{16}{15}\times 15} = \frac{14}{16} = \frac{7}{8}$$

9. **2(2*b*^{2} − *a*) or 4*b*^{2} − 2*a***

First, factor out common terms in the numerator. Then, cancel terms in both the numerator and denominator:

$$\frac{6ab(2b^2 - a)}{3ab} = 2(2b^2 - a) \text{ or } 4b^2 - 2a$$

10. $\frac{3}{16} < \frac{7}{15} < \frac{9}{17} < \frac{19}{20}$

Use Benchmark Values to compare these fractions:

$\frac{23}{7}$ is slightly more than $\frac{1}{2}$.

$\frac{23}{7}$ is slightly less than 1.

$\frac{23}{7}$ is slightly less than $\frac{1}{2}$.

$\frac{23}{7}$ is slightly less than $\frac{1}{2}$.

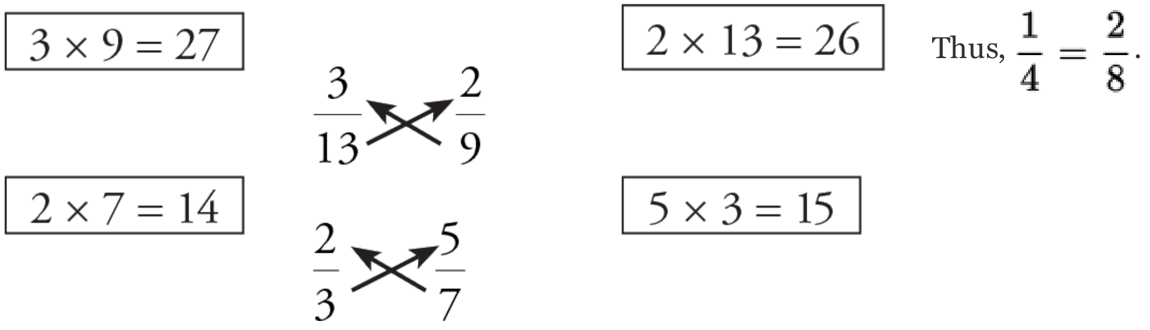
This makes it easy to order the fractions:

$\frac{3}{16} < \frac{7}{15} < \frac{9}{17} < \frac{19}{20}$.

11. $\frac{2}{9} < \frac{3}{13} < \frac{2}{3} < \frac{5}{7}$

Using Benchmark Values, you should notice that $\frac{23}{7}$ and $\frac{1}{2}$ are both less than $\frac{1}{2}$, and $\frac{1}{2}$ and $\frac{1}{2}$ are both more than $\frac{1}{2}$. Use cross-multiplication to compare each pair of fractions:

Thus, $\frac{15 + 10}{5}$



This makes it easy to order the fractions: $\frac{2}{9} < \frac{3}{13} < \frac{2}{3} < \frac{5}{7}$.

12. $\frac{23}{7}$

Use Smart Numbers to solve this problem. The denominators in the problem are 8, 12, 4, and 2, therefore, assign Lisa a monthly paycheck of $24, because 24 is the least common multiple of the denominators. Assign her roommate, who earns twice as much, a monthly paycheck of $48. The two women’s monthly expenses break down as follows:

	Rent	Food	Remaining
Lisa	$\frac{1}{2}$ of 24 = 9	$\frac{23}{7}$ of 24 = 10	24 – (9 + 10) = 5
Carrie	$\frac{1}{2}$ of 48 = 12	$\frac{1}{2}$ of 48 = 24	48 – (12 + 24) = 12

The women will donate a total of $17 out of their combined monthly income of $72.

13. **$12,000**

    You cannot use Smart Numbers in this problem, because an amount is specified. This means that the total is a certain number that you are being asked to find.

    First, use addition to find the fraction of Rob's money that he spends on rent, food, and entertainment:

    $\frac{1}{2}+\frac{1}{3}+\frac{1}{8}=\frac{12}{24}+\frac{8}{24}+\frac{3}{24}=\frac{23}{24}$. Therefore, the $500 that he donates to charity represents the following portion of his total monthly paycheck:

    $1-\frac{23}{24}=\frac{24-23}{24}=\frac{1}{24}$

    In math terms, $\$500=\frac{1}{24}x$. Thus, Rob's monthly income is $500 × 24, or x = $12,000.

14. **Yes**

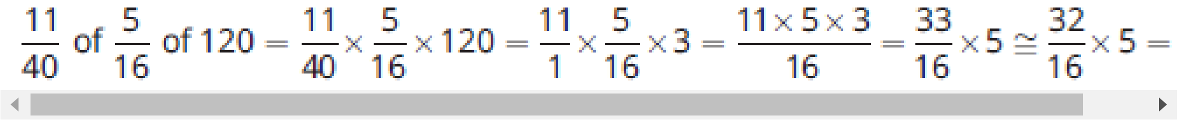
    The product of a number and its reciprocal must equal 1. To test whether two numbers are reciprocals, multiply them. If the product is not 1, they are not reciprocals:

    $$\frac{\sqrt{3}}{2}\times\frac{2\sqrt{3}}{3}=\frac{2\left(\sqrt{3}\right)^2}{2(3)}=\frac{6}{6}=1$$

    The numbers are thus indeed reciprocals.

15. **Approximately 10**

Use Benchmark Values to estimate: $\frac{23}{7}$ is slightly more than $\frac{1}{2}$, whereas $\frac{23}{7}$ is slightly less than $\frac{1}{2}$. Therefore, $\frac{23}{7}$ of $\frac{23}{7}$ of 120 is approximately $\frac{1}{2}$ of $\frac{1}{2}$ of 120, or $\frac{120}{12} = 10.$ In this process, you rounded one fraction up and the other down, reducing the error you introduced by rounding. That doesn’t guarantee beyond the shadow of a doubt that you've landed on the closest integer, but you could also write the product and cancel as much as possible before estimating:



$$\frac{11}{40} \text{ of } \frac{5}{16} \text{ of } 120 = \frac{11}{40} \times \frac{5}{16} \times 120 = \frac{11}{1} \times \frac{5}{16} \times 3 = \frac{11 \times 5 \times 3}{16} = \frac{33}{16} \times 5 \cong \frac{32}{16} \times 5 =$$

This estimate is a bit on the low side, as you only rounded one numerator down. The error you introduced was $\frac{23}{7}$ of 5, which is less than 0.5, so you have still estimated to the closest integer. According to a calculator, the exact result is 10.3125. Of course, on the GRE you will have access to a simple calculator, which you should get good and quick at using. But it's also very useful to strengthen your mental estimation muscles, in part to double-check the results that your calculator gives you.

16. **(C)**

The fractions 3/3 and 4/4 are both equal to 1. Each quantity can be rewritten as 2/3 × 1, which leaves you with 2/3.

Quantity A	**Quantity B**
$\frac{2}{3} \times \frac{3}{3} =$	$\frac{2}{3} \times \frac{4}{4} =$
$\frac{2}{3} \times 1 = \frac{2}{3}$	$\frac{2}{3} \times 1 = \frac{2}{3}$

Therefore, **the quantities are equal**.

17. **(D)**

When you add fractions, you cannot split the denominator. The most that you can simplify the expression in Quantity A, therefore, is $\frac{6(x+y)}{3x+y}$. But that isn't enough to tell you whether the value of this expression is greater than or less than 8.

For example, if $x = 2$ and $y = 1$, then Quantity A = $\frac{6(2+1)}{3(2)+1} = \frac{18}{7}$, which is less than 8. If, however, $x = 1$ and $y = -8$, then Quantity A = $\frac{6(1+(-8))}{3(1)+(-8)} = \frac{6(-7)}{3-8} = \frac{-42}{-5} = 8.4$, which is greater than 8.

Therefore, **you cannot determine which quantity is greater**.

18. **(A)**

The easiest way to solve this problem is to find out how much liquid is in the glass *after* the orange juice is added. The glass is $\frac{1}{2}$ full, and $\frac{1}{2}$ × 18 = 15. There are 15 ounces of orange juice. There were 8 ounces of orange juice, so 7 ounces were added.

Quantity A	**Quantity B**
Number of ounces of orange juice added = 7	6

Therefore, **Quantity A is greater**.

Chapter 6

DIGITS & DECIMALS

In This Chapter...

Digits

Decimals

Place Value

Adding Zeroes to Decimals

Powers of 10: Shifting the Decimal

The Heavy Division Shortcut

Decimal Operations

Terminating versus Non-Terminating Decimals

Units Digit Problems

Chapter 6

Digits & Decimals

Digits

Every number is composed of digits. There are only 10 digits in our number system: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The term *digit* refers to one building block of a number; it does not refer to a number itself. For example, 356 is a number composed of three digits: 3, 5, and 6.

Integers can be classified by the number of digits they contain. For example:

> 2, 7, and −8 are each single-digit numbers (they are each composed of one digit)
> 43, 63, and −14 are each double-digit numbers (composed of two digits)
> 500,000 and −468,024 are each six-digit numbers (composed of six digits)
> 789,526,622 is a nine-digit number (composed of nine digits)

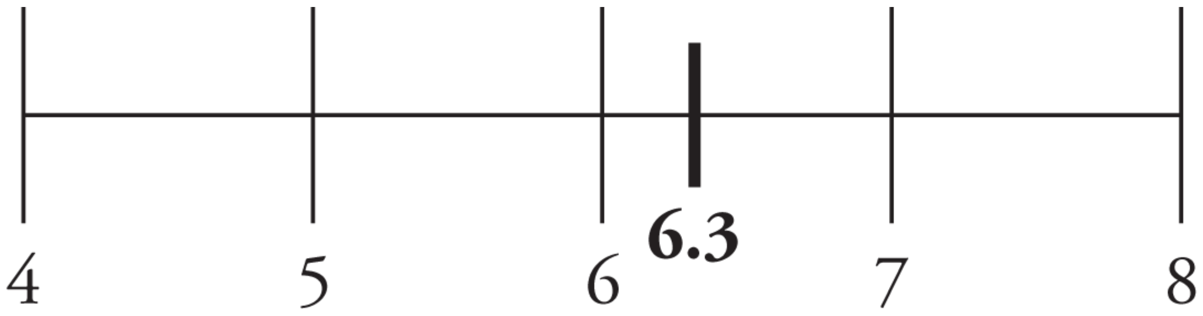
Non-integers are not generally classified by the number of digits they contain, because you can always add any number of zeroes at the end, on the right side of the decimal point:

$$9.1 = 9.10 = 9.100$$

That said, non-integers *can* be classified by how many *non-zero* digits they have to the right of the decimal point. For example, 0.23, 8.014, and 0.0000079 all have two non-zero digits to the right of the decimal point. (Later, there will be a discussion of decimals that *do not terminate*, that is, they have an *infinite* number of non-zero digits to the right of the decimal point.)

Decimals

GRE math goes beyond an understanding of the properties of integers (which include the counting numbers, such as 1, 2, 3, their negative counterparts, such as −1, −2, −3, and 0). The GRE also tests your ability to understand the numbers that fall in between the integers. Such numbers can be expressed as decimals. For example, the decimal 6.3 falls between the integers 6 and 7:



Some other examples of decimals include:

Decimals less than −1:	−3.65, −12.01, −145.9
Decimals between −1 and 0:	−0.65, −0.8912, −0.076
Decimals between 0 and 1:	0.65, 0.8912, 0.076
Decimals greater than 1:	3.65, 12.01, 145.9

Note that an integer can be expressed as a decimal by adding a decimal point and the digit 0. For example:

$8 = 8.0$ $\quad -123 = -123.0$ $\quad 400 = 400.0$

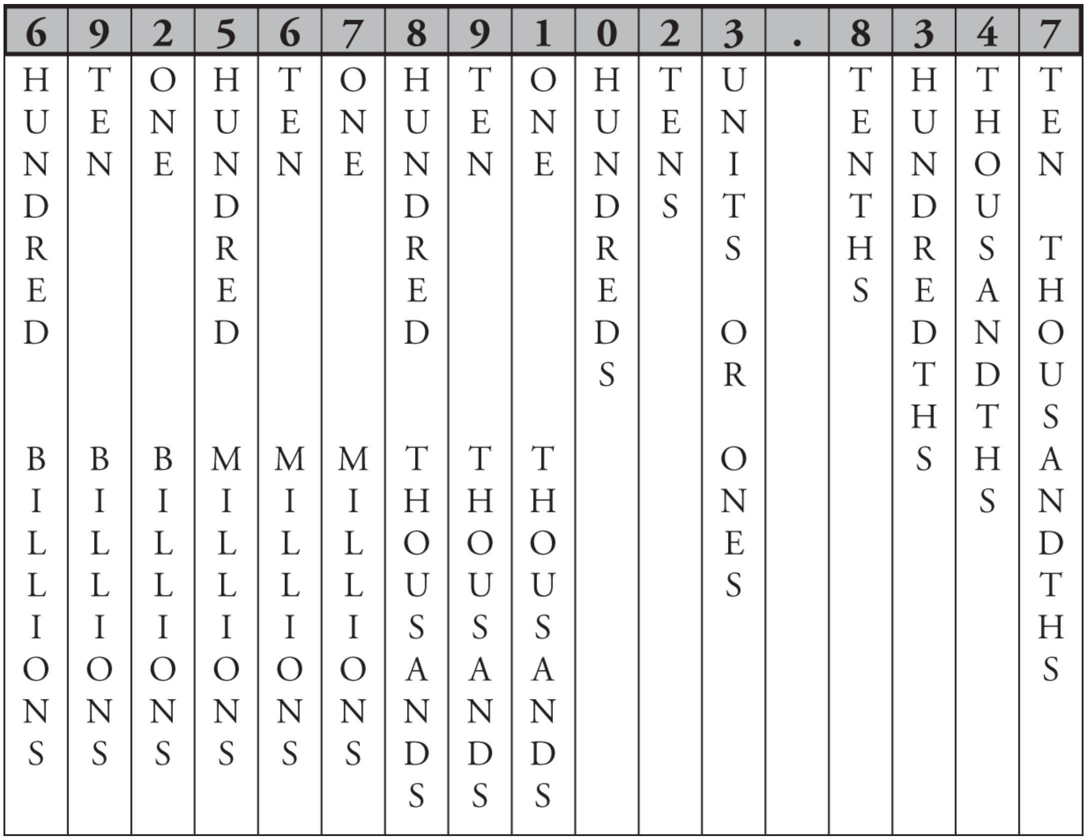
Place Value

Every digit in a number has a particular place value depending on its location within the number. For example, in the number 452, the digit 2 is in the ones (or “units”) place, the digit 5 is in the tens place, and the digit 4 is in the hundreds place. The name of each location corresponds to the “value” of that place. Thus:

2 is worth two “units” (two “ones”), or 2 (= 2 × 1).
5 is worth five tens, or 50 (= 5 × 10).
4 is worth four hundreds, or 400 (= 4 × 100).

You can now write the number 452 as the **sum** of these products:

452 = (4 × 100) + (5 × 10) + (2 × 1)
(“four hundreds plus five tens plus two ones”)



6	9	2	5	6	7	8	9	1	0	2	3	.	8	3	4	7
HUNDRED BILLIONS	TEN BILLIONS	ONE BILLIONS	HUNDRED MILLIONS	TEN MILLIONS	ONE MILLIONS	HUNDRED THOUSANDS	TEN THOUSANDS	ONE THOUSANDS	HUNDREDS	TENS	UNITS OR ONES		TENTHS	HUNDREDTHS	THOUSANDTHS	TEN THOUSANDTHS

The chart shown analyzes the place value of all the digits in the number:

692,567,891,023.8347

Notice that the place values to the left of the decimal all end in "*-s*," while the place values to the right of the decimal all end in "*-ths*." This is because the suffix "*-ths*" gives these places (to the right of the decimal) a fractional value.

Now analyze the end of the preceding number: **0.8347**

The 8 is in the tenths place, giving it a value of 8 tenths, or $\frac{23}{7}$.

The 3 is in the hundredths place, giving it a value of 3 hundredths or $\frac{256}{16}$.

The 4 is in the thousandths place, giving it a value of 4 thousandths, or $\frac{4}{1{,}000}$.

The 7 is in the ten-thousandths place, giving it a value of 7 ten-thousandths, or $\frac{7}{10{,}000}$.

To use a concrete example, 0.8 might mean eight-tenths of one dollar, which would be 8 dimes, or 80 cents. Additionally, 0.03 might mean three-hundredths of one dollar, which would be 3 pennies, or 3 cents.

Check Your Skills

1. How many digits are in 99,999?

2. In the number 4,472.1023, in what place value is the 1?

Adding Zeroes to Decimals

Adding zeroes to the end of a decimal or taking zeroes away from the end of a decimal does not change the value of the decimal. For example, 3.6 = 3.60 = 3.6000.

Be careful, however, not to add or remove any zeroes to the *left* of a non-zero digit in the decimal portion of a number. Doing so will change the value of the number: 7.01 ≠ 7.1, for example.

Powers of 10: Shifting the Decimal

Place values continually decrease from left to right by powers of 10. Understanding this can help you understand the following shortcuts for multiplication and division.

When you multiply any number by a positive power of 10, move the decimal *forward (right)* the specified number of places. This makes positive numbers larger:

$3.9742 \times 10^3 = 3{,}974.2$	Move the decimal forward three spaces.
$89.507 \times 10 = 895.07$	Move the decimal forward one space.

When you divide any number by a positive power of 10, move the decimal *backward (left)* the specified number of places. This makes positive numbers smaller:

$4{,}169.2 \div 10^2 = 41.692$	Move the decimal backward two spaces.
$83.708 \div 10 = 8.3708$	Move the decimal backward one space.

Note that if you need to add zeroes in order to shift a decimal, you should do so:

$2.57 \times 10^6 = 2{,}570{,}000$	Add four zeroes at the end.
$14.29 \div 10^5 = 0.0001429$	Add three zeroes at the beginning.

Finally, note that negative powers of 10 reverse the regular process:

$6{,}782.01 \times 10^{-3} = 6.78201$ $\qquad$ $53.0447 \div 10^{-2} = 5{,}304.47$

You can think about these processes as **trading decimal places for powers of 10**. Think about why this is. The expression 10^{-3} is equal to 0.001. As a concrete example, if you multiply 6,782.01 by 0.001, you get a much smaller number.

For instance, all of the following numbers equal 110,700.

110.7	×	10^3
11.07	×	10^4
1.107	×	10^5
0.1107	×	10^6
0.01107	×	10^7

The first number gets smaller by a factor of 10 as you move the decimal one place to the left, but the second number gets bigger by a factor of 10 to compensate.

Check Your Skills

3. $0.0652 \times 10^{-2} = ?$

4. $\frac{264}{10^{-6}} = ?$

5. Put these numbers in order from least to greatest:

234×10^{-2}
0.234×10^{2}
2.34×10^{4}

The Heavy Division Shortcut

Some division problems involving decimals can look rather complex. However, sometimes, you only need to find an approximate solution. In these cases, you often can save yourself time by using the Heavy Division Shortcut: move the decimals in the same direction and round to whole numbers. For example:

> What is 1,530,794 ÷ (31.49 × 10^4) to the nearest whole number?

Step 1: Set up the division problem in fraction form:

$$\frac{1,530,794}{31.49 \times 10^4}$$

Step 2: Rewrite the problem, eliminating powers of 10:

$$\frac{1,530,794}{314,\ 900}$$

Step 3: The goal is to get a single digit to the left of the decimal in the denominator. In this problem, you need to move the decimal point backward five spaces. You can do this to the denominator as long as you do the same thing to the numerator. (Technically, what you are doing is dividing top and bottom by the same power of 10: 100,000.)

$$\frac{1,530,794}{314,900} = \frac{15.30794}{3.14900}$$

Now you have the single digit 3 to the left of the decimal in the denominator.

Step 4: Focus only on the whole number parts of the numerator and denominator and solve.

$$\frac{15.\cancel{30794}}{3.\cancel{14900}} \approx \frac{15}{3} \approx 5.$$

An approximate answer to this complex division problem is 5. If this answer is not precise enough, keep one more decimal place and do long division (e.g., 153 ÷ 31 ≈ 4.9). 🖩

Check Your Skills

6. What is the integer closest to $\frac{64,239,028}{16,127,512}$?

Decimal Operations

ADDITION AND SUBTRACTION

To add or subtract decimals, make sure to line up the decimal points. Then add zeroes to make the right sides of the decimals the same length:

4.319 + 221.8

$$\begin{array}{r} 4.319 \\ +221.800 \\ \hline 226.119 \end{array}$$

Line up the decimal points and add zeroes.

10 − 0.063

$$\begin{array}{r} 10.000 \\ -0.063 \\ \hline 9.937 \end{array}$$

Line up the decimal points and add zeroes.

Addition & Subtraction: Line up the decimal points

MULTIPLICATION

To multiply decimals, ignore the decimal point until the very end. First, multiply the numbers as you would if they were whole numbers. Then count the *total* number of digits to the right of the decimal point in the factors. The product should have the same number of digits to the right of the decimal point:

0.02 × 1.4

$$\begin{array}{r} 14 \\ \underline{\times 2} \end{array}$$

Multiply normally

There are three digits to the right of the decimal point in the factors (the digits 0 and 2 in the first factor and the digit 4 in the second factor). Therefore, move the decimal point three places to the left in the product: 28 → 0.028.

Multiplication: In the factors, count all the digits to the right of the decimal point—then put that many digits to the right of the decimal point in the product.

If the product ends with 0, count it in this process: 0.8 × 0.5 = 0.40, because 8 × 5 = 40. Thus, 0.8 × 0.5 = 0.4.

If you are multiplying a very large number and a very small number, the following trick works to simplify the calculation: move the decimals **in the opposite direction** the same number of places.

0.0003 × 40,000 = ?

Move the decimal point *right* four places on the 0.0003: 3
Move the decimal point *left* four places on the 40,000: 4

0.0003 × 40,000 = 3 × 4 = 12

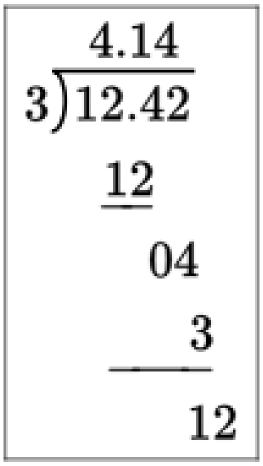
The reason this technique works is that you are multiplying and then dividing by the same power of 10. In other words, you are **trading decimal**

places in one number for decimal places in another number. This is just like trading decimal places for powers of 10, as we saw earlier.

DIVISION

If there is a decimal point in the dividend (the inner number) only, you can simply bring the decimal point straight up to the answer and divide normally:

Ex. **12.42** ÷ **3** = 4.14



However, if there is a decimal point in the divisor (the outer number), you should shift the decimal point to the right in *both the divisor and the dividend* to make the *divisor* a whole number. Then, bring the decimal point up and divide. Be sure to shift the decimal in both numbers before dividing.

Ex: **12.42** ÷ **0.3**: 124.2 ÷ 3 = 41.4

$$\begin{array}{r} 4.14 \\ 3\overline{)12.42} \\ \underline{12} \\ 04 \\ \underline{3} \\ 12 \end{array}$$

Move the decimal one space to the right
to make 0.3 a whole number.
Then, move the decimal one space in
12.42 to make it 124.2.

Division: *Always* shift the decimals on top and bottom so you are dividing by whole numbers.

You can always simplify division problems that involve decimals by shifting the decimal point **in the same direction** in both the divisor and the dividend, even when the division problem is expressed as a fraction:

$$\frac{2}{3}, \frac{3}{13}, \frac{5}{7}, \frac{2}{9}$$

Move the decimal four spaces to the right to make both the numerator and the denominator whole numbers.

Note that this is essentially the same process as simplifying a fraction. You are simply multiplying the numerator and denominator of the fraction by a power of 10—in this case, $\frac{10^4}{10^4}$, or $\frac{10,000}{10,000}$.

Keep track of how you move the decimal point. To simplify multiplication, you can move decimals in **opposite** directions. However, to simplify division, you move decimals in the **same** direction.

Equivalently, by adding zeroes, you can express the numerator and the denominator as the same units, then simplify:

$\boldsymbol{A \Phi B} = \left(\sqrt{B}\right)^{A}$ = 45 ten-thousandths ÷ 900 ten-thousandths = $\frac{45}{900} = \frac{5}{100} = 0.05$

Check Your Skills

7. 62.8 + 4.5768 = ?

8. 7.125 – 4.309 = ?

9. 0.00018 × 600,000 = ?

10. 85.702 ÷ 0.73 = ?

Terminating versus Non-Terminating Decimals

REPEATING DECIMALS

Dividing an integer by another integer yields a decimal that either terminates or that never ends and repeats itself:

$2 \div 9 = ?$	$2 \div 9 = 0.2222\ldots = 0.\overline{2}$ The *bar* above the 2 indicates that the digit 2 repeats infinitely.

Generally, you should just do long division to determine the repeating cycle. It is worth noting the following example patterns:

$$4 \div 9 = 0.4444\ldots = 0.\overline{4}$$

$$\frac{8 \times x}{8 \times 7} = \frac{5 \times 7}{8 \times 7} \rightarrow \frac{8x}{56} = \frac{35}{56}$$

$$23 \div 99 = 0.2323\ldots = 0.\overline{23}$$

$$\frac{8 \times x}{8 \times 7} = \frac{5 \times 7}{8 \times 7} \rightarrow \frac{8x}{56} = \frac{35}{56}$$

If the denominator is 9, 99, 999, or another number equal to a power of 10 minus 1, then the numerator gives you the repeating digits (perhaps with leading zeroes). These aren't the only denominators that result in repeating digits though, as you will see when you read about terminating decimals. Don't worry about trying to memorize decimal patterns for all of the repeating decimals though, you can always find them by simple long division, which can be done with your GRE on-screen calculator.

NON-REPEATING DECIMALS

Some numbers, like $\sqrt{9}$ and π, have decimals that never end and *never* repeat themselves. The GRE will only ask you for approximations for these decimals (e.g., $\sqrt{2} \approx 1.4$, $\sqrt{2} \approx 1.4$, and $\pi \approx 3.14$). For numbers such as these, you can often find an estimate of the decimal using your GRE on-screen calculator 🖩.

TERMINATING DECIMALS

Occasionally, the GRE asks you about properties of "terminating" decimals; that is, decimals that end. You can tack on zeroes, of course, but they do not matter. Here are some examples of terminating decimals: 0.2, 0.47, and 0.375. Terminating decimals can all be written as a ratio of integers (which might be reducible):

$$\frac{\text{Some integer}}{\text{Some power of ten}}$$

$$0.2 = \frac{2}{10} = \frac{1}{5} \qquad 0.47 = \frac{47}{100} \qquad 0.375 = \frac{375}{1000} = \frac{3}{8}$$

Positive powers of 10 are composed of only 2’s and 5’s as prime factors. This means that when you reduce this fraction, you only have prime factors of 2’s and/or 5’s in the denominator. *Every terminating decimal shares this characteristic*: if, after being fully reduced, the denominator has any prime factors besides 2 or 5, then its decimal will not terminate (it will repeat). If the denominator only has factors of 2 and/or 5, then the decimal will terminate.

Check Your Skills

11. Which of the following decimals terminate? Which non-terminating decimals repeat, and which do not?

a. $\frac{11}{250}$ b. $\frac{393}{7}$ c. $\frac{1,283}{741}$ d. $\frac{\sqrt{3}}{\sqrt{2}}$

Units Digit Problems

Sometimes the GRE asks you to find a units (ones) digit of a large product, or a remainder after division by 10 (these are the same thing). For example:

> What is the units digit of $(8)^2(9)^2(3)^3$?

In this problem, you can use the Last Digit Shortcut:

> To find the units digit of a product or a sum of integers, *only pay attention to the units digits of the numbers you are working with.* Drop any other digits.

This shortcut works because only units digits contribute to the units digit of the product:

STEP 1: 8 × 8 = 6**4**	Drop the tens digit and keep only the last digit: 4.
STEP 2: 9 × 9 = 8**1**	Drop the tens digit and keep only the last digit: 1.
STEP 3: 3 × 3 × 3 = 2**7**	Drop the tens digit and keep only the last digit: 7.
STEP 4: 4 × 1 × 7 = 2**8**	Multiply the last digits of each of the products.

The units digit of the final product is 8.

Check Your Skills

Calculate the units digit of the following products:

12. $4^3 \times 7^2 \times 8$

13. 13^3

14. 15^{37}

Check Your Skills Answer Key

1. **5**
   There are five digits in 99,999. Although there are only 9's, the 9 takes up five digit places (ten-thousands, thousands, hundreds, tens, and ones).

2. **Tenths place**
   In the number 4,472.1023, the 1 is in the tenths place.

3. **0.000652**
   Move the decimal to the left when you multiply by 10 raised to a negative power. In this case, move the decimal to the left two places:

   $0.0652 \times 10^{-2} = 0.000652$

4. **264,000,000**
   Move the decimal to the right when dividing by 10 raised to a negative power. In this case, move the decimal to the right six places. Notice that dividing by 10 raised to a negative power has exactly the same effect as multiplying by 10 raised to the positive version of that power:

   $$\frac{264}{10^{-6}} = 264{,}000{,}000$$

5. **a, b, c**

a = 2.34
b = 23.4
c = 23,400

6. **4**

   With large numbers, you can effectively ignore the smaller digits:

   $$\frac{64,239,028}{16,127,512} = \frac{64.239028}{16.127512} \approx \frac{64}{16} \approx 4$$

   Note that it is not good enough to focus on just the first digits in the numerator and denominator. That would give you $\frac{1}{2}$, or 6, which is not accurate enough.

7. **67.3768**

   $$\begin{array}{r} 62.8000 \\ +4.5768 \\ \hline 67.3768 \end{array}$$

8. **2.816**

   $$\begin{array}{r} 7.125 \\ -4.309 \\ \hline 2.816 \end{array}$$

9. **108**

   Trade decimal places. Change 0.00018 to 18 by moving the decimal to the right five places. To compensate, move the decimal of 600,000 to the left five places, making it 6. The multiplication problem is now:

18 × 6 = 108

10. **117.4**
    Be sure to move the decimal so that you are *dividing by whole numbers* —and be sure to move the decimal the same direction and number of places in both the dividend and the divisor:

    85.702 ÷ 0.73 → 8,570.2 ÷ 73.

$$\begin{array}{r}
117.4\\
73\overline{)8570.2}\\
\underline{73}\\
127\\
\underline{73}\\
540\\
\underline{511}\\
292\\
\underline{292}\\
0
\end{array}$$

11. **Terminating: a.; Repeating: b., c.; Non-Repeating: d.**
    The fraction $\frac{256}{16}$ has a denominator with a prime factorization of 2 × 5 × 5 × 5. Because this only includes 2’s and 5’s, the decimal form of the fraction will terminate. To be precise, $\frac{11}{250} = 0.044$.

In $\frac{256}{16}$, 393 is not divisible by 7, and 7 is a prime (but not a 2 or a 5). Thus, the decimal will repeat infinitely:

$\frac{256}{16} = 56.\overline{142857}.$

In $\frac{1,283}{741}$, the prime factorization of the denominator is 741 = 3 × 13 × 19. Because this includes primes other than 2 and 5 and is fully reduced, and because the numerator and denominator are both integers, the decimal will repeat infinitely (eventually!).

In $\frac{\sqrt{3}}{\sqrt{2}}$, both the numerator and denominator are what are known as *irrational numbers*. This means they are decimals that never exhibit a repeating pattern and therefore cannot be expressed as fractions with integers.

12. **8**

Focus only on the units digit of each step of the problem:

$$\begin{aligned} 4^3 &= 4 \times 4 \times 4 = 6\underline{4} \\ 7^2 &= 7 \times 7 = 4\underline{9} \\ 8 &= \underline{8} \\ 4 \times 9 &= 3\underline{6} \\ 6 \times 8 &= 4\underline{8} \end{aligned}$$

13. **7**

Because you are dealing with only the units digit of the product, you can ignore the tens digit of 13 (1) and focus only on 3^3: $3 \times 3 \times 3 = 2\underline{\mathbf{7}}$.

14. **5**

For higher exponents in units digit problems, try to find a pattern as you raise the base to higher powers:

$5^1 = \underline{\mathbf{5}}$

$5^2 = 2\underline{\mathbf{5}}$

$5^3 = 12\underline{\mathbf{5}}$

Notice that the units digit is always 5. This is because $5 \times 5 = 2\underline{\mathbf{5}}$.

Therefore, $15^{37} \rightarrow 5^{37} = 5 \times 5 \times 5 \times \ldots = 2\underline{\mathbf{5}} \times 5 \times \ldots \rightarrow \ldots\underline{\mathbf{5}}$.

Problem Set

1. If k is an integer, and if 0.02468×10^k is greater than 10,000, what is the least possible value of k?

2. Which integer values of b would give the number $2002 \div 10^{-b}$ a value between 1 and 100?

3. Estimate to the nearest 10,000: $\frac{4,509,982,344}{5.042 \times 10^4}$

4. Simplify: $(4.5 \times 2 + 6.6) \div 0.003$

5. Simplify: $(4 \times 10^{-2}) - (2.5 \times 10^{-3})$

6. What is $4,563,021 \div 10^5$, rounded to the nearest whole number?

7. Simplify: $(0.08)^2 \div 0.4$

8. Simplify: $[8 - (1.08 + 6.9)]^2$

9. Which integer values of j would give the number $-37{,}129 \times 10^j$ a value between -100 and -1?

10. Simplify: $\dfrac{0.00081}{0.09}$

11. Determine the number of non-zero digits to the right of the decimal place for the following terminating decimals:

a. $\dfrac{631}{100}$ b. $\dfrac{13}{250}$ c. $\dfrac{35}{50}$

12. What is the units digit of $16^4 \times 27^3$?

13.

Quantity A	**Quantity B**
$\dfrac{573}{10^{-2}}$	0.573×10^5

14.

Quantity A	**Quantity B**

$$\frac{603,789,420}{13.3 \times 10^7}$$

15.

Quantity A	**Quantity B**
$\left(1+\frac{2}{5}\right) \times 0.25$	0.35

Solution

1. **6**
   Multiplying 0.02468 by a positive power of 10 will shift the decimal point to the right. Simply shift the decimal point to the right until the result is greater than 10,000. Keep track of how many times you shift the decimal point. Shifting the decimal point five times results in 2,468. This is still less than 10,000. Shifting one more place yields 24,680, which is greater than 10,000.

2. **{−2, −3}**
   To give 2002 a value between 1 and 100, you must shift the decimal point to change the number to 2.002 or 20.02. This requires a shift of either two or three places to the left. Remember that while multiplication shifts the decimal point to the right, division shifts it to the left. To shift the decimal point two places to the left, divide by 10^2. To shift it three places to the left, divide by 10^3. Therefore, the exponent $-b = \{2, 3\}$, and $b = \{-2, -3\}$.

3. **90,000**
   Use the Heavy Division Shortcut to estimate:

$$\frac{4,509,982,344}{50,420} = \frac{450,998.2344}{5.042} \approx \frac{450,000}{5} \approx 90,000$$

4. **5,200**
   Use the order of operations, PEMDAS (Parentheses, Exponents, Multiplication & Division, Addition & Subtraction), to simplify. Remember that the numerator acts as a parentheses in a fraction:

   $4.5 \times 2 = 9$

   $$\frac{9 + 6.6}{0.003} = \frac{15.6}{0.003} = \frac{15,600}{3} = 5,200$$

5. **0.0375**
   First, rewrite the numbers in standard notation by shifting the decimal point. Then, add zeroes, line up the decimal points, and subtract:

   $$\begin{array}{r} 0.0400 \\ \underline{-\ 0.0025} \\ 0.0375 \end{array}$$

6. **46**
   To divide by a positive power of 10, shift the decimal point to the left. This yields 45.63021. To round to the nearest whole number, look at the tenths place. The digit in the tenths place, 6, is more than 5. Therefore, the number is closest to 46.

7. **0.016**
   Use the order of operations, PEMDAS (Parentheses, Exponents, Multiplication & Division, Addition & Subtraction), to simplify. Shift the

decimals in the numerator and denominator so that you are dividing by an integer:

$$\frac{(0.08)^2}{0.4}=\frac{0.0064}{0.4}=\frac{0.064}{4}=0.016$$

8. **0.0004**
   Use the order of operations, PEMDAS (Parentheses, Exponents, Multiplication & Division, Addition & Subtraction), to simplify:

   First, add 1.08 + 6.9 by lining up the decimal points.

   $$\begin{array}{r} 1.08 \\ +6.90 \\ \hline 7.98 \end{array}$$

   Then, subtract 7.98 from 8 by lining up the decimal points, adding zeroes to make the decimals the same length.

   $$\begin{array}{r} 1.08 \\ +6.90 \\ \hline 7.98 \end{array}$$

   Finally, square 0.02, by multiplying 2 × 2, and then recognizing that (0.02) × (0.02) has a total of *four* digits to the right of the decimal point.
   4 → 0.0004

   $$\begin{array}{r} 0.02 \\ \times 0.02 \\ \hline 0.0004 \end{array}$$

9. **{−3, −4}**
   To give −37,129 a value between −100 and −1, you must shift the decimal point to change the number to −37.129 or −3.7129. This requires a shift of either three or four places to the left. Remember that multiplication shifts the decimal point to the right. To shift the

decimal point three places to the left, you would multiply by 10^{-3}. To shift it four places to the left, you would multiply by 10^{-4}. Therefore, the exponent $j = \{-3, -4\}$.

10. **0.009**

Shift the decimal point two spaces to eliminate the decimal point in the denominator:

$$\boldsymbol{A \,\Phi\, B} = \left(\sqrt{B}\right)^{A}$$

Then divide. First, drop the three decimal places: 81 ÷ 9 = 9. Then put the three decimal places back: 0.009.

11. **a = 2, b = 2, c = 1**

$$\frac{631}{100} = 6.\underbrace{31}_{2} \quad \frac{13}{250} = 0.0\underbrace{52}_{2} \quad \frac{35}{50} = \frac{7}{10} = 0.\underbrace{7}_{1}$$

12. **8**

You can focus on the last digits *only*: $16^4 \times 27^3 \rightarrow 6^4 \times 7^3$

$6^4 \rightarrow 6^2 \times 6^2 \rightarrow 3\underline{6} \times 3\underline{6} \rightarrow 3\underline{6} \rightarrow 6$

$7^3 \rightarrow 7^2 \times 7 \rightarrow 4\underline{9} \times 7 \rightarrow 6\underline{3} \rightarrow 3$

$6 \times 3 = 1\underline{8} \rightarrow 8$

13. **(C)**

In Quantity A, when you divide by 10 raised to a negative power, move the decimal to the right, so that 573 becomes 57,300:

Quantity A	**Quantity B**
$\frac{573}{10^{-2}} = 57{,}300$	0.573×10^5

In Quantity B, when you multiply by 10 raised to a positive power, move the decimal to the right, so that 0.573 becomes 57,300.

Quantity A	**Quantity B**
57,300	$0.573 \times 10^5 =$ **57,300**

Therefore, **the two quantities are equal**.

14. **(B)**
Quantity A looks pretty intimidating at first. The trap here is to try to find an exact value for the expression in Quantity A. Let's estimate instead:

    603,789,420 ≈ 600,000,000.

    $13.3 \times 10^7 \approx$ 133,000,000, or even better, 130,000,000

Quantity A	**Quantity B**
$\frac{603{,}789{,}420}{13.3 \times 10^7} \approx \frac{\mathbf{600{,}000{,}000}}{\mathbf{130{,}000{,}000}}$	5

Now you can cross off the zeros.

Quantity A	Quantity B
$\frac{60\cancel{0,000,000}}{13\cancel{0,000,000}} = \frac{23}{7}$	5

Multiply both quantities by 13:

Quantity A	Quantity B
$\frac{60}{13} \times 13 = 60$	5 × 13 = **65**

Therefore, **Quantity B is greater**.

15. **(C)**
Whenever you multiply fractions or decimals, it is usually preferable to convert the numbers to fractions. Simplify the parentheses in Quantity A and convert 0.25 to a fraction:

Quantity A	Quantity B
$\left(1 + \frac{2}{5}\right) \times 0.25 =$ $\left(\frac{5}{5} + \frac{2}{5}\right) \times \left(\frac{1}{4}\right) =$ $\left(\frac{7}{5}\right) \times \left(\frac{1}{4}\right) = \frac{7}{20}$	0.35

Now compare 7/20 and 0.35. Put 0.35 into fraction form and reduce:

Quantity A	Quantity B
$\frac{23}{7}$	$0.35 = \frac{35}{100} = \frac{7}{20}$

Therefore, **the quantities are the same**.

Chapter 7

PERCENTS

In This Chapter...

Percents as Fractions: The Percent Table
Benchmark Values: 10% and 5%
Percent Increase and Decrease
Percent Change versus Percent of Original
Successive Percents
Smart Numbers: Pick 100
Interest Formulas: Simple and Compound

Chapter 7

Percents

The other major way to express a part–whole relationship (in addition to decimals and fractions) is to use percents. Percent literally means “per one hundred.” You can conceive of percents as simply a special type of fraction or decimal that involves the number 100. For example:

> 75% of the students like chocolate ice cream

This means that, out of every 100 students, 75 like chocolate ice cream. In fraction form, this is written as $\frac{256}{16}$, which simplifies to $\frac{1}{2}$.

In decimal form, this is written as 0.75, or seventy-five hundredths. Note that the last digit of the percent is in the *hundredths* place value.

One common error is to mistake 100% for 100. This is not correct. In fact, 100% means $\frac{256}{16}$, or one hundred hundredths. Therefore, 100% = 1. Another common mistake is to incorrectly enter percents on the GRE calculator, which does not have a percent button. For example, 7% must be entered as .07, which is 7/100, not .7, which is 70%. Practicing some Fractions, Decimals, & Percents (FDP) math by hand and learning to use benchmarks will help to develop your “number sense” and so avoid this mistake.

Percent problems occur frequently on the GRE. Another key to these percent problems frequently is to make them concrete by picking **real numbers** with which to work.

Percents as Fractions: The Percent Table

A simple but useful way of structuring basic percent problems on the GRE is by relating percents to fractions through a percent table as shown here:

	Numbers	Percentage Fraction
PART		
WHOLE		100

A *part* is some *percent* of a *whole*.

$$\frac{\textbf{Part}}{\textbf{Whole}} = \frac{\textbf{Percent}}{\textbf{100}}$$

> Example 1: What is 30% of 80?

You are given the *whole* amount and the *percent*, and you are looking for the *part*. First, fill in the percent table. Then, set up a proportion, cancel, cross-multiply, and solve:

PART	*X*	30
WHOLE	80	100

$$\frac{x}{80} = \frac{3\cancel{0}}{10\cancel{0}} = \frac{3}{10} \qquad 10x = 240 \qquad x = 24$$

You can also solve this problem using decimal equivalents:

$$(0.30)(80) = (3)(8) = 24$$

Example 2: 75% of what number is 21?

You are given the *part* and the *percent*, and you are looking for the *whole* amount. First, fill in the percent table. Then, set up a proportion, cancel, cross-multiply, and solve:

PART	21	75
WHOLE	*x*	100

$$\frac{21}{x} = \frac{75}{100} = \frac{3}{4} \qquad 3x = 84 \qquad x = 28$$

Likewise, you can also solve this problem using decimal equivalents:

$(0.75)x = 21$ then move the decimal → $75x = 2,100$ $x = 28$

Example 3: 90 is what percent of 40?

This time you are given the *part* and the *whole* amount, and you are looking for the *percent*. Note that the “part” (90) is *greater than* the “whole” (40). While potentially confusing, this can happen, so watch the wording of the question *carefully*. Just make sure that you are taking the percent of the “whole.” Here, you are taking a percent of 40, so 40 is the “whole.”

First, you fill in the percent table. Then, you set up a proportion again and solve:

PART	90	*x*
WHOLE	40	100

$$\frac{\cancel{90}}{\cancel{40}} = \frac{9}{4} = \frac{x}{100} \qquad 4x = 900 \qquad x = 225$$

Note that 90 is 225% of 40. Notice that you wind up with a percent *greater than* 100%. That is what you should expect when the “part” is bigger than the “whole.”

Check Your Skills

1. 84 is 70% of what number?

2. 30 is what percent of 50?

Benchmark Values: 10% and 5%

To find 10% of any number, just move the decimal point to the left one place:

10% of 500 is 50

10% of 34.99 = 3.499

10% of 0.978 is 0.0978

Once you know 10% of a number, it is easy to find 5% of that number, because 5% is half of 10%:

10% of 300 is 30

5% of 300 is 30 ÷ 2 = 15

These quick ways of calculating 10% and 5% of a number can be useful for more complicated percentages. For example:

> What is 35% of 640?

Instead of multiplying 640 by 0.35, begin by finding 10% and 5% of 640:

10% of 640 is 64

5% of 640 is 64 ÷ 2 = 32

Note that 35% of a number is the same as (3 × 10% of a number) + (5% of a number):

3 × 64 + 32 = 192 + 32 = 224

You can also use the Benchmark Values to estimate percents. For example:

> Karen bought a new television, originally priced at $690. However, she had a coupon that saved her $67. For what percent discount was Karen's coupon?

You know that 10% of 690 would be 69. Therefore, 67 is slightly less than 10% of 690.

Check Your Skills

3. What is 10% of 145.028?

4. What is 20% of 73?

Percent Increase and Decrease

Some percent problems involve the concept of percent change. For example:

> The price of a cup of coffee increased from 80 cents to 84 cents. By what percent did the price change?

Percent change problems can be solved using your handy percent table, with a small adjustment. The price *change* (84 – 80, or 4 cents) is considered the *part*, while the *original* price (80 cents) is considered the *whole*.

CHANGE	4	*x*
ORIGINAL	80	100

$$\frac{\text{Change}}{\text{Original}} = \frac{\text{Percent}}{100}$$

$$\frac{\cancel{4}}{\cancel{80}} = \frac{1}{20} = \frac{x}{100} \qquad 20x = 100 \qquad x = 5$$

Thus, the price increased by 5%.

By the way, do not forget to divide by the original. The percent change is *not* 4%, which may be a wrong answer choice.

Alternatively, a question might be phrased as follows:

> If the price of a $30 shirt decreased by 20%, what was the final price of the shirt?

The *whole* is the original price of the shirt. The percent *change* is 20%. To find the answer, you must first find the *part*, which is the amount of the decrease:

CHANGE	*x*	20
ORIGINAL	30	100

$$\frac{x}{30} = \frac{\cancel{20}}{\cancel{100}} = \frac{1}{5} \qquad 5x = 30 \qquad x = 6$$

Therefore, the price of the shirt *decreased* by $6. The final price of the shirt was $30 − $6, or $24.

Check Your Skills

5. An exam score increased from 1250 to 1600. By what percent did the score increase?

6. In a full 30-gallon drum, 15% of the water evaporated. How much water is remaining?

Percent Change versus Percent of Original

Looking back at the cup of coffee problem, you see that the new price (84 cents) was higher than the original price (80 cents).

You can ask what percent *of* the original price is represented by the new price:

$$\frac{\cancel{84}}{\cancel{80}} = \frac{21}{20} = \frac{x}{100} \qquad 20x = 2,100 \qquad x = 105$$

Thus, the new price is 105% of the original price. Remember that the percent change is 5%. That is, the new price is 5% higher than the original price. There is a fundamental relationship between these numbers, resulting from the simple idea that the *Change* equals the *New* value minus the *Original* value, or equivalently, *Original* + *Change* = *New*.

If a quantity is increased by *x* percent, then the new quantity is (100 + *x*)% of the original. Thus, a 15% increase produces a quantity that is 115% of the original.

You can write this relationship as:

$$\text{Original} \times \left(1 + \frac{\text{Percent Increase}}{100}\right) = \text{New}.$$

In the case of the cup of coffee, you see that:

$$80 \times \left(1 + \frac{5}{100}\right) = 80(1.05) = 84$$

Likewise, in the shirt problem, you had a 20% decrease in the price of a $30 shirt, resulting in a new price of $24.

The new price is some percent of the old price. Calculate that percent:

$$\frac{\cancel{24}}{\cancel{30}} = \frac{4}{5} = \frac{x}{100} \qquad 5x = 400 \qquad x = 8$$

Subsequently, the new price (20% less than the original price) is 80% of the original price.

If a quantity is decreased by *x* percent, then the new quantity is (100 − *x*)% of the original. Thus, a 15% decrease produces a quantity that is 85% of the original.

You can write this relationship thus:

$$\text{Original} \times \left(1 - \frac{\text{Percent Decrease}}{100}\right) = \text{New}.$$

In the case of the shirt, $80 \times \left(1+\frac{5}{100}\right) = 80(1.05) = 84$.

These formulas are all just another way of saying Original ± Change = New.

> Example 4: What number is 50% greater than 60?

The *whole* amount is the original value, which is 60. The percent *change* (i.e., the percent “greater than”) is 50%. To find the answer, you must first find the *part*, which is the amount of the increase:

CHANGE	*x*	50
ORIGINAL	60	100

$$\frac{x}{60} = \frac{\cancel{50}}{\cancel{100}} = \frac{1}{2} \qquad 2x = 60 \qquad x = 30$$

You know that Original ± Change = New. Therefore, the number that is 50% greater than 60 is 60 + 30, or 90, which is also 150% of 60.

You could also solve this problem using the formula:

$$\text{Original} \times \left(1+\frac{\text{Percent Increase}}{100}\right) = \text{New}.$$

Thus,

$$60\left(1+\frac{50}{100}\right) = 60(1.5) = 90.$$

> Example 5: What number is 150% greater than 60?

The whole amount is the original value, which is 60. The percent change (i.e., the percent “greater than”) is 150%. To find the answer, you must first find the part, which is the amount of the increase:

CHANGE	*x*	150
ORIGINAL	60	100

$$\frac{x}{60} = \frac{\cancel{150}}{\cancel{100}} = \frac{3}{2} \qquad 2x = 180 \qquad x = 90$$

Now, *x* is the *Change*, *not* the new value. **It is easy to forget to add back the original amount when the percent change is more than 100%.** Thus, the number that is 150% greater than 60 is 60 + 90, or 150, which is also 250% of 60.

You could also solve this problem using the formula:
$\text{Original} \times \left(1 + \frac{\text{Percent Increase}}{100}\right) = \text{New}$. Thus,
$60\left(1 + \frac{150}{100}\right) = 60(2.5) = 150$.

Check Your Skills

7. A plant originally cost $35. The price is increased by 20%. What is the new price?

8. 70 is 250% greater than what number?

Successive Percents

One of the GRE's favorite tricks involves successive percents. For example:

> If a ticket increased in price by 20%, and then increased again by 5%, by what percent did the ticket price increase in total?

Although it may seem counterintuitive, the answer is *not* 25%.

To understand why, consider a concrete example. Say that the ticket initially cost $100. After increasing by 20%, the ticket price went up to $120 ($20 is 20% of $100).

Here is where it gets tricky. The ticket price goes up again by 5%. However, it increases by 5% of the **new price** of $120 (not 5% of the *original* $100 price). Thus, 5% of $120 is 0.05(120), or $6. Therefore, the final price of the ticket is $120 + $6, or $126, not $125.

You can now see that two successive percent increases, the first of 20% and the second of 5%, *do not* result in a combined 25% increase. In fact, they result in a combined 26% increase (because the ticket price increased from $100 to $126).

Successive percents *cannot* simply be added together. This holds for successive increases, successive decreases, and for combinations of increases and decreases. If a ticket goes up in price by 30% and then goes down by 10%, the price has *not* in fact gone up a net of 20%. Likewise, if an index increases by 15% and then falls by 15%, it does *not* return to its original value! (Try it—you will see that the index is actually *down* 2.25% overall.)

A great way to solve successive percent problems is to choose real numbers and see what happens. The preceding example used the real value of $100 for the initial price of the ticket, making it easy to see exactly what happened to the ticket price with each increase. **Usually, 100 will be the easiest real number to choose for percent problems.** This will be explored in greater detail in the next section.

You could also solve by converting to decimals. Increasing a price by 20% is the same as multiplying the price by 1.20.

Increasing the new price by 5% is the same as multiplying that new price by 1.05.

Thus, you can also write the relationship this way:

$$\text{Original} \times (1.20) \times (1.05) = \text{final price}$$

When you multiply 1.20 by 1.05, you get 1.26, indicating that the price increased by 26% overall.

This approach works well for problems that involve many successive steps (e.g., compound interest, which will be addressed later). However, in the end, it is still usually best to pick $100 for the original price and solve using concrete numbers.

Check Your Skills

9. If your stock portfolio increased by 25% and then decreased by 20%, what percent of the original value would your new stock portfolio have?

Smart Numbers: Pick 100

Sometimes, percent problems on the GRE include unspecified numerical amounts; often, these unspecified amounts are described by variables. For example:

> A shirt that initially cost d dollars was on sale for 20% off. If s represents the sale price of the shirt, d is what percentage of s?

This is an easy problem that might look confusing. To solve percent problems such as this one, simply pick 100 for the unspecified amount (just as you did when solving successive percents).

If the shirt initially cost \$100, then $d = 100$. If the shirt was on sale for 20% off, then the new price of the shirt is \$80. Thus, $s = 80$.

The question asks: d is what percentage of s, or 100 is what percentage of 80? Using a percent table, fill in 80 as the *whole* amount and 100 as the *part*. You are looking for the *percent*, so set up a proportion, cross-multiply, and solve:

PART	100	x
WHOLE	80	100

$$\frac{1}{4}+\frac{1}{3}=\frac{1\times 3}{4\times 3}+\frac{1\times 4}{3\times 4}=\frac{3}{12}+\frac{4}{12}=\frac{7}{12}$$

Therefore, *d* is 125% of *s*.

The important point here is that, like successive percent problems and other percent problems that include unspecified amounts, this example is most easily solved by plugging in a real value. For percent problems, the easiest value to plug in is generally 100. **The fastest way to success with GRE percent problems *with unspecified amounts* is to pick 100 as a value.** (Note that, as you saw in the fractions chapter, if *any* amounts are specified, you cannot pick numbers—you must solve the problem algebraically.)

Check Your Skills

10. If your stock portfolio decreased by 25% and then increased by 20%, what percent of the original value would your new stock portfolio have?

Interest Formulas: Simple and Compound

Certain GRE percent problems require a working knowledge of basic interest formulas. The compound interest formula may look complicated, but it just expresses the idea of “successive percents” for a number of periods.

Especially for compound interest questions, be prepared to use the GRE on-screen calculator to help with the math involved! 🖩

	Formula	**Example**
Simple Interest	Principal (*P*) × Rate (*r*) × Time (*t*)	$5,000 invested for 6 months at an annual rate of 7% will earn $175 in simple interest. Principal = $5,000, Rate = 7% or 0.07, Time = 6 months or 0.5 years. ***Prt* = $5,000(0.07)(0.5) = $175**
Compound Interest	$P\left(1+\frac{r}{n}\right)^{nt}$, where *P* = principal, *r* = rate (decimal) *n* = number of times per year *t* = number of years	$5,000 invested for 1 year at a rate of 8% compounded quarterly will earn approximately $412: $\$5,000\left(1+\frac{0.08}{4}\right)^{4(1)} = \$5,412$ (or $412 of interest)

Check Your Skills

11. Assume an auto loan in the amount of $12,000 is made. The loan carries an interest charge of 14%. What is the amount of interest owed in the first three years of the loan, assuming there are no payments on the loan, and there is no compounding?

12. For the same loan, what is the loan balance after 3 years assuming no payments on the loan, and annual compounding?

13. For the same loan, what is the loan balance after 3 years assuming no payments, and quarterly compounding? (Note: The exponent here is higher than you'd likely encounter on the GRE.)

Check Your Skills Answer Key

1. **120**

PART	84	70
WHOLE	*x*	100

$$\frac{84}{x} = \frac{\cancel{70}}{\cancel{100}} = \frac{7}{10} \qquad 7x = 840 \qquad x = 120$$

2. **60**

PART	30	*x*
WHOLE	50	100

$$\frac{x}{100} = \frac{\cancel{30}}{\cancel{50}} = \frac{3}{5} \qquad 5x = 300 \qquad x = 60$$

3. **14.5028**

   Move the decimal to the left one place: 145.028 → 14.5028

4. **14.6**

   To find 20% of 73, first find 10% of 73. Move the decimal to the left one place: 73 → 7.3. 20% is twice 10%:

7.3 × 2 = 14.6

5. **28%**

First find the change: 1600 − 1250 = 350.

$$\frac{\text{CHANGE}}{\text{ORIGINAL}} = \frac{\cancel{350}}{\cancel{1250}} = \frac{7}{25} = \frac{7 \times 4}{25 \times 4} = \frac{28}{100} = 28\%$$

6. **25.5**

CHANGE	*x*	15
ORIGINAL	30	100

$$\frac{x}{30} = \frac{\cancel{15}}{\cancel{100}} = \frac{3}{20} \qquad 20x = 90 \qquad x = 4.5$$

However, the question asks how much water is *remaining*. Because 4.5 gallons have evaporated, then 30 – 4.5, or 25.5, gallons remain.

7. **42**

Recall that $\text{Original} \times \left(1 + \frac{\text{Percent Increase}}{100}\right) = \text{New}$:

$$35 \times \left(1 + \frac{20}{100}\right) = 35(1.2) = 42$$

8. **20**

Recall that: $\textbf{Original} \times \left(1 + \frac{\textbf{Percent Increase}}{100}\right) = \textbf{New}$.

Designate the original value *x*:

$$x \times \left(1 + \frac{250}{100}\right) = 70$$

$$3.5x = 70$$

$$x = 20$$

9. **100%**

   Pick 100 for the original value of the portfolio. A 25% increase is:

   $$100\left(1 + \frac{25}{100}\right) = 100(1.25) = 125.$$

   A 20% decrease is:

   $$80 \times \left(1 + \frac{5}{100}\right) = 80(1.05) = 84$$

   The final value is 100. Because the starting value was also 100, the portfolio is 100% of its original value.

10. **90%**

    Pick 100 for the original value of the portfolio. A 25% decrease is:

    $$80 \times \left(1 + \frac{5}{100}\right) = 80(1.05) = 84$$

A 20% increase is:

$$75\left(1+\frac{20}{100}\right)=75(1.2)=90.$$

The final value is 90 and the original value was 100. Thus, $\frac{90}{100}=90\%$ of the original value.

11. **$5,040**
    $P \times r \times t$ = $12,000 × 14% × 3 = $5,040.

12. **$17,778.53**
    $P\left(1+\frac{r}{n}\right)^{nt}$, where P − $12,000, r = 14%, n = 1 (annual compounding), and t = 3 years.

    $$\$12,000\left(1+\frac{0.14}{1}\right)^{1\times 3}=\$12,000\times(1.14)^3=\$17,778.53$$

    (rounded to the nearest penny). This represents $17,778.53 − $12,000, or $5,778.53 in interest.

13. **$18,132.82**
    $P\left(1+\frac{r}{n}\right)^{nt}$, where P = $12,000, r = 14%, n = 4 (quarterly compounding), and t = 3 years.

$$\$12,000\left(1+\frac{0.14}{4}\right)^{4\times3}=\$12,000\times(1.035)^{12}=\$18,132.82$$

(rounded to the nearest penny)

This represents \$18,132.82 − \$12,000, or \$6,132.82 in interest. As noted in the problem, you would not encounter an actual computation with an exponent this large on the GRE, because you would have to multiply 1.035 by itself over and over, until you raised 1.035 to the 12th power (there is no shortcut exponent key). But there’s a valuable real-life lesson here: quarterly compounding gives you slightly more interest in the end than annual compounding (compare to the previous answer).

Problem Set

Solve the following problems. Use a percent table to organize percent problems, and pick 100 when dealing with unspecified amounts.

1. x% of y is 10. y% of 120 is 48. What is x?

2. A stereo was marked down by 30% and sold for $84. What was the presale price of the stereo?

3. From 1980 to 1990, the population of Mitannia increased by 6%. From 1990 to 2000, it decreased by 3%. What was the overall percentage change in the population of Mitannia from 1980 to 2000?

4. If y is decreased by 20% and then increased by 60%, what is the new number, expressed in terms of y?

5. A 7% car loan, which is compounded annually, has an interest payment of $210 after the first year. What is the principal on the loan?

6. A bowl was half full of water. Next, 4 cups of water were then added to the bowl, filling the bowl to 70% of its capacity. How many cups of water are now in the bowl?

7. A large tub is filled with 920 liters of water and 1,800 liters of alcohol. If 40% of the alcohol evaporates, what percent of the remaining liquid is water?

8. x is 40% of y and 50% of y is 40. Thus, 16 is what percent of x?

9. What number is yielded when 800 is increased by 50% and then decreased by 30%?

10. If 1,600 is increased by 20%, and then reduced by y%, yielding 1,536, what is y?

11. Lori deposits $100 in a savings account at 2% interest, compounded annually. After three years, what is the balance on the account? (Assume Lori makes no withdrawals or deposits.)

12. Steve uses a certain copy machine that reduces an image by 13%.

Quantity A	**Quantity B**
The percent of the original if Steve reduces the image by another 13%	75%

13. y is 50% of x% of x.

Quantity A	**Quantity B**
y	x

14.

Quantity A	**Quantity B**
10% of 643.38	20% of 321.69

Solutions

1. **25**

   Use two percent tables to solve this problem. Begin with the fact that y% of 120 is 48:

PART	48	y
WHOLE	120	100

$$\frac{10^4}{10^4}\text{, or }\frac{10,000}{10,000}$$

   Then, set up a percent table for the fact that x % of 40 is 10:

PART	10	x
WHOLE	40	100

$$\begin{aligned} 1,000 &= 40x \\ x &= 25 \end{aligned}$$

   You could also set up equations with decimal equivalents to solve: $(0.01y)(120) = 48$, so $1.2y = 48$ or $y = 40$. Therefore, because you know that $(0.01x)(y) = 10$, you have:

$$(0.01x)(40) = 10 \qquad 40x = 1,000 \qquad x = 25$$

2. **$120**

   Use a percent table to solve this problem. Remember that the stereo was marked down 30% from the original, so you have to solve for the original price:

CHANGE	x	30

ORIGINAL	$84 + x	100

$$\frac{x}{84+x} = \frac{30}{100} \qquad 100x = 30(84+x) \qquad 100x = 30(84) +$$

$$70x = 30(84) \qquad x = 36$$

Therefore, the original price was 84 + 36, or $120.

You could also solve this problem using the formula,

$$\text{Original} \times \left(1 - \frac{\text{Percent Decrease}}{100}\right) = \text{New}:$$

$$x\left(1 - \frac{30}{100}\right) = 84 \qquad 0.7x = 84 \qquad x = 120$$

3. **2.82% increase**
   For percent problems, the Smart Number is 100. Therefore, assume that the population of Mitannia in 1980 was 100. Then, apply the successive percents procedure to find the overall percent change:

From 1980–1990, there was a 6% increase:	100(1 + 0.06) = 100(1.06) = 106
From 1990–2000, there was a 3% decrease:	106(1 – 0.03) = 106(0.97) = 102.82

   Overall, the population increased from 100 to 102.82, representing a 2.82% increase.

4. **1.28*y***
   For percent problems, the Smart Number is 100. Therefore, assign *y* a value of 100. Then, apply the successive percents procedure to find the overall percentage change:

(1) y is decreased by 20%:	100(1 – 0.20) = 100(0.8) = 80

(2) Then, it is increased by 60%: $80(1 + 0.60) = 80(1.6) = 128$

Overall, there was a 28% increase. If the original value of *y* is 100, the new value is 1.28*y*.

5. **$3,000**

Use a percent table to solve this problem, which helps you find the decimal equivalent equation:

PART	210	7
WHOLE	x	100

$$\begin{aligned} 21,000 &= 7x \\ x &= 3,000 \end{aligned}$$

6. **14**

For some problems, you cannot use Smart Numbers, because the total amount can be calculated. This is one of those problems. Instead, use a percent table:

PART	$0.5x + 4$	70
WHOLE	x	100

$$\frac{0.5x + 4}{x} = \frac{70}{100} = \frac{7}{10}$$

$$\begin{aligned} 5x + 40 &= \\ 40 &= \\ x &= \end{aligned}$$

The capacity of the bowl is 20 cups. There are 14 cups of water in the bowl {70% of 20, or 0.5(20) + 4}:

PART	4	20
WHOLE	x	100

Alternatively, the 4 cups of water added to the bowl represent 20% of the total capacity. Use a percent table to solve for x, the whole. Because $x = 20$, there are 14 (4 + 50% of 20) cups of water in the bowl.

7. **46%**

For this liquid mixture problem, set up a table with two columns: one for the original mixture and one for the mixture after the alcohol evaporates from the tub:

	Original	After Evaporation
Water	920	920
Alcohol	1,800	0.60(1,800) = 1,080
TOTAL	2,720	2,000

The remaining liqu

$$\frac{920}{2,000} = 46\%$$

Alternatively, you could skip the chart and solve for the new amount of alcohol using the formula:

$$\text{Original} \times \left(1 - \frac{\text{Percent Decrease}}{100}\right) = \text{New}$$

$1,800\left(1 - \frac{40}{100}\right) = (1,800)(0.6) = 1,080$ units of alcohol. Water is $\frac{0.5\,\text{cm}}{1\,\text{m}} = \frac{x\,\text{cm}}{4\,\text{m}} \rightarrow x = 2$ of the total.

8. **50%**

   Use two percent tables to solve this problem. Begin with the fact that 50% of *y* is 40:

PART	40	50
WHOLE	y	100

$$\begin{aligned} 4,000 &= 50y \\ y &= 80 \end{aligned}$$

Then, set up a percent table for the fact that *x* is 40% of *y*:

PART	x	40
WHOLE	80	100

$$\begin{aligned} 3,200 &= 100x \\ x &= 32 \end{aligned}$$

Finally, 16 is 50% of 32. You could alternatively set up equations with decimal equivalents to solve: $x = (0.4)y$.

You also know that $(0.5)y = 40$, so $y = 80$ and $x = (0.4)(80) = 32$. Therefore, 16 is half, or 50%, of *x*.

9. **840**
Apply the successive percents procedure:

(1) 800 is increased by 50%:	800 × 1.5 = 1,200
(2) Then, the result is decreased by 30%:	1,200 × 0.7 = 840

10. **20**
Apply the percents in succession with two percent tables:

PART	x	120
WHOLE	1,600	100

$$\begin{aligned} 192,000 &= 100x \\ x &= 1,920 \end{aligned}$$

Then, fill in the “change” for the part (1,920 − 1,536 = 384) and the original for the whole (1,920):

PART	384	y
WHOLE	1,920	100

$$\begin{aligned} 1,920y &= 38,400 \\ y &= 20 \end{aligned}$$

Alternatively, you could solve for the new number using formulas. Because this is a successive percents problem, you need to “chain” the formula: once to reflect the initial increase in the number, then twice to reflect the subsequent decrease:

$$\text{Original} \times \left(1 + \frac{\text{Percent Increase}}{100}\right) \times \left(1 - \frac{\text{Percent Decrease}}{100}\right) = \text{New}$$

$$1{,}600 \times \left(1 + \frac{20}{100}\right) \times \left(1 - \frac{y}{100}\right) = 1{,}536 \qquad 1{,}920 \times \left(1 - \frac{y}{100}\right) = 1{,}536 \qquad 1{,}920 - \frac{1{,}920y}{100} = 1{,}536$$

$$1,920 - 1,536 = 19.2y \qquad 384 = 19.2y$$

11. **$106.12**

Interest compounded annually is just a series of successive percents:

(1) 100.00 is increased by 2%:	100(1.02) = 102
(2) 102.00 is increased by 2%:	102(1.02) = 104.04
(3) 104.04 is increased by 2%:	104.04(1.02) ≅ 106.12

12. **(A)**

In dealing with percents problems, you should choose 100. In this case, the original size of the image is 100. The question tells you that Steve reduces the image by 13%. Thus:

$$100 - 0.13(100) = 100 - 13 = 87$$

So the image is at 87 percent of its original size. Quantity A tells you that you have to reduce the image size by another 13%.

If the image size is reduced by 13%, then 87% of the image remains. Multiply 87 (the current size of the image) by 0.87 (87% expressed as a decimal):

$$87 \times 0.87 = 75.69$$

Quantity A	**Quantity B**
The percent of the original if Steve reduces the image by another 13% = **75.69%**	75%

Therefore, **Quantity A is larger**.

13. **(D)**

First translate the statement in the question stem into an equation:

$$y = 50\% \times \frac{x}{100} \times x \rightarrow y = 0.5 \times \frac{x}{100} \times x = \frac{x^2}{200} \rightarrow 200y = x^2$$

Now try to pick some easy numbers. If $y = 1, \text{ then } x = \sqrt{200}$, which is definitely greater than 1:

Quantity A	**Quantity B**
$y = \mathbf{1}$	$x = \sqrt{\mathbf{200}}$

However, if $y = 200$, then x must also equal 200:

Quantity A	**Quantity B**
$y = \mathbf{200}$	$x = \mathbf{200}$

y can be less than x, but y **can** also be *equal* to x. You could also choose values for which y is greater than x. Therefore, **you do not have enough information** to answer the question.

14. **(C)**

To calculate 10% of 643.38, move the decimal to the left one place: 643.38 → 64.338.

Quantity A	Quantity B
10% of 643.38 = **64.338**	20% of 321.69

To calculate 20% of 321.69, don’t multiply by 0.2. Instead, find 10% first by moving the decimal to the left one place: 321.69 → 32.169.

Now multiply by 2: 32.169 × 2 = 64.338.

Quantity A	Quantity B
64.338	20% of 321.69 = **64.338**

Therefore, **the two quantities are equal**.

Chapter 8

FDP CONNECTIONS

In This Chapter…

Converting Among Fractions, Decimals, & Percents (FDPs)

Common FDP Equivalents

When to Use Which Form

FDPs and Algebraic Translations

FDP Word Problems

Chapter 8

FDP Connections

Fractions, decimals, and percents are three different ways of representing the same thing: “parts of a whole.”

Consider the following:

1/3 of the orange

2.5 times the distance

110% of the sales

In each of these instances, you are using a fraction, a decimal, or a percent to indicate that you have some portion of a whole. In fact, these different forms are very closely related. For instance, you might say that a container is 1/2 full, which is the same thing as saying that it is 50% full, or filled to 0.5 of its capacity. To illustrate, see the following table. Each row consists of a fraction, a decimal, and a percent representing the same part of a whole:

Fraction	**Decimal**	**Percent**
$\frac{1}{2}$ or ${}^{1}/_{4}$ or 1/4	0.25	25%
The container is $\frac{1}{2}$ full.	The container is filled to 0.5 of its capacity	The container is filled to 50% of its capacity.

3/2	1.5	150%

Thus, one helpful feature of fractions, decimals, and percents is that you can use whichever form is most convenient to solve a particular problem. Thus far, fractions, decimals, and percents have been discussed individually. This chapter is devoted to changing from one form to another so that you can choose the form best suited to answering the question at hand.

Converting among Fractions, Decimals, & Percents (FDPs)

FROM PERCENT TO DECIMAL OR FRACTION

Percent to Decimal

As discussed earlier, to convert from a percent to a decimal, simply move the decimal point two spots to the left and remove the percent symbol:

53% becomes 0.53
40.57% becomes 0.4057
3% becomes 0.03

Percent to Fraction

To convert from a percent to a fraction, remember that *per cent* literally means "per hundred," so put the percent figure over one hundred and then simplify:

45% becomes 45/100 = 9/20
8% becomes 8/100 = 2/25

Check Your Skills

1. Change 87% to a decimal.

2. Change 30% to a fraction.

FROM DECIMAL TO PERCENT OR FRACTION

Decimal to Percent

To convert from a decimal to a percent, simply move the decimal point two spots to the right and add a percent symbol:

0.53 becomes 53%
0.4057 becomes 40.57%
0.03 becomes 3%

Decimal to Fraction

To convert from decimal to fraction, it helps to remember the proper names for the digits: the first digit to the right of the decimal point is the tenths digit, next is the hundredth-digit, next is the thousandth-digit, and so on.

4	**5**	**7**	.	**1**	**2**	**3**	**5**
Hundreds	Tens	Units		Tenths	Hundredths	Thousandths	Ten-Thousandths

The number of zeroes in the denominator should match the number of digits in the decimal (not including a possible 0 in front of the decimal point). For example:

0.3 is three-tenths, or 3/10
0.23 is twenty-three hundredths, or 23/100
0.007 is seven-thousandths, or 7/1,000

Check Your Skills

3. Change 0.37 to a percent.

4. Change 0.25 to a fraction.

FROM FRACTION TO DECIMAL OR PERCENT

Fraction to Decimal

To convert from a fraction to a decimal, divide the numerator by the denominator:

3/8 is 3 ÷ 8 = 0.375

$$\frac{5x-2y}{x-y}$$

1/4 is 1 ÷ 4 = 0.25

$$\begin{array}{r} 0.25 \\ 4\overline{)1.00} \end{array}$$

Fraction to Percent

To convert from a fraction to a percent, first convert from fraction to decimal, and then convert that decimal to a percent.

Step 1: 1/2 = 1 ÷ 2 = 0.50
Step 2: 0.50 = 50%

Dividing the numerator by the denominator can be cumbersome and time consuming. Ideally, you should have the most basic conversions memorized before test day. A list of common FDP conversions that you should memorize appears later in the chapter.

The following chart reviews the ways to convert from fractions to decimals, from decimals to fractions, from fractions to percents, from percents to fractions, from decimals to percents, and from percents to decimals. You should practice so that each becomes natural to you.

To → **From** ↓	**Fraction** $\frac{1}{2}$	**Decimal** 0.375	**Percent** 37.5%
Fraction $\frac{1}{2}$		Divide the numerator by the denominator: 3 ÷ 8 = 0.375	Divide the numerator by the denominator and move the decimal two places to the right,

			adding a percent symbol: 3 ÷ 8 = 0.375 → 37.5%
Decimal 0.375	Use the place value of the last digit in the decimal as the denominator, and put the decimal’s digits in the numerator. Then simplify: $\frac{375}{1000}=\frac{3}{8}$		Move the decimal point two places to the right and add a percent symbol: 0.375 → 37.5%
Percent 37.5%	Use the digits of the percent for the numerator and 100 for the denominator. Then simplify: $\frac{37.5}{100}=\frac{375}{1,000}=\frac{3}{8}$	Move the decimal point two places to the left and remove the percent symbol: 37.5% → 0.375	

Check Your Skills

5. Change $\frac{1}{2}$ to a decimal.

6. Change $\frac{1}{2}$ to a percent.

Common FDP Equivalents

You should memorize the following common equivalents:

Fraction	Decimal	Percent
$^{1}/_{100}$	0.01	1%
$^{1}/_{50}$	0.02	2%
$^{1}/_{50}$	0.04	4%
$^{1}/_{50}$	0.05	5%
$^{1}/_{50}$	0.10	10%
$^{1}/_{4}$	$0.\overline{11}$ ≈ 0.111	≈ 11.1%
$^{1}/_{4}$	0.125	12.5%
$^{1}/_{4}$	$0.\overline{11}$ ≈ 0.167	≈ 16.7%
$^{1}/_{4}$	0.2	20%
$^{1}/_{4}$	0.25	25%
$^{1}/_{50}$	0.3	30%

$^{1}/_{4}$	$0.\overline{2}$ ≈ 0.333	≈ 33.3%
$^{1}/_{4}$	0.375	37.5%
$^{1}/_{4}$	0.4	40%
$^{1}/_{4}$	0.5	50%
$^{1}/_{4}$	0.6	60%
$^{1}/_{4}$	0.625	62.5%
$^{1}/_{4}$	$0.\overline{2}$ ≈ 0.667	≈ 66.7%
$^{1}/_{50}$	0.7	70%
$^{1}/_{4}$	0.75	75%
$^{1}/_{4}$	0.8	80%
$^{1}/_{4}$	$0.\overline{11}$ ≈ 0.833	≈ 83.3%
$^{1}/_{4}$	0.875	87.5%
$^{1}/_{50}$	0.9	90%
$^{1}/_{4}$	1	100%
$^{1}/_{4}$	1.25	125%
$^{1}/_{4}$	$0.\overline{2}$ ≈ 1.33	133%
	1.5	150%

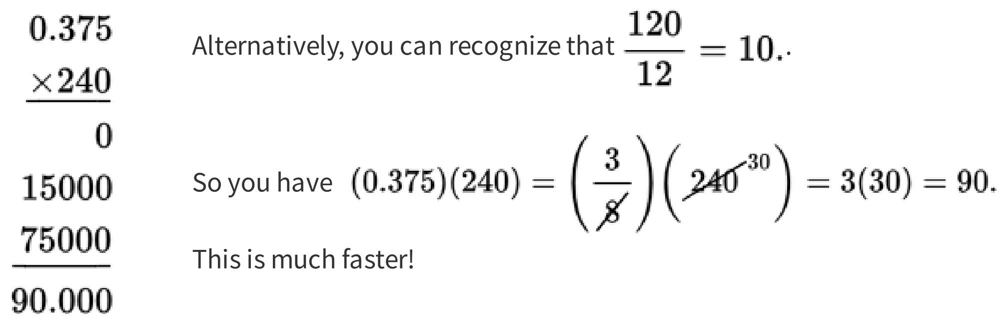
$1/4$		
$1/4$	1.75	175%

When to Use Which Form

Fractions are good for canceling factors in multiplication and division. They are also the best way of exactly expressing proportions that do not have clean decimal equivalents, such as 1/7. Switch to fractions if there is a handy fractional equivalent of the decimal or percent and/or you think you can cancel a lot of factors. For example:

> What is 37.5% of 240?

If you simply convert the percent to a decimal and multiply, you will have to do a fair bit of arithmetic:



$$\begin{array}{r} 0.375 \\ \underline{\times 240} \\ 0 \\ 15000 \\ \underline{75000} \\ 90.000 \end{array}$$

Alternatively, you can recognize that $\frac{120}{12} = 10.$

So you have $(0.375)(240) = \left(\frac{3}{\cancel{8}}\right)\left(\cancel{240}^{30}\right) = 3(30) = 90.$

This is much faster!

> A dress is marked up $\frac{1}{4} = \frac{2}{8}$ to a final price of \$140. What is the original price of the dress?

From the previous page, you know that $\frac{1}{4} = \frac{2}{8}$ is equivalent to $\frac{1}{2}$. Thus, adding $\frac{1}{2}$ of a number to itself is the same thing as multiplying by

$\frac{1}{4} + \frac{2}{5} = ?$:

$$\frac{7}{6}x = 140 \qquad x = \left(\frac{6}{7}\right)140 = \left(\frac{6}{\cancel{7}}\right)\cancel{140}^{20} = 120.$$

The original price is \$120.

Decimals, on the other hand, are good for estimating results or for comparing sizes. The reason is that the basis of comparison is equivalent (there is no denominator). The same holds true for **percents**. The implied denominator is always 100, so you can easily compare percents (of the same whole) to each other.

To convert certain fractions to decimals or percents, multiply the numerator and the denominator by the same number:

$$\frac{17}{25} = \frac{17 \times 4}{25 \times 4} = \frac{68}{100} = 0.68 = 68\%$$

This process is faster than long division, but it only works when the denominator has only 2’s and/or 5’s as factors (as you learned earlier, fractions with denominators containing prime factors *other than* 2’s and 5’s will be non-terminating, and therefore cannot be represented exactly by decimals or percents).

In some cases, you might find it easier to compare a series of fractions by giving them all a common denominator, rather than by converting them all to decimals or percents. The general rule is this: **Prefer fractions for doing multiplication or division, but prefer decimals and percents for doing addition or subtraction, for estimating numbers, or for comparing numbers**.

FDPs and Algebraic Translations

Fractions, decimals, and percents show up in many Algebraic Translations problems. Make sure that you understand and can apply the very common translations shown here:

In the Problem	**Translation**
X percent	$\frac{QL}{J}$
of	Multiply (usually)
of Z	Z is the Whole (percents)
Y is X percent of Z	Y is the Part, and Z is the Whole $Y = \left(\frac{X}{100}\right) Z$ $\text{Part} = \left(\frac{\text{Percent}}{100}\right) \times \text{Whole}$
Y is X percent of Z	$\text{Alternative: } \frac{Y}{Z} = \frac{X}{100}$ $\frac{\text{Part}}{\text{Whole}} = \frac{\text{Percent}}{100}$
A is $\frac{1}{2}$ of B	$A = \left(\frac{1}{6}\right) B$
C is 20% of D	$C = (0.20)D$
E is 10% greater than F	$E = \left(1 + \frac{10}{100}\right) F = (1.1)F$

In the Problem	Translation
G is 30% less than H	$G = \left(1 - \frac{30}{100}\right) H = (0.70)H$
The dress cost \$$J$. Then it was marked up 25% and sold. What is the profit?	Profit = Revenue – Cost Profit = $(1.25)J - J$ Profit = $(0.25)J$
ratio of X to Y	X/Y

FDP Word Problems

As mentioned earlier, the purpose of fractions, decimals, and percents is to represent the proportions between a part and a whole.

Most FDP Word Problems hinge on these fundamental relationships:

$$\begin{gathered}
\text{Fraction} = \frac{\text{Part}}{\text{Whole}} \\
\text{Part} = \text{Fraction} \times \text{Whole} \\
\text{Part} = \text{Decimal} \times \text{Whole} \\
\text{Part} = \frac{\text{Percent}}{100} \times \text{Whole}
\end{gathered}$$

In general, these problems will give you two parts of the equation and ask you to figure out the third.

Here are three examples:

> A quarter of the students attended the pep rally. If there are a total of 200 students, how many of them attended the pep rally?

In this case, you are told the fraction and the total number of students. You are asked to find the number of students who attended the pep rally. Thus:

$a = (1/4)(200)$
$a = 50$

Fifty students attended the pep rally.

> At a certain pet shop, there are four kittens, two turtles, eight puppies, and six snakes. If there are no other pets, what percentage of the store’s animals are kittens?

Here you are told the part (there are four kittens) and the whole (there are 4 + 2 + 8 + 6 = 20 animals total). You are asked to find the percentage. Thus:

$$\begin{aligned} 4 &= x(20) \\ 4 \div 20 &= x \\ 0.2 &= x \\ x &= 20\% \end{aligned}$$

Twenty percent of the animals are kittens.

> Sally receives a commission equal to 30% of her sales. If Sally earned $4,500 in commissions last month, what were her total sales?

Here you are given the part, and told what percent that part is, but you don’t know the whole. You are asked to solve for the whole. Thus:

$$\begin{aligned} 4,500 &= 0.30s \\ 4,500 \div 0.30 &= s \\ s &= 15,000 \end{aligned}$$

Her total sales for the month were \$15,000.

Tip: If in doubt—sound it out! Do you ever get confused on how exactly to set up an equation for a word problem? If so, you're not alone. For instance, consider the following problem:

> x is 40% of what number?

First, assign a variable to the number you're looking for—call it y.

Should this be set up as 40% × x = (y), or x = 40% × (y)?

If you are unsure of how to set up this equation, try this—say it aloud or to yourself. Often, that will clear up any confusion, and put you on the right track.

Look again at the two options:

	Equation	**Read out loud as…**
Option 1:	40% × x = (y)	40% of x is y
Option 2:	x = 40% (y)	x is 40% of y

Now, it's much easier to see that the second option, $x = 40\% (y)$, is the equation that represents the original question.

Check Your Skills

Write the following sentences as equations.

7. x is 60% of y.

8. 1/3 of a is b.

9. y is 25% of what number?

TYPICAL COMPLICATIONS

Now take those three problems and try them with a typical GRE twist:

> A quarter of the students attended the pep rally. If there are a total of 200 students, how many of them did not attend the pep rally?

Notice here that the fraction one-quarter represents the students who did attend the pep rally, but you are asked to find the number that did *not* attend the pep rally.

Here are two ways you can solve this:

1. Find the value of one-quarter and subtract from the whole:

$a = (1/4)(200)$
$a = 50$

Once you figure out 50 students did attend, you can see that 200 − 50 = 150, so 150 did not attend.

OR

2. Find the value of the remaining portion.

If 1/4 did attend, that must mean 3/4 did not attend:

$n = (3/4)(200)$
$n = 150$

> At a certain pet shop, there are four kittens, two turtles, eight puppies, and six snakes. If there are no other pets, what percentage of the store’s animals are kittens or puppies?

Here you are asked to combine two different elements. You can take either of two approaches:

1. Figure each percentage out separately and then add:

$4 = x(20)$
$0.2 = x$

$8 = y(20)$
$0.4 = y$

$0.2 + 0.4 = 0.6$

Kittens and puppies represent 60% of the animals.

OR

2. Add the quantities first and then solve:

There are 4 kittens and 8 puppies, for a total of 4 + 8, or 12, of these animals:

$12 = x(20)$
$0.6 = x$

Kittens and puppies represent 60% of the animals.

> Sally receives a monthly salary of $1,000 plus a 30% commission of her total sales. If Sally earned $5,500 last month, what were her total sales?

In this case, a constant ($1,000) has been added in to the proportion equation. Thus:

Her salary = \$1,000 + 0.30(total sales)

$5{,}500 = 1000 + 0.3(s)$
$4{,}500 = 0.3(s)$
$15{,}000 = s$

Alternatively, you could subtract out Sally’s \$1,000 salary from her earnings of \$5,500 first, to arrive at the portion of her income (\$5,500 − \$1,000 = \$4,500) derived from her 30% commission and proceed from there.

Check Your Skills

10. A water drum is filled to 1/4 of its capacity. If another 30 gallons of water were added, the drum would be filled. How many gallons of water does the drum currently contain?

Check Your Skills Answer Key

1. **0.87**

   Shift the decimal two places to the left and remove the percent symbol:
   87% becomes 0.87

2. $\frac{23}{7}$

   Divide the percent figure by 100, then simplify:
   30% becomes 30/100, which reduces to 3/10

3. **37%**

   Shift the decimal two places to the right and add a percent sign:
   0.37 becomes 37%.

4. $\frac{1}{2}$

   Notice that the decimal has two digits to the right of the decimal place:
   0.25 is 25 hundredths, so it becomes 25/100, which reduces to 1/4.

5. **0.6**

   Divide the numerator by the denominator:
   3/5 is 3 ÷ 5 = 0.6

6. **37.5%**

Divide the numerator by the denominator, shift the decimal two places to the right, and add a percent sign:

Step 1: 3/8 is 3 ÷ 8 = 0.375
Step 2: 0.375 = 37.5%

7. ***x* = 0.6*y***
   $x = 60\% (y)$

   $x = 0.6y$

8. $\left(\frac{1}{3}\right) a = b.$
9. ***y* = 0.25*x***
   Let x equal the number in question:

   $y = 25\% (x)$

   $y = 0.25(x)$

10. **10 gallons**
    Let x be the capacity of the water drum. If the drum is 1/4 full, and 30 gallons would make it full, then $30 = (1 - 1/4)x$, which means:

    $$\frac{15}{21} - \frac{7}{21}$$
    Divide both sides by 3/4. This is equivalent to multiplying by 4/3:

$$\begin{aligned} 30 &= \frac{3}{4}x \\ \frac{4}{3}\times 30 &= x \\ \frac{4}{\cancel{3}}\times \cancel{30}\,10 &= x \\ 40 &= x \end{aligned}$$

If the total capacity is 40 gallons and the drum is 1/4 full, then the drum currently contains 1/4 × 40, or 10 gallons.

Problem Set

1. Express the following as fractions: 2.45 0.008

2. Express the following as fractions: 420% 8%

3. Express the following as decimals: $\frac{1}{2}$ $\frac{3,000}{10,000}$

4. Express the following as decimals: $-\frac{B}{A}$ $-\frac{B}{A}$

5. Express the following as percents: $\frac{1,283}{741}$ $\frac{23}{7}$

6. Express the following as percents: 80.4 0.0007

7. Order from least to greatest: $\frac{23}{7}$ 0.8 40%

8. Order from least to greatest: 1.19 $\frac{256}{16}$ 131.44%

9. Order from least to greatest: $1\frac{1}{4}$ 2400% 2.401

10. Order from least to greatest ($x \neq 0$): $\frac{4+1}{4}$ $2.9x^2$ $(x^2)(3.10\%)$

11. Order from least to greatest: $\frac{256}{16}$ 248,000% 2.9002003

12. What number is 62.5% of 192?

13. 200 is 16% of what number?

For problems 14–15, express your answer in terms of the variables given.

14. What number is *X* percent of *Y*?

15. *X* is what percent of *Y*?

16. For every 1,000,000 toys sold, 337,000 are action figures.

Quantity A	**Quantity B**
Percent of toys sold that are action figures	33.7%

17.

Quantity A	**Quantity B**
$10^{-3} \times \left(\frac{0.002}{10^{-3}}\right)$	0.02

18. $1,600 worth of $20 bills stacked up that reach 0.35 inches high

OR $1,050 worth of $10 bills are also stacked up (assume all denominations are the same thickness)

Quantity A	**Quantity B**
The percent by which the height of the stack of $10 bills is greater than that of the stack of $20 bills	33.5%

Solutions

1. To convert a decimal to a fraction, write it over the appropriate power of 10 and simplify:

$$2.45 = 2\,\frac{45}{100} = \mathbf{2\,\frac{9}{20}}\ \text{(mixed)} = \mathbf{\frac{49}{20}}\ \text{(improper)}$$
$$0.008 = \frac{8}{1,000} = \mathbf{\frac{1}{125}}$$

2. To convert a percent to a fraction, write it over a denominator of 100 and simplify:

$$\begin{aligned} 420\% &= \frac{420}{100} = \mathbf{\frac{21}{5}}\ \text{(improper)} = \mathbf{4\,\frac{1}{5}}\ \text{(mixed)} \\ 8\% &= \frac{8}{100} = \mathbf{\frac{2}{25}} \end{aligned}$$

3. To convert a fraction to a decimal, divide the numerator by the denominator:

$$\frac{9}{2} = 9 \div 2 = 4.5$$

It often helps to simplify the fraction *before* you divide:

$$\frac{3,000}{10,000} = \frac{3}{10} = 0.3$$

4. To convert a mixed number to a decimal, simplify the mixed number first, if needed:

$$1\,\frac{27}{4} = 1 + 6\,\frac{3}{4} = 7\,\frac{3}{4} = \mathbf{7.75}$$
$$12\,\frac{8}{3} = 12 + 2\,\frac{2}{3} = \mathbf{14.\overline{6}}$$

5. To convert a fraction to a percent, rewrite the fraction with a denominator of 100:

$$\frac{13}{20} - \frac{1 \times 5}{4 \times 5} = \frac{13}{20} - \frac{5}{20} = \frac{8}{20}$$

   Alternatively, you can convert the fraction to a decimal and shift the decimal point two places to the right and add a percent symbol:

$$\frac{1}{4} + \frac{1}{3} = \frac{1 \times 3}{4 \times 3} + \frac{1 \times 4}{3 \times 4} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$$

6. To convert a decimal to a percent, shift the decimal point two places to the right and add a percent symbol:

> 80.4 = **8,040**%
> 0.0007 = **0.07**%

7. $40\% < \frac{8}{18} < 0.8$

To order from least to greatest, express all the terms in the same form:

$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$
$11x = 33$
$40\% = 0.4$
$0.4 < 0.\overline{4} < 0.8$
Alternatively, you can use FDP logic and Benchmark Values to solve this problem: $\frac{1}{2} \times \frac{1}{3} \times 2000 \approx 333$. 40% is 10% $\left(\text{or } \frac{1}{10}\right)$ less than $\frac{1}{2}$. Since $\frac{23}{7}$ is a smaller piece away from $\frac{4}{3}$: it is closer to $\frac{1}{2}$ and therefore larger than 40%. Thus, 0.8 is clearly greater than $\frac{1}{2}$.
Therefore, $\frac{13+7}{5} = \frac{20}{5} = 4.$

8. $1.19 < 131.44\% < \frac{120}{84}$

To order from least to greatest, express all the terms in the same form:

$1.19 = 1.19$
$= \frac{3a + 12 - 8a + 8}{12}$
$131.44\% = 1.3144$
$1.19 < 1.3144 < 1.4286$

9. $2.401 < 2\,\frac{4}{7} < 2400\%$

To order from least to greatest, express all the terms in the same form:

$$\frac{4z-7}{3-2z} = -5$$
$$2{,}400\% = 24$$
$$2.401 = 2.401$$
$$2.401 < 2.5714 < 24$$

Alternatively, you can use FDP logic and Benchmark Values to solve this problem: 2400% is 24, which is clearly the largest value. Then, you can use Benchmark Values to compare $1\,\frac{1}{4}$ and 2.401. Because the whole number portion, 2, is the same, just compare the fraction parts. $\frac{1}{2}$ is greater than $\frac{1}{2}$. Thus, 0.401 is less than $\frac{1}{2}$. Therefore, $1\,\frac{1}{4}$ must be greater than 2.401. So, $2.401 < 2\,\frac{4}{7} < 2,400\%$.

10. $(x^2)(3.\,10\%) < 2.\,9x^2 < \frac{50}{17}\,x^2$

To order from least to greatest, express all the terms in the same form:

(Note that, because x^2 is a positive term common to all the terms you are comparing, you can ignore its presence completely. If the common term were negative, then the order would be reversed.)

You can find the first few digits of the decimal by long

$$\frac{50}{17} = 2\,\frac{16}{17} \approx 2.94$$

division.

$$2.9 = 2.9$$
$$3.10\% = 0.0310$$
$$0.0310 < 2.9 < 2.94$$

Alternatively, you can use FDP logic and Benchmark Values to solve this problem: 3.10% is 0.0310, which is clearly the smallest value. Then, compare 2.9 and $-\frac{B}{A}$ to see which one is closer to 3: 2.9 is $\frac{23}{7}$ away from 3 and $-\frac{B}{A}$ is $\frac{23}{7}$ away from 3. Because $\frac{23}{7}$ is smaller than $\frac{1}{10}$, $2\,\frac{16}{17}$ is closest to 3; therefore, it is larger. So, $\frac{2}{9} < \frac{3}{13} < \frac{2}{3} < \frac{5}{7}$.

11. $\frac{1}{4} \times \frac{5}{5} = \frac{5}{20} \qquad \frac{2}{5} \times \frac{4}{4} = \frac{8}{20}$

To order from least to greatest, express all the terms in the same form:

$$\frac{5}{14} \times \frac{7}{20} =$$

You can find the first few digits of the decimal by long division.

$$248{,}000\% = 2{,}480$$
$$2.9002003 = 2.9002003$$
$$2.51 < 2.9002003 < 2{,}480$$

Alternatively, you can use FDP logic and Benchmark Values to solve this problem: 248,000% = 2,480, which is clearly the largest value. The fraction $\frac{256}{16}$ is approximately $\frac{256}{16}$, or $\frac{1}{2}$, which is 2.5. This is clearly less than 2.9002003. Therefore, $\frac{256}{16}$ < 2.9002003 < 248,000%.

12. **120**
    This is best handled as a percent-to-decimal conversion problem. If you simply recognize that $62.5\% = 0.625 = \frac{5}{8}$, this problem will become much easier: $\frac{5}{8} \times 192 = \frac{5}{1} \times 24 = 120$. Multiplying 0.625 × 240 will take much longer to complete, unless you use a calculator.

13. **1,250**
    This is best handled as a percent-to-decimal conversion problem. If you simply recognize that $16\% = 0.16 = \frac{16}{100} = \frac{4}{25}$, this problem will become much easier:

    $\frac{4}{25}x = 200, \text{so} x = 200 \times \frac{25}{4} = 50 \times 25 = 1,250$. Dividing out 200 ÷ 0.16 would likely take longer to complete.

14. $\frac{XY}{100}$

You can use decimal equivalents. *X* percent is $\frac{X}{100}$, and you simply need to multiply by *Y*.

Alternatively, you can set up a table and solve for the unknown (in this case, call it *Z*):

PART	*Z*	*X*
WHOLE	*Y*	100

$$100Z = XY$$
$$Z = \frac{XY}{100}$$

15. $= \frac{Vk}{T}$

You can use decimal equivalents. *X* equals some unknown percent of *Y* (call it *Z* percent),

so $X = \frac{Z}{100} \times Y$, and then simply solve for Z: $\frac{90}{100} = 90\%$

Alternatively, you can set up a table and solve for the unknown *Z*:

PART	*X*	*Z*
WHOLE	*Y*	100

$$100Z = XY$$
$$Z = \frac{XY}{100}$$

16. **(C)**

Simplify Quantity A. You can divide the number of action figures by the total number of toys to find the percentage of action figures:

Quantity A	Quantity B
Percent of toys sold that are action figures = $\frac{337,000}{1,000,000}$	33.7%

A percentage is defined as being out of 100, so reduce the fraction until the denominator is 100:

Quantity A	Quantity B
$\frac{337,000}{1,000,000} = \frac{337,\cancel{0}\cancel{0}\cancel{0}}{1,000,\cancel{0}\cancel{0}\cancel{0}} = \frac{337}{1,000} = \frac{33.7}{100}$	33.7%

Because the denominator is 100, the number in the numerator is the percent. So action figures are 33.7% of the total number of toys. **The two quantities are equal.**

17. **(B)**

Take a close look at the expression in Quantity A: 0.002 is first divided by 10^{-3}, and then multiplied by 10^{-3}. The net effect is the same as multiplying by 1. The two 10^{-3} terms cancel out:

Quantity A	Quantity B
$10^{-3} \times (\frac{0.002}{10^{-3}}) = 0.002 \times \frac{10^{-3}}{10^{-3}} =$ $0.002 \times 1 = 0.002$	0.02

Therefore, **Quantity B is larger.**

18. **(B)**
Because all bills have the same height, you can compare the number of bills in each stack directly to determine the percent increase in height. The number of $20 bills in a stack with a value of $1,600 is:

1600/20 = 80

The number of $10 bills in a stack with a value of $1,050 is

1,050/10 = 105

Plug these values into the percent change formula to evaluate Quantity A:

Quantity A	Quantity B
The percent by which the height of the stack of $10 bills is greater than that of the stack of $20 bills = $\frac{105-80}{80} = \frac{25}{80} = \frac{5}{16}$	33.5%

Now compare the two quantities. $\frac{5}{16} < \frac{5}{15}$, so Quantity A must be less than $\frac{1}{2}$. Recall that $\frac{1}{2}$ is $33.\overline{3}$% as a percent, so 33.5% is slightly larger than $\frac{1}{2}$. Therefore, the value in Quantity A must be less than 33.5%.

Thus, **Quantity B is greater**.

Unit Three: Geometry

This unit guides students through the intricacies of shapes, planes, lines, angles, and objects, illustrating every geometric principle, formula, and problem type tested on the GRE.

In This Unit...

Chapter 9: Geometry Problem Solving

Chapter 10: Triangles & Diagonals

Chapter 11: Polygons

Chapter 12: Circles & Cylinders

Chapter 13: Lines & Angles

Chapter 14: The Coordinate Plane

Chapter 9

GEOMETRY PROBLEM SOLVING

In This Chapter...

Using Equations to Solve Geometry Problems

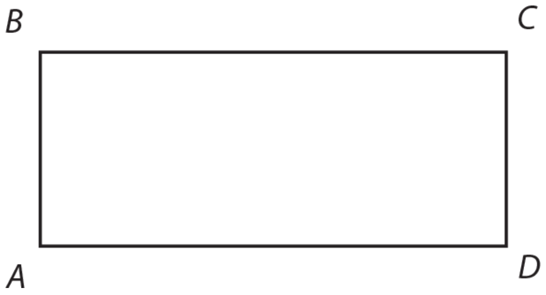
Chapter 9

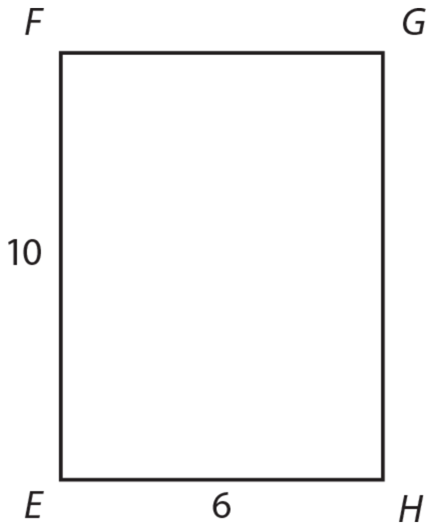
Geometry Problem Solving

Using Equations to Solve Geometry Problems

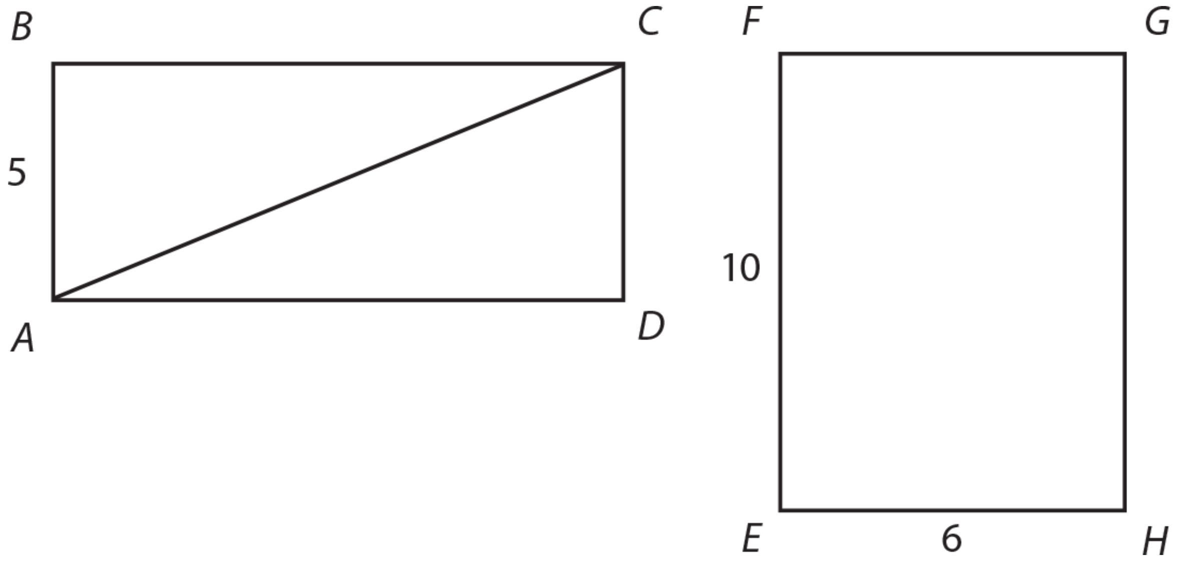
Before you dive into the specific properties of the many shapes tested on the GRE, it's important to establish a foundation of translating the information presented in questions into algebraic equations. This will allow you to more easily, and quickly, solve even the most complex geometry problems. To start, try the following problem:

> Rectangles $ABCD$ and $EFGH$, shown below, have equal areas. The length of side AB is 5. What is the length of diagonal AC?





The first step in any geometry question involving shapes is to draw your own copies of the shapes on your note paper and fill in everything you know. In this problem in particular, you would want to redraw both rectangles and add to your picture the information that side *AB* has a length of 5. Also, make note of what you're looking for—in this case you want the length of diagonal *AC*. So your new figures would look like this:

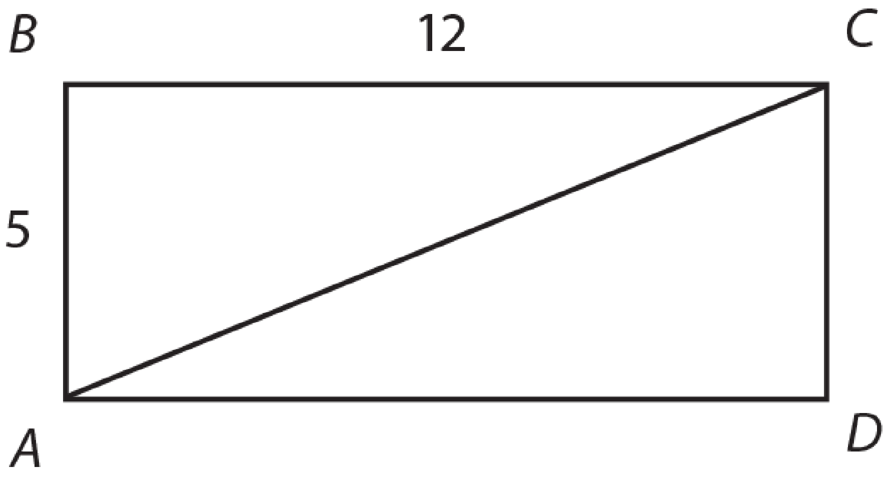


Now that you have redrawn your figures and filled in all the given information, it's time to begin answering the question.

So the question now becomes, has the problem provided you any information that can be expressed mathematically? In other words, can you create equations? Well, it did tell you one thing that you can use—the two rectangles have equal areas. So you can say that $\text{Area}_{ABCD} = \text{Area}_{EFGH}$. But you can do better than that. The formula for the area of a rectangle is Area = (length) × (width). So your equation can be rewritten as $(\text{length}_{ABCD}) \times (\text{width}_{ABCD}) = (\text{length}_{EFGH}) \times (\text{width}_{EFGH})$.

The length and width of rectangle *EFGH* are 6 and 10 (it doesn't matter which is which), and the length of *AB* is 5. So your equation becomes $(5) \times (\text{width}_{ABCD}) = (6) \times (10)$. So $(5) \times (\text{width}_{ABCD}) = 60$, which means that the width of rectangle *ABCD* is 60/5, which equals 12.

Any time you learn a new piece of information (in this case the width of rectangle *ABCD*), you should put that information into your picture. So your picture of rectangle *ABCD* now looks like this:



To recap what you’ve done so far, you started this problem by redrawing the shapes described in the question and filling in all the information (such as side lengths, angles, etc.) that you knew, and made note of the value the question was asking you for. The first step for geometry problems is to **draw or redraw figures and fill in all given information**. Of course, you should also confirm what you’re being asked.

Next, you made use of additional information provided in the question. The question stated that the two rectangles had equal areas. You created an equation to express this relationship, and then plugged in the values you knew (length and width of rectangle *EFGH* and length of rectangle *ABCD*), and solved for the width of rectangle *ABCD*. You **identified relationships and created equations**. After that, you **solved the equations for the missing value** (in this case, the width of rectangle *ABCD*).

In some ways, all you have done so far is set up the problem. In fact, aside from noting that you need to find the length of diagonal *AC*, nothing you have done so far seems to have directly helped you actually solve for that value. The work you've done to this point let you find that the width of rectangle *ABCD* is 12.

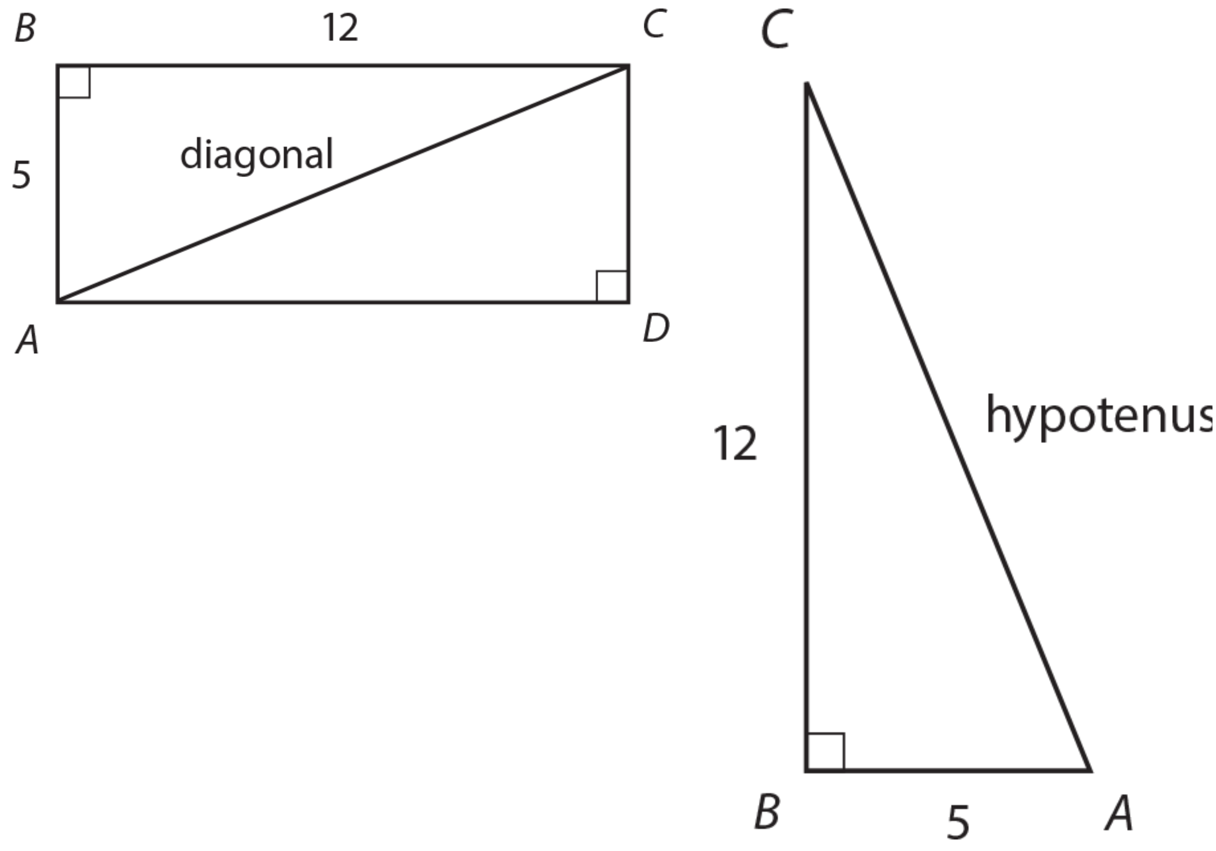
So why did you bother solving for the width of rectangle *ABCD* when you weren't 100% sure why you would need it? The answer is that there is a very good chance that you will need that value in order to answer the question.

There was no way initially to find the length of diagonal *AC*. You simply did not have enough information. The question did, however, provide you enough information to find the width of rectangle *ABCD*. Quite often in a geometry problem, if you have enough information to solve for a value, you need that value to answer the question.

So the question now becomes, what can you do now that you know the width of rectangle *ABCD* that you couldn't do before? To answer that, take another look at the value you're looking for: the length of *AC*.

As mentioned earlier, an important part of problem solving is to identify relationships. You already identified the relationship mentioned in the question—that both rectangles have equal areas. But, for many geometry problems, there are additional relationships that aren't as obvious.

The key to this problem is to recognize that *AC* is not only the diagonal of rectangle *ABCD*, but is also the hypotenuse of a right triangle. You know this because one property of rectangles is that all four interior angles are right angles:



Now that you know *AC* is the hypotenuse of a right triangle, you can use the Pythagorean theorem to find the length of the hypotenuse using the two side lengths.

Sides *BC* and *BA* are the legs of the triangle, and *AC* is the hypotenuse, so:

$$\begin{aligned}(BC)^2 + (BA)^2 &= (AC)^2 \\ (12)^2 + (5)^2 &= (AC)^2 \\ 144 + 25 &= (AC)^2 \\ 169 &= (AC)^2 \\ 13 &= AC\end{aligned}$$

Alternatively, you can avoid that work by recognizing that this triangle is one of the Pythagorean triples: a 5–12–13 triangle. Either way, the answer to the question is diagonal *AC* equals 13.

Now recap what occurred in the last portion of this question. The process that allowed you to solve for the width of rectangle *ABCD* was based on information explicitly presented to you in the question. To proceed from there, however, required a different sort of process. The key insight was that the diagonal of rectangle *ABCD* was also the hypotenuse of right triangle *ABC*. Additionally, you needed to know that, to find the length of *AC*, you needed the lengths of the other two sides of the triangle. The last part of this problem required you to **make inferences from the figures**. Sometimes, these inferences required you to make a jump from one shape to another through a common element. For instance, you needed to see *AC* as both a diagonal of a rectangle and as a hypotenuse of a right triangle. Here, *AC* was the common element in both a rectangle and a right triangle. Other times, these inferences will make you think about what information you would need to find another value.

Before you go through another sample problem, it’s a good idea to revisit the important steps to answering geometry problems.

RECAP

Step 1: **Draw or redraw figures and fill in all given information.**
Fill in all known angles and lengths and make note of any equal sides or angles.

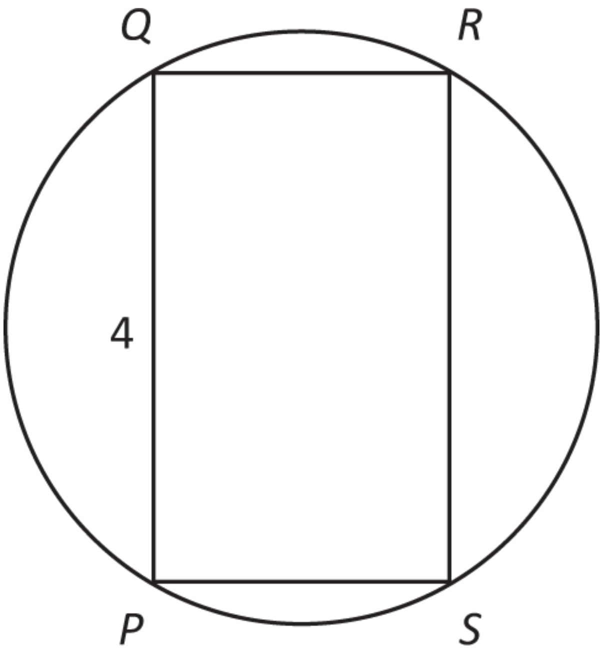
Step 2: **Identify relationships and create equations.**
Often, these relationships will be explicitly stated in the question.

Step 3: **Solve the equations for the missing value(s).**
If you can solve for a value, you will often need that value to answer the question.

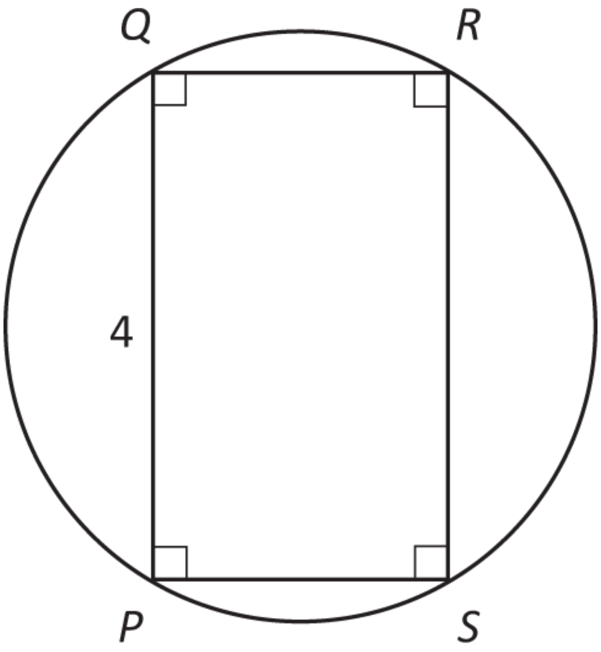
Step 4: **Make inferences from the figures.**
You will often need to make use of relationships that are not explicitly stated.

Now that you’ve got the basic process down, try another problem. Try it on your own first, then look at the steps used to solve it:

> Rectangle $PQRS$ is inscribed in circle O pictured below. If the circumference of circle O is 5π, what is the area of rectangle $PQRS$?



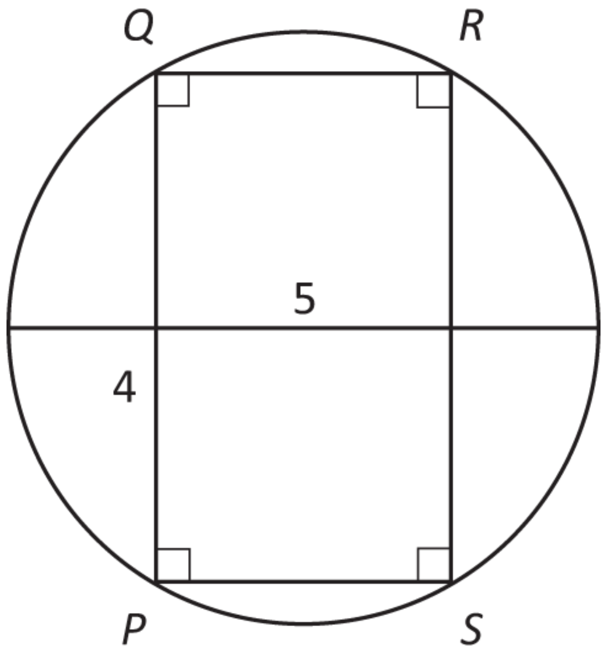
The first thing you need to do is to **redraw the figure** on whatever note paper you are using and **fill in all the given information**. The question didn't explicitly give you the value of any side lengths or angles, but it did say that *PQRS* is a rectangle. That means all four internal angles are right angles. So when you redraw the figure, it might look like this:



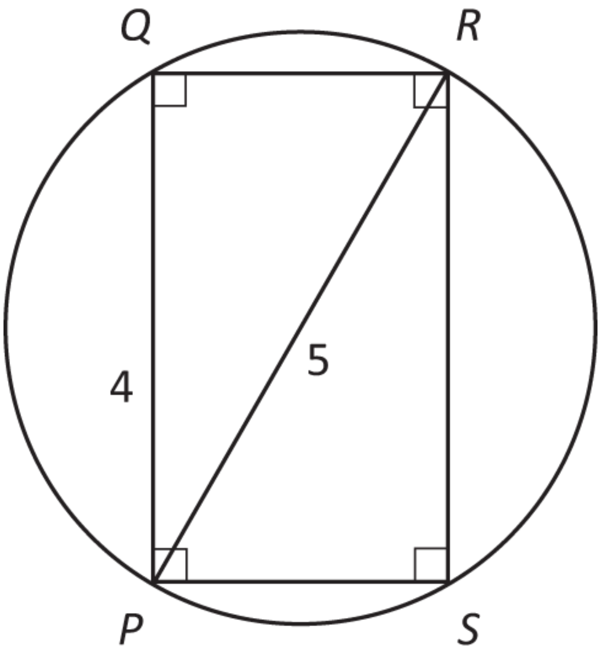
Now it’s time to **identify relationships and create equations**. The question stated that the circumference of circle *O* is 5π, and the formula for circumference is circumference equals 2πr, so $5\pi = 2\pi r$. Now that you know the circumference, there’s only one unknown (r), so you should **solve the equation for the missing value** and find the radius, which turns out to be 2.5. You also know that $d = 2r$, so the diameter of circle *O* is 5.

As with the previous problem, you are now left with the question: Why did you find the radius and diameter? You were able to solve for them, which is a very good clue that you need one of them to answer the question. Now is the time to **make inferences from the figures**.

Ultimately, this question is asking for the area of rectangle *PQRS*. What information do you need to find that value? You have the length of *QP*, which means that if you can find the length of either *QR* or *PS*, you can find the area of the rectangle. So you need to somehow find a connection between the rectangle and the radius or diameter. Put a diameter into the circle:



That didn’t really seem to help much, because you still have no way to make the connection between the diameter and the rectangle. It’s important to remember, though, that *any* line that passes through the center is a diameter. What if you drew the diameter so that it passed through the center but touched the circle at points P and R? You know that the line connecting points P and R will be a diameter because you know that the center of the circle is also the center of the rectangle. Your circle now looks like this:

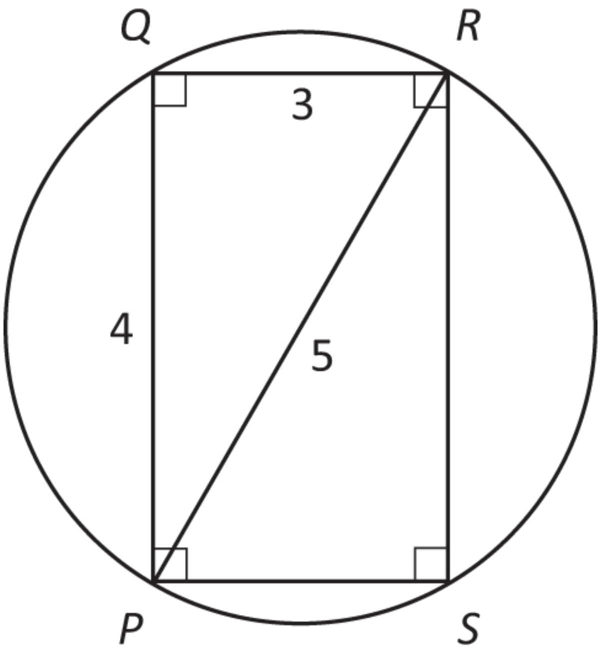


What was the advantage of drawing the diameter so that it connected points *P* and *R*? Now the diameter of the circle is also the diagonal of the rectangle. The circle and the rectangle have a common element. **Whenever possible, draw new elements such that they relate one shape to another.**

Where do you go from here? You still need the length of either *QR* or *PS*. Do you have a way to get either one of those values? As a matter of fact, you do. *PQR* is a right triangle. It's not oriented the way you are used to seeing it, but all the important elements are there. It's a triangle, and one of its internal angles is a right angle. Additionally, you know the lengths of two of the sides: *PQ* and *PR*. That means you can use the Pythagorean theorem to find the length of the third side, *QR*:

$$\begin{aligned}(QR)^2 + (PQ)^2 &= (PR)^2\\(QR)^2 + (4)^2 &= (5)^2\\(QR)^2 + 16 &= 25\\(QR)^2 &= 9\\QR &= 3\end{aligned}$$

Alternatively, you could have recognized the Pythagorean triple: triangle *PQR* is a 3–4–5 triangle. Either way, you arrive at the conclusion that the length of *QR* is 3. Your circle now looks like this:



Now you have what you need to find the area of rectangle *PQRS*: Area = (length) × (width) = (4) × (3) = 12. So the answer to the question is 12.

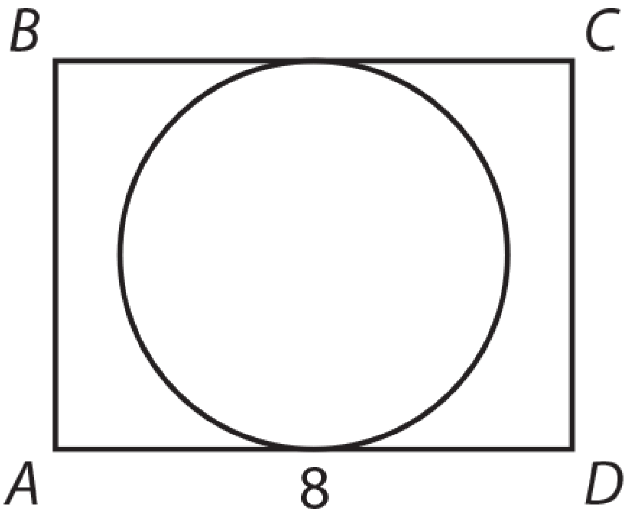
What did you need to do to arrive at that answer? For starters, you needed to make sure that you had an accurate figure to work with, and that you populated that figure with all the information that you had been given.

Next, you had to realize that knowing the circumference of the circle allowed you to find the diameter of the circle.

After that came what is often the most difficult part of the process—you had to make inferences based on the figure. The key insight in this problem was that you could draw a diameter in your figure that could also act as the diagonal of the rectangle. As if that wasn't difficult enough, you then had to recognize that *PQR* was a right triangle, even though it was rotated in a way that made this difficult to see. It is these kinds of insights that are going to be crucial to success on the GRE—recognizing shapes when they're presented in an unfamiliar format and finding connections between different shapes.

Check Your Skills

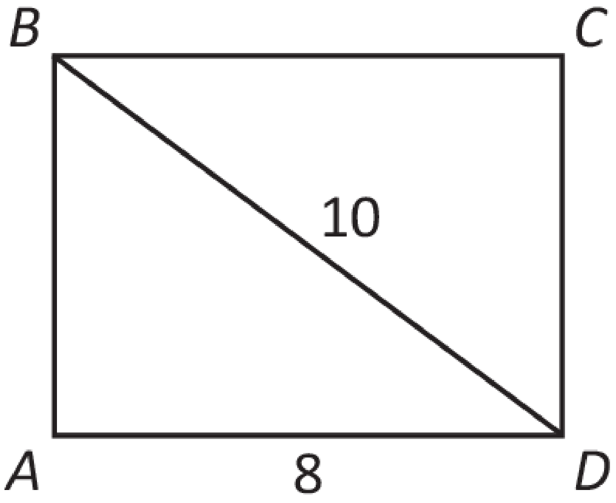
1. In rectangle *ABCD*, the distance between *B* and *D* is 10. What is the area of the circle inside the rectangle, if the circle is tangent to both *AD* and *BC*?


B
C
A
8
D

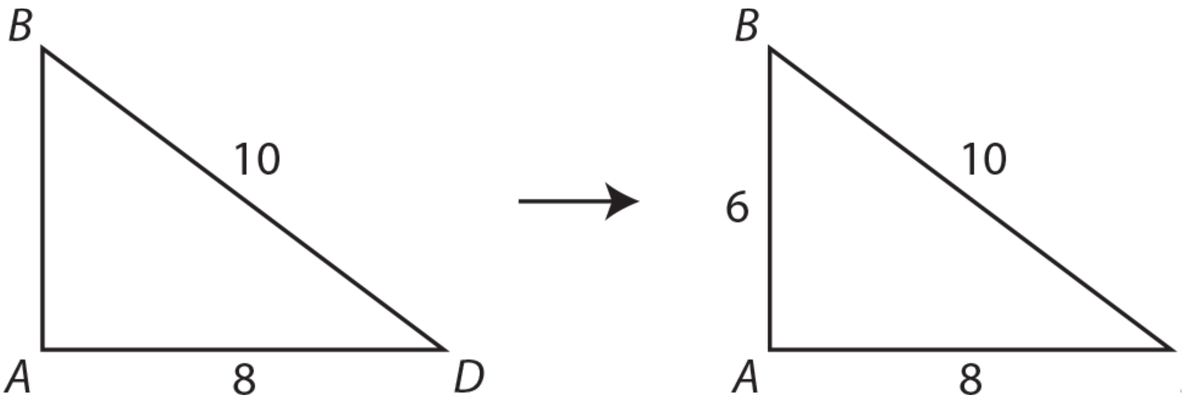
Check Your Skills Answer Key

1. **9*π***

   Consider only the rectangle for a moment. Diagonal *BD* cuts the rectangle into two right triangles, and the length of this diagonal is given:

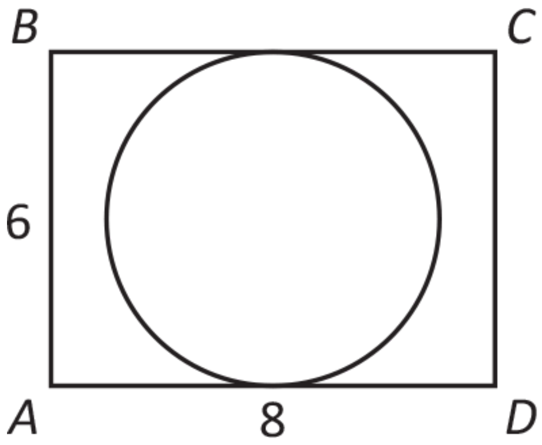
   

   Now look at right triangle *ABD*. The line segment *BD* functions not only as the diagonal of rectangle *ABCD* but also as the hypotenuse of right triangle *ABD*. So now find the third side of triangle *ABD*, either using the Pythagorean theorem or recognizing a Pythagorean triple (6–8–10):

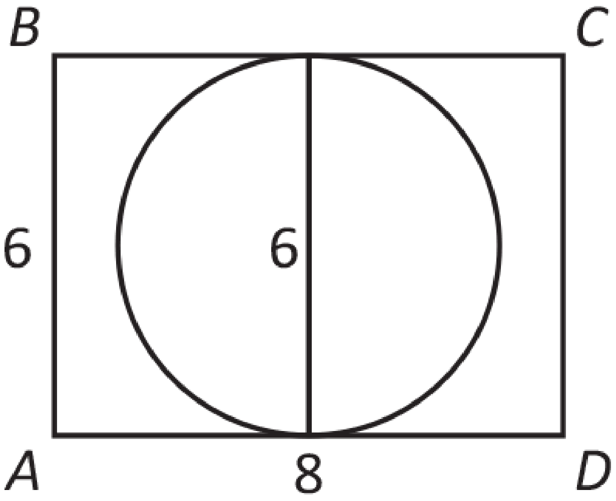


$$\begin{aligned}(AB)^2 + 8^2 &= 10^2\\ (AB)^2 + 64 &= 100\\ (AB)^2 &= 36\\ AB &= 6\end{aligned}$$

Now consider the circle within this 6 by 8 rectangle:



Since the circle touches both AD and BC, its diameter must be 6.



Finally, find the radius and compute the area:

$d = 6 = 2r$

$3 = r$

$\text{Area} = \pi r^2$

$\text{Area} = \pi 3^2$

$\text{Area} = 9\pi$

Problem Set

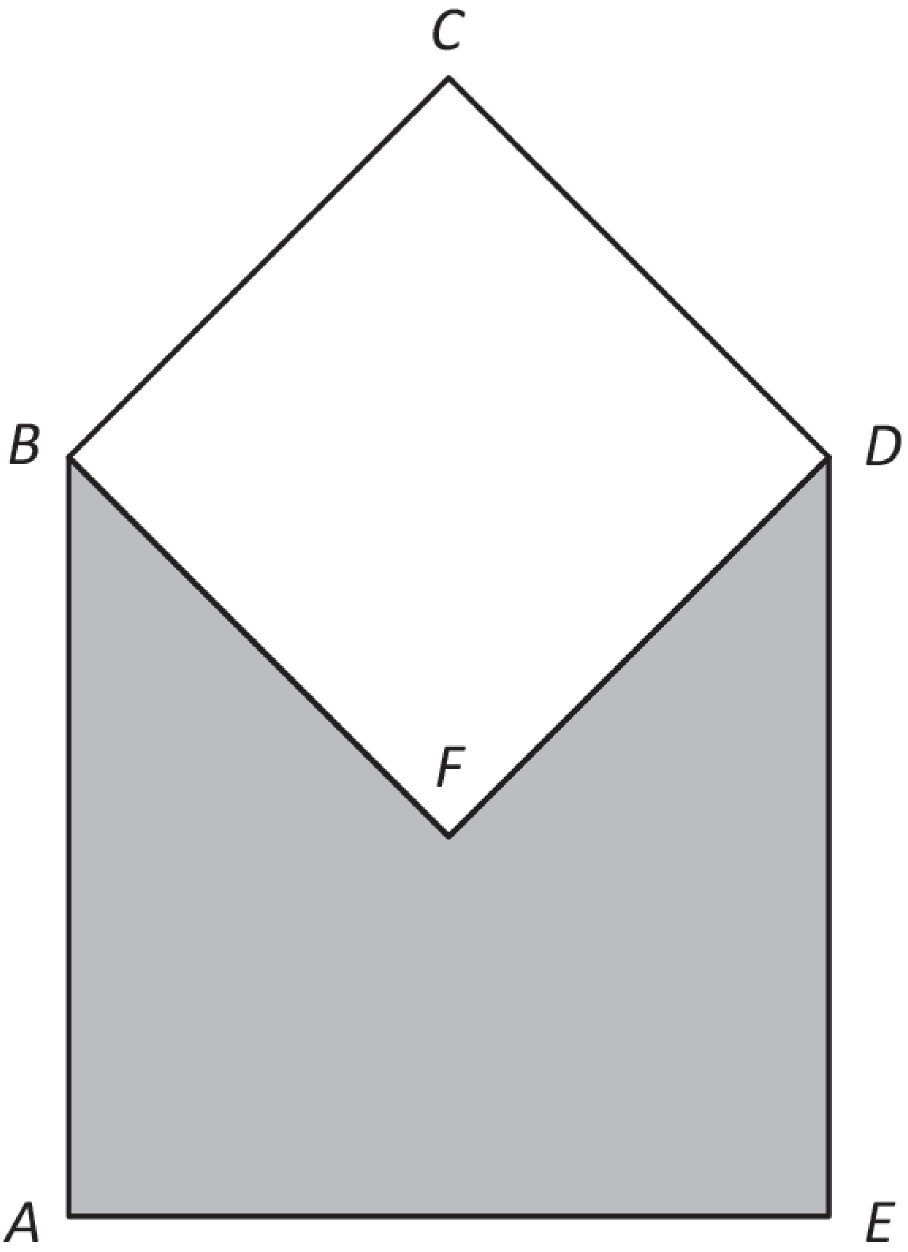
1. The “aspect ratio” of a rectangular TV screen is the ratio of its width to its height.

Quantity A	**Quantity B**
The area of a rectangular TV screen with an aspect ratio of 4:3 and a diagonal of 25"	The area of a rectangular TV screen with an aspect ratio of 16:9 and a diagonal of 25"

2. Ten 8-foot-long poles will be arranged in a rectangle to surround a flower bed.

Quantity A	**Quantity B**
The area in square feet of the flower bed	300

3.



BCDF and *ABDE* are squares.

Quantity A	**Quantity B**
Twice the area of the shaded region	Three times the area of *BCDF*

Solutions

1. **(A)**

   For the TV in Quantity A, the aspect ratio of 4:3 means the width is 4*x* and the height is 3*x*, where *x* is some unknown multiplier. By Pythagorean theorem, the diagonal is:

   $\sqrt{a^2+b^2}=\sqrt{(4x)^2+(3x)^2}=\sqrt{16x^2+9x^2}=\sqrt{25x^2}=5x$ You know that the diagonal is 25 inches, so *x* is 5. The width of the TV is (4)(5), which is 20, and the height is (3)(5), which is 15. Thus, the area is *wh* = (20)(15), which equals 300.

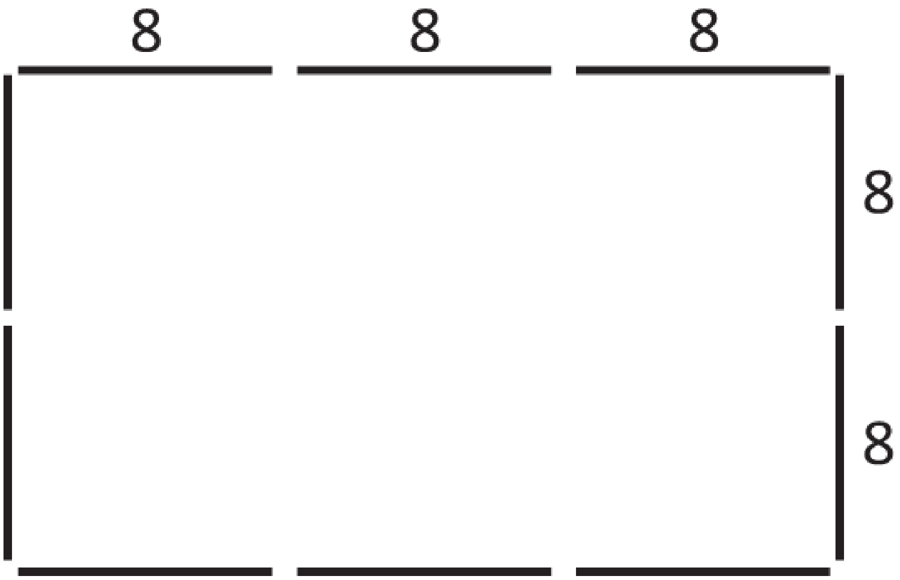
   For the TV in Quantity B, the aspect ratio of 16:9 means the width is 16*y* and the height is 9*y*, where *y* is some unknown multiplier. By Pythagorean theorem, the diagonal is:

   $\sqrt{a^2+b^2}=\sqrt{(16y)^2+(9y)^2}=\sqrt{256y^2+81y^2}=\sqrt{337y^2}\approx 18.3576y$ (use the calculator). You know that the diagonal is 25 inches, so $y\approx\frac{25}{18.3576}\approx 1.3618$. The width of the TV is approximately (16)(1.3618), which is 21.7888, and the height is approximately (9)(1.3618), which is 12.2562. Thus, the area is (width)(height) = (21.7888)(12.2562), which equals 267.05.

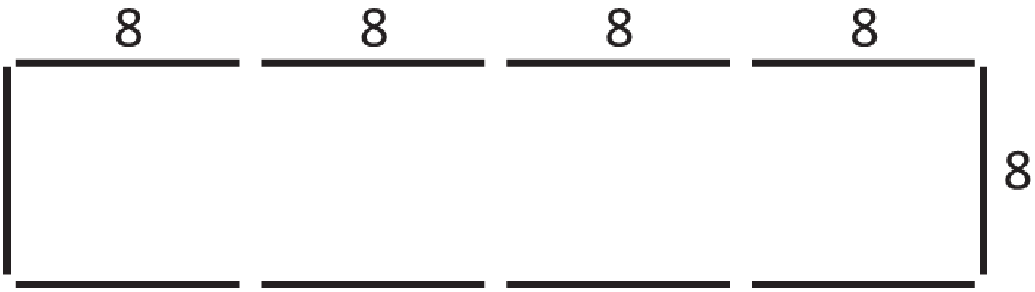
   Thus, **Quantity A is greater**.

Incidentally, you will learn a shortcut for this problem in Chapter 4 of this guide (in the section “Maximum Area of Polygons”). For a TV with either a fixed perimeter or a fixed diagonal, which similarly depends on the width and height, the area is maximized when the aspect ratio is 1. Because the aspect ratio 4:3 (equivalent to $\frac{15+10}{5}$) is closer to 1 than the aspect ratio 16:9 (equivalent to $\frac{5}{16} < \frac{5}{15}$, the TV with the 4:3 aspect ratio has a greater area.

2. **(D)**
   First, **draw the figure and fill in all given information**. The flower bed might look like this:



Or like this:



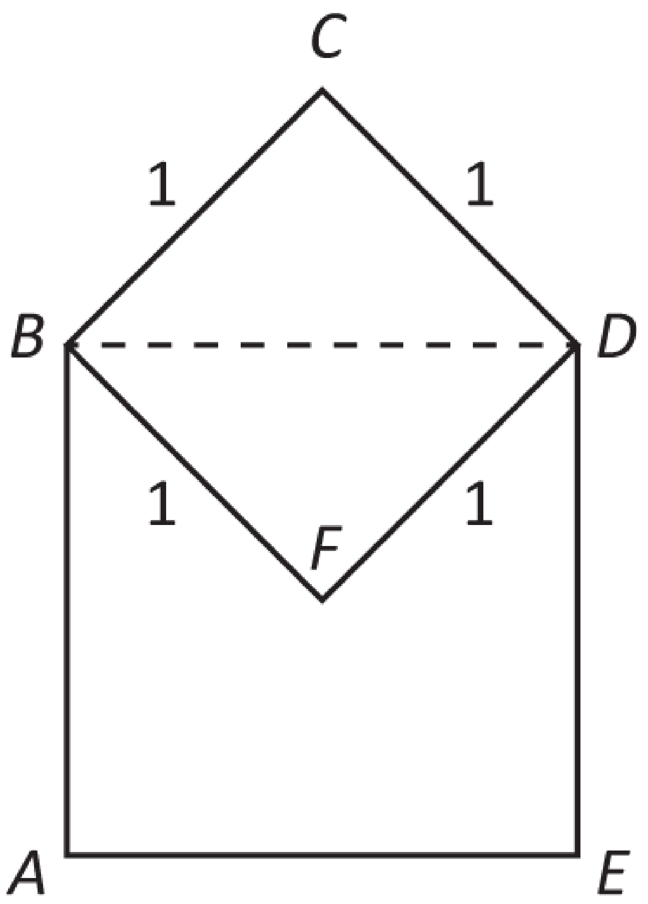
Then, **make inferences from the figures**. The area of a rectangle equals the width times the height.

The top flower bed has an area of (2 × 8)(3 × 8) = (16)(24) = 384, which is greater than 300.

The bottom flower bed has an area of (1 × 8)(4 × 8) = (8)(32) = 256, which is less than 300.

Therefore, **the relationship cannot be determined from the information given**.

3. **(C)**
   No lengths are given, therefore, you are free to pick some easy numbers to work with. **Draw the figure and fill in all given information**, which here is that the figures are squares. For the sides of the small square, 1 is an easy number, as it makes the area of square *BCDF* equal (1)(1), which is 1:



Next, identify relationships. The figure implies that the dashed line *BD* is both an edge of square *ABDE* and the diagonal of square *BCDF*, making it the hypotenuse of equal right triangles *BCD* and *BDF*. By Pythagorean theorem on right triangle *BCD*, you get the following:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ BD^2 &= 1^2 + 1^2 \\ BD^2 &= 2 \\ BD &= \sqrt{2} \end{aligned}$$

Thus, the area of square *ABDE* is $\sqrt{8} \times \sqrt{8} = 8$.

Make inferences from the figure: The shaded area is the area of square *ABDE* minus half the area of square *BCDF*. The shaded area is $\frac{1}{4} + \frac{2}{5} = ?$.

Quantity A is 2 times the shaded area, or $2 \times \frac{3}{2} = 3$.

Quantity B is 3 times the area of *BCDF*, or 3 × 1 = 3.

Thus, **the two quantities are equal**.

Chapter 10

TRIANGLES & DIAGONALS

In This Chapter…

The Basic Properties of a Triangle

Perimeter and Area

Right Triangles

Pythagorean Triples

Isosceles Triangles and the 45–45–90 Triangle

Equilateral Triangles and the 30–60–90 Triangle

Diagonals of Other Polygons

Chapter 10

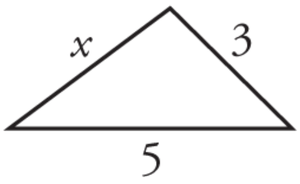
Triangles & Diagonals

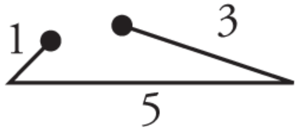
The Basic Properties of a Triangle

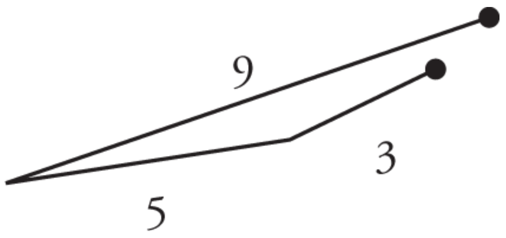
Triangles show up all over the GRE. You’ll often find them hiding in problems that seem to be about rectangles or other shapes. Of the basic shapes, triangles are perhaps the most challenging to master. One reason is that several properties of triangles are tested.

Following are some general comments on triangles.

The sum of any two side lengths of a triangle will always be greater than the third side length. This is because the shortest distance between two points is a straight line. At the same time, the third side length will always be greater than the difference of the other two side lengths. The following pictures illustrate these two points:







What is the largest number x could be? What’s the smallest? Observe what happens when you try to make $x = 9$ or $x = 1$:

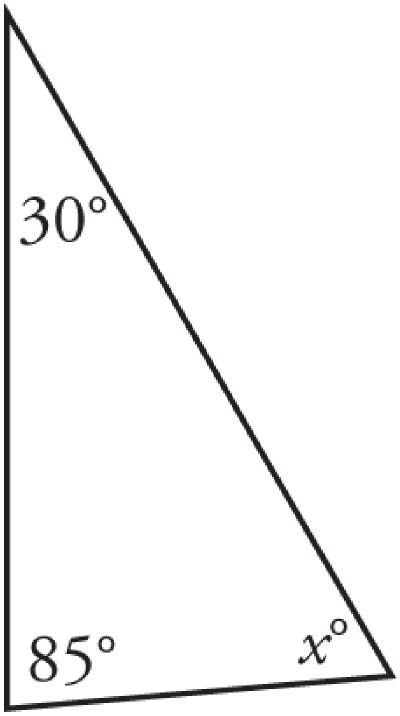
> x must be less than $3 + 5 = 8$
> x must be greater than $5 - 3 = 2$
> $2 < x < 8$

Check Your Skills

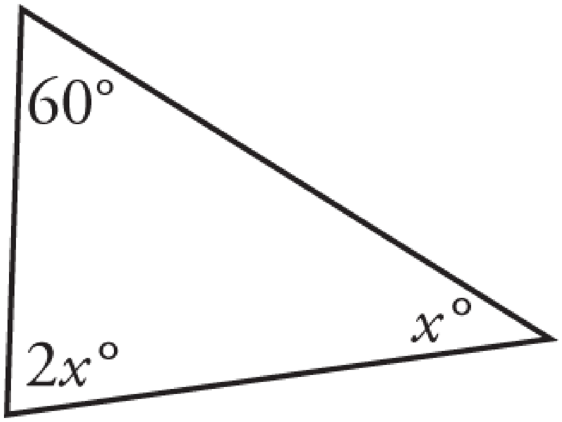
1. Two sides of a triangle have lengths of 5 and 19. Can the third side have a length of 13?

2. Two sides of a triangle have lengths of 8 and 17. What is the range of possible values of the length of the third side?

The internal angles of a triangle must add up to 180°. This rule can sometimes allow you to make inferences about angles of unknown size. It means that if you know the measures of two angles of the triangle, you can determine the measure of the third angle. Take a look at this triangle:



The three internal angles must add up to 180°, so you know that 30 + 85 + *x* = 180. Solving for *x* tells you that *x* = 65, so the third angle is 65°. The GRE can also test your knowledge of this rule in more complicated ways. Take a look at this triangle:



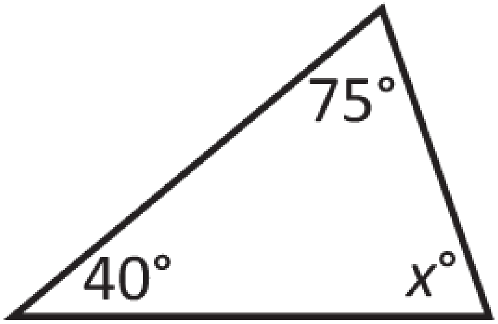
In this situation, you only know one of the angles. The other two are given in terms of *x*. Even though you only know one angle, you can still determine the other two. Again, you know that the three angles will add up to 180°, so $60 + x + 2x = 180$. That means that $3x = 120$, so $x = 40$. Thus, the angle labeled x° has a measure of 40° and the angle labeled $2x$° has a measure of 80°.

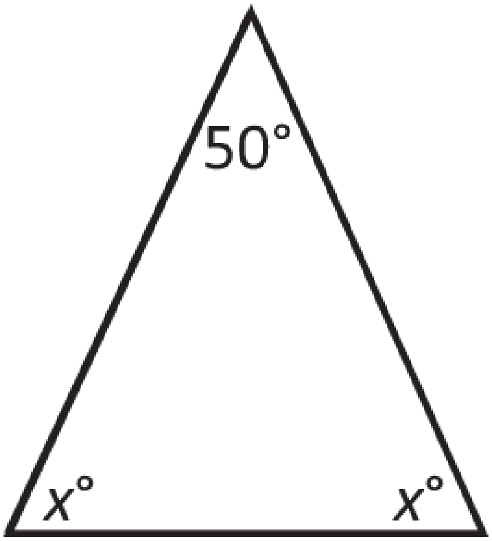
The GRE will not always draw triangles to scale, so don't try to guess angles from the picture, which could be distorted. Instead, solve for angles mathematically.

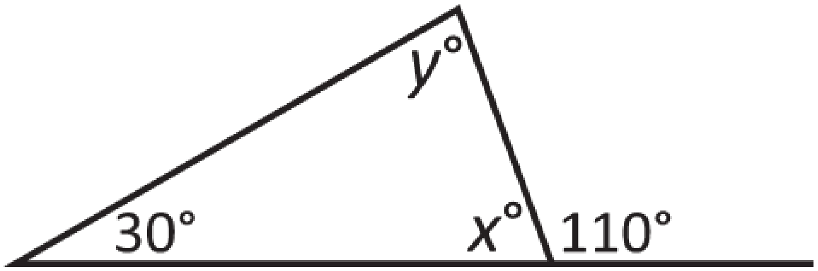
Check Your Skills

Find the missing angle(s).

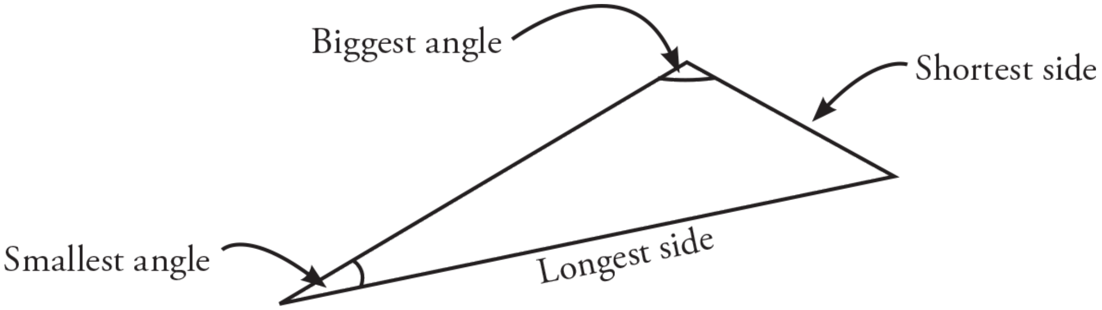
3.



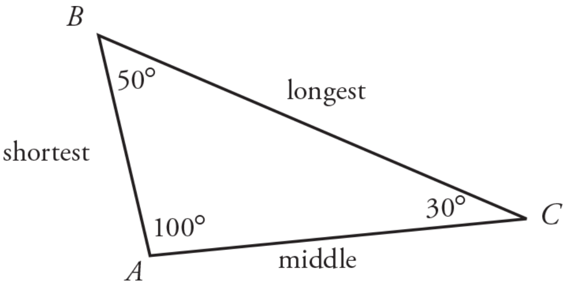
4. 

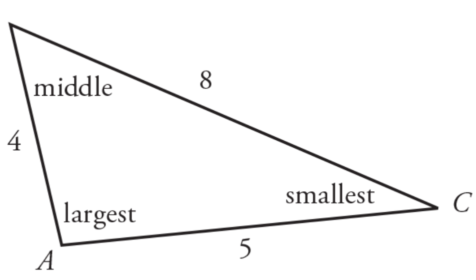
5. 

Internal angles of a triangle are important on the GRE for another reason. **Sides correspond to their opposite angles**. This means that **the longest side is opposite the largest angle, and the smallest side is opposite the smallest angle**. Think about an alligator opening its mouth, bigger and bigger … as the angle between its upper and lower jaws increases, the distance between the front teeth on the bottom and top would get longer and longer. This is illustrated as follows:



One important thing to remember about this relationship is that it works both ways. If you know the sides of the triangle, you can make inferences about the angles. If you know the angles, you can make inferences about the sides, as shown here.





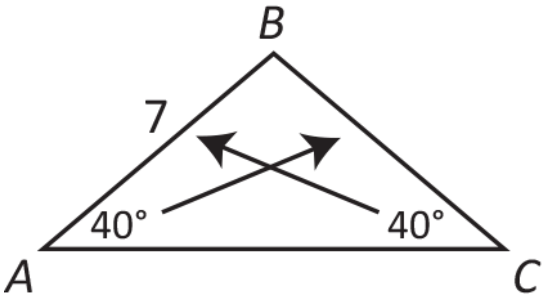
Although you can determine from the angle measures which sides are longer or shorter, you cannot determine how *much* longer or shorter. For instance, in the triangle above on the left, angle *BAC* is twice as large as angle *ABC*, but that does *not* mean that *BC* is twice as long as *AC*.

Things get interesting when a triangle has sides that are the same length or angles that have the same measure. You can classify triangles by the number of equal sides that they have:

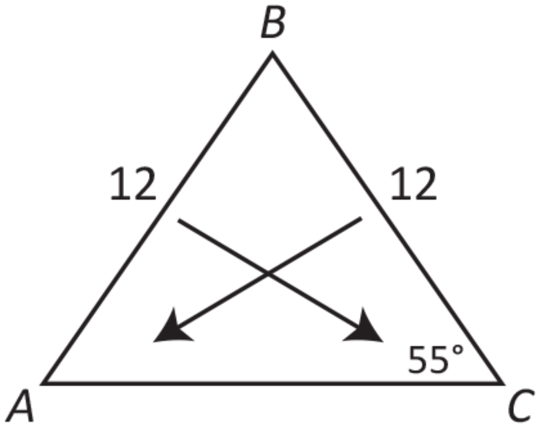
- A triangle that has two equal angles and two equal sides (opposite the equal angles) is an **isosceles triangle**.

- A triangle that has three equal angles (all 60°) and three equal sides is an **equilateral triangle**.

Once again, it is important to remember that this relationship between equal angles and equal sides works in both directions. Take a look at these isosceles triangles, and think about what additional information you can infer from them:

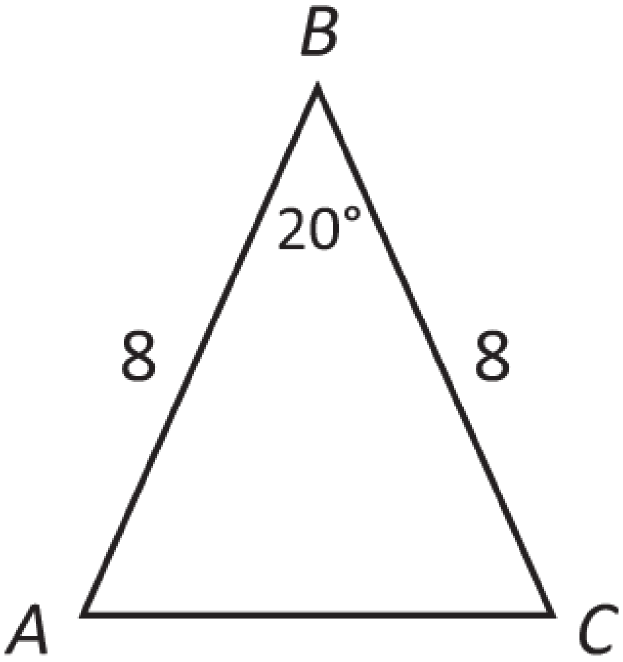


Infer: $BC = 7$



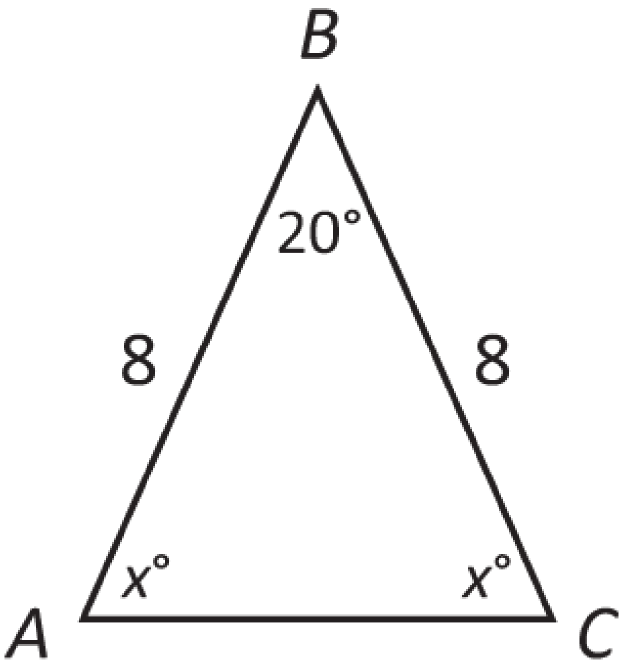
Infer: $\angle A = 55°$

The GRE loves isosceles triangles and uses them in a variety of ways. The following is a more challenging application of the equal sides/equal angles rule:



Take a look at the triangle and see what other information you can fill in. Specifically, do you know the degree measure of either angle *BAC* or angle *ACB*?

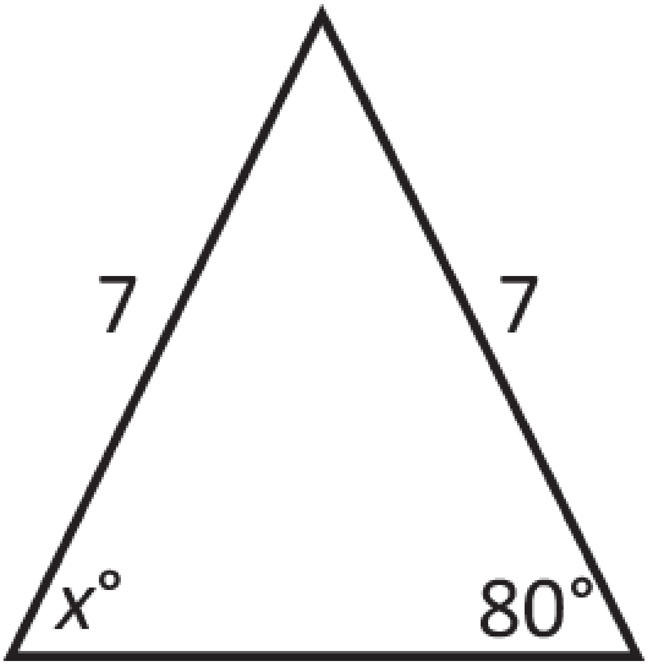
Because side *AB* is the same length as side *BC*, you know that angle *BAC* has the same degree measure as angle *ACB*. For convenience, you could label each of those angles as *x*° on your diagram:

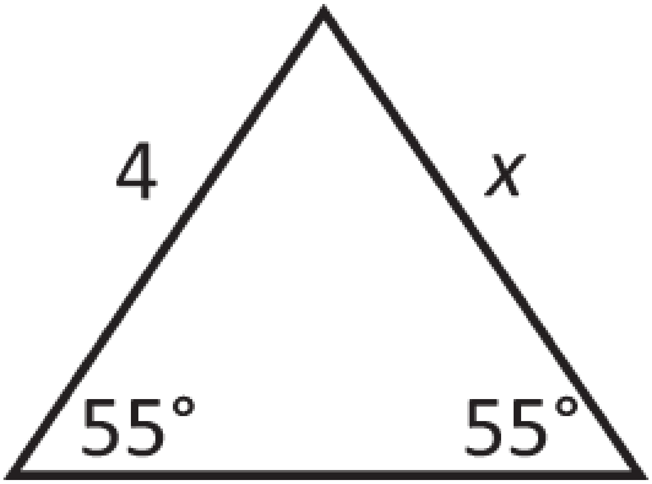


You also know that the three internal angles will add up to 180°, so $20 + x + x = 180$. Thus, $2x = 160$, and $x = 80$. So angle *BAC* and angle *ACB* each equal 80°. You can't find the side length *AC* without more advanced math, but the GRE wouldn't ask you for the length of *AC* for that very reason.

Check Your Skills

Find the value of x.

6. 

7. 

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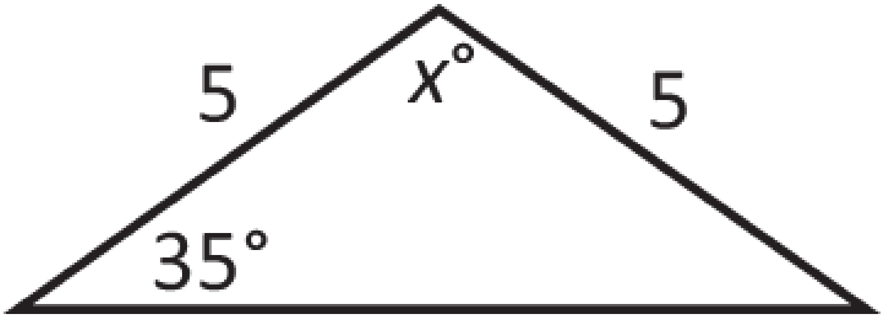
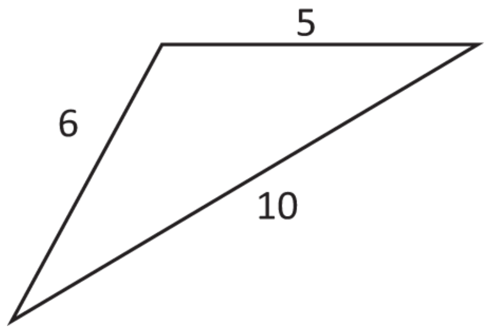
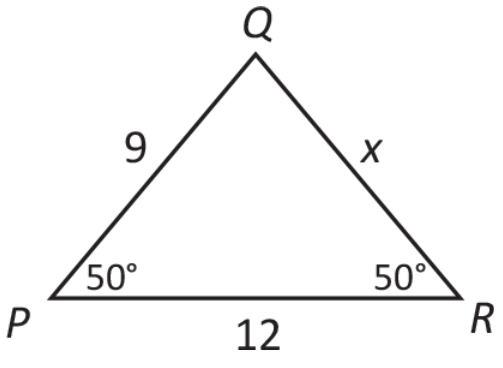


Perimeter and Area

The **perimeter** of a triangle is the sum of the lengths of all three sides.



In this triangle, the perimeter is: 5 + 6 + 10 = 21. This is a relatively simple property of a triangle, so often it will be used in combination with another property. Try this next problem. What is the perimeter of triangle *PQR*?

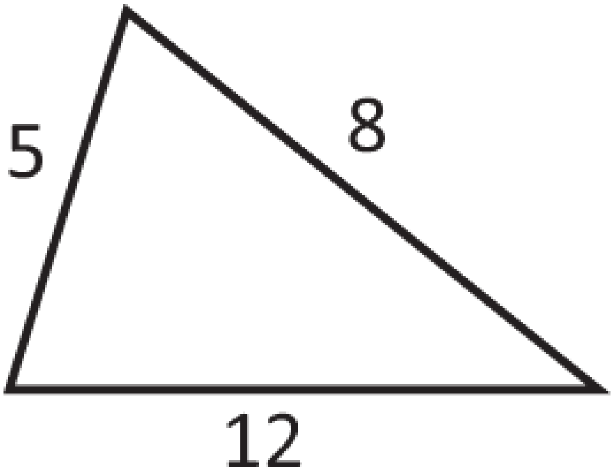


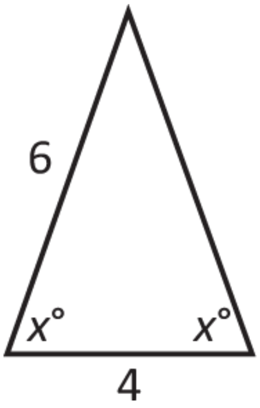
To solve for the perimeter, you will need to determine the value of *x*. Because angles *QPR* and *PRQ* are both 50°, you know that their opposite sides will have equal lengths. That means sides *PQ* and *QR* must have equal lengths, so you can infer that side *QR* has a length of 9. The perimeter of triangle *PQR* is: 9 + 9 + 12 = 30.

Check Your Skills

Find the perimeter of each triangle.

9.



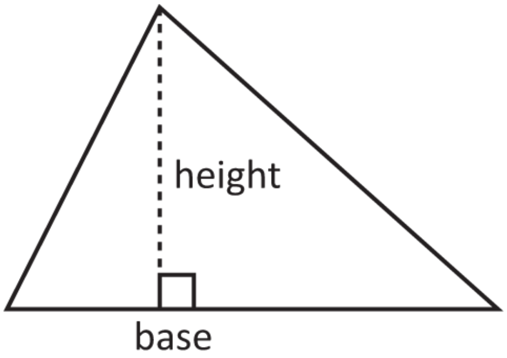
10. 

Note: Figures not drawn to scale.

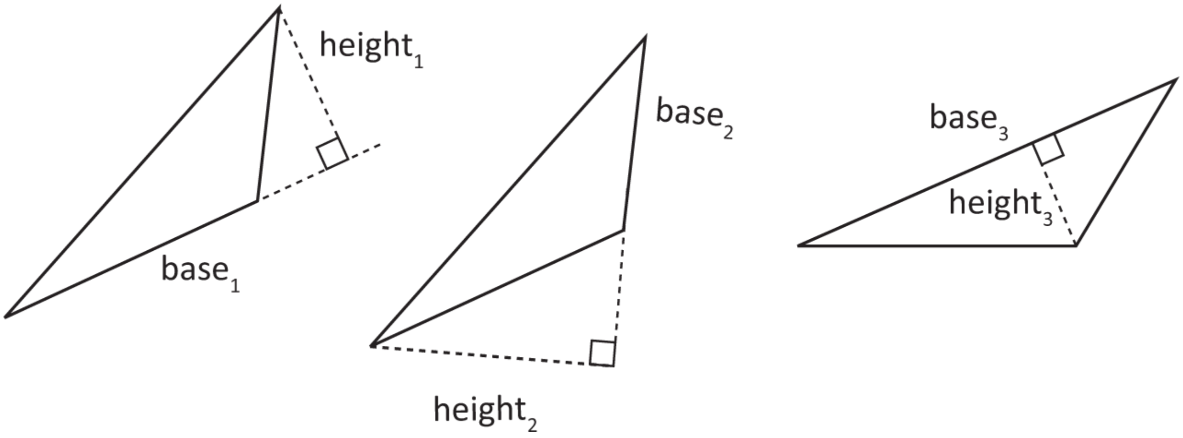
You need to be ready to solve geometry problems without depending on exactly accurate figures.

The final property of a triangle to review is area. You may be familiar with the equation $\textbf{Area} = \frac{1}{2}\,(\textbf{base}) \times (\textbf{height})$.

One very important thing to understand about the area of a triangle (and area in general) is the relationship between the base and the height. The base and the height MUST be perpendicular to each other. In a triangle, one side of the triangle is the base, and the height is formed by dropping a line from the opposite point of the triangle straight down toward the base, so that it forms a 90° angle with the base. The small square located where the height and base meet (shown in the following figure) is a very common symbol used to denote a right angle.



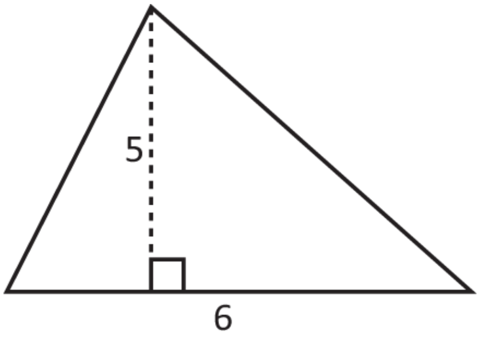
An additional challenge on the GRE is that problems will ask you about familiar shapes but present them to you in orientations you are not accustomed to. Even the area of a triangle can be affected. Most people generally think of the base of the triangle as the bottom side of the triangle, but, in reality, any side of the triangle could act as a base. In fact, depending on the orientation of the triangle, there may not actually be a bottom side. The following three triangles are all the same triangle, but each one has a different side as the base, and the corresponding height is drawn in.

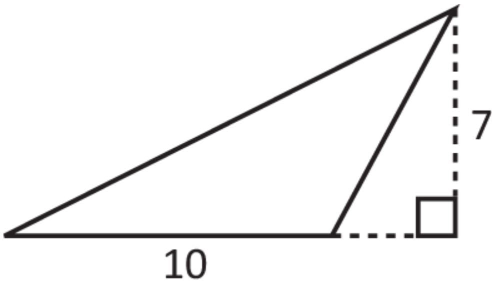


As it turns out, not only can any side be the base, but the height might be drawn outside the triangle! The only thing that matters is that the base and the height are perpendicular to each other.

Check Your Skills

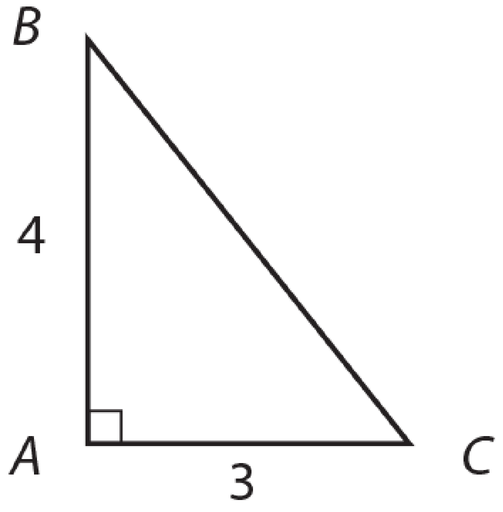
Determine the areas of the following triangles.

11. 

12. 

Right Triangles

There is one more class of triangle that is very common on the GRE: the **right triangle**. A right triangle is any triangle in which one of the angles is a right angle. The reason such triangles are so important will become more clear as you attempt to answer the next question:

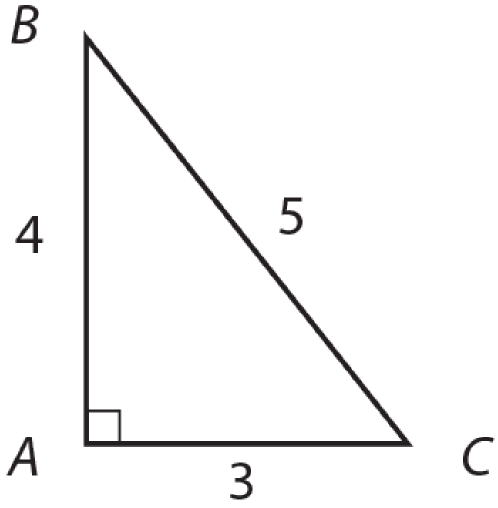
> What is the perimeter of triangle *ABC*?
>
> 
>

Normally, you would be unable to answer this question. You only have two sides of the triangle, but you need all three sides to calculate the perimeter.

The reason you can answer this question is that right triangles have an additional property that the GRE likes to make use of: there is a consistent relationship among the lengths of its sides. This relationship is known as the **Pythagorean theorem**. For *any* right triangle, the relationship is

$a^2 + b^2 = c^2$, where a and b are the lengths of the sides forming the right angle, also known as **legs**, and c is the length of the side opposite the right angle, also known as the **hypotenuse**.

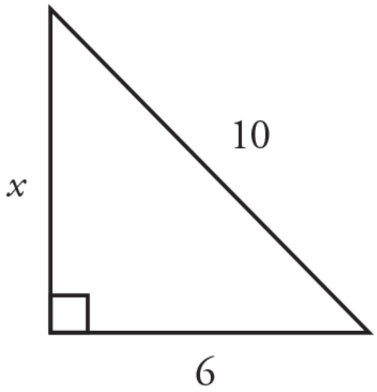
In the previous triangle, sides AB and AC are a and b (it doesn't matter which is which) and side BC is c. Thus, $(3)^2 + (4)^2 = (BC)^2 = 9 + 16 = (BC)^2$, so $25 = (BC)^2$, which makes the length of side BC equal to 5. The triangle really looks like this:



Finally, the perimeter is: 3 + 4 + 5 = 12.

Pythagorean theorem: $a^2 + b^2 = c^2$

<



What is x?

$$\begin{aligned} a^2 + b^2 &= c^2 \\ x^2 + 6^2 &= 10^2 \\ x^2 + 36 &= 100 \\ x^2 &= 64 \\ x &= 8 \end{aligned}$$

Pythagorean Triples

As mentioned in the previous section, right triangles show up in many problems on the GRE, and many of these problems require the Pythagorean theorem. However, there is a shortcut that you can use in many situations to make the calculations easier.

The GRE favors a certain subset of right triangles in which all three sides have lengths that are integer values. The previous triangle was an example of that. The lengths of the sides were 3, 4, and 5—all integers. This group of side lengths is a **Pythagorean triple**—in this case a 3–4–5 triangle. Although there is an infinite number of Pythagorean triples, a few are likely to appear on the test and should be memorized. For each triple, the first two numbers are the lengths of the sides that *form the right angle*, and the third (and largest) number is the *length of the hypotenuse*. They are:

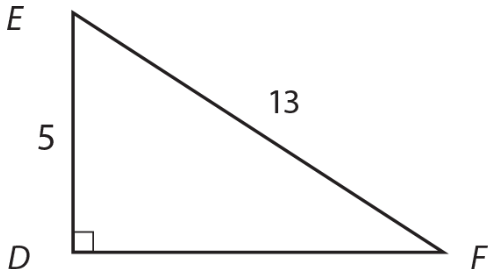
Common Combinations	**Key Multiples**
3–4–5	6–8–10
The most popular of all right triangles:	9–12–15
$3^2 + 4^2 = 5^2$ (9 + 16 = 25)	12–16–20
5–12–13	
Also quite popular on the GRE:	10–24–26
$5^2 + 12^2 = 13^2$ (25 + 144 = 169)	
8–15–17	

This one appears less frequently: $8^2 + 15^2 = 17^2$ (64 + 225 = 289)	None

Warning: Even as you memorize these triangles, don't assume that all triangles fall into these categories. When using common combinations to solve a problem, be sure that the triangle is a right triangle, and that the largest side (hypotenuse) corresponds to the largest number in the triple. For example, if you have a right triangle with one side measuring 3 and the hypotenuse measuring 4, *do not* conclude that the remaining side is 5.

That being said, try a practice question to see how memorizing these triples can save you time on the GRE:

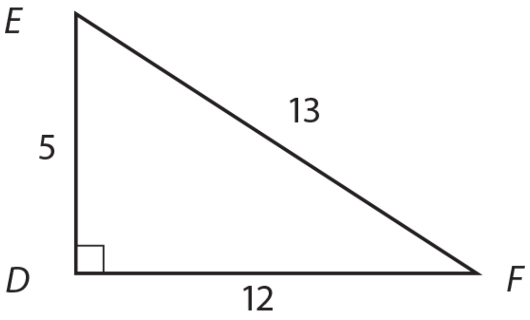
> What is the area of triangle *DEF*?



What do you need to find the area of triangle *DEF*? The formula for the area of this triangle is $\text{area} = \frac{1}{2}\,(\text{base}) \times (\text{height})$, so you need a base and a height. This is a right triangle, so sides *DE* and *DF* are perpendicular to each other, which means that if you can figure out the length of side *DF*, you can calculate the area.

The question then becomes, how do you find the length of side *DF*? First, realize that you can *always* find the length of the third side of a right triangle if you know the lengths of the other two sides. That’s because you know the Pythagorean theorem. In this case, the formula would look like this: $(DE)^2 + (DF)^2 = (EF)^2$. You know the lengths of two of those sides, so you could rewrite the equation as $(5)^2 + (DF)^2 = (13)^2$. Solving this equation, you get $25 + (DF)^2 = 169$, so $(DF)^2 = 144$, which means *DF* is 12. But these calculations are unnecessary; once you see a right triangle in which one of the legs has a length of 5 and the hypotenuse has a length of 13, you should recognize the Pythagorean triple. The length of the other leg must be 12.

However you find the length of side *DF*, your triangle now looks like this:

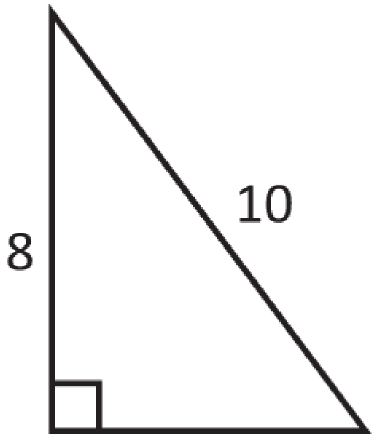


Now you have what you need to find the area of triangle *DEF*:
$\frac{1}{2} \div \frac{1}{4} = \frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$. Note that in a right triangle, you can consider one leg the base and the other leg the height.

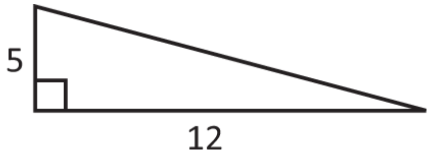
Check Your Skills

For questions 13–14, find the length of the third side of the triangle.

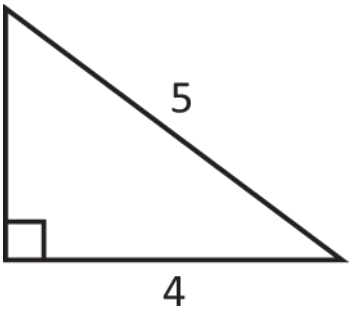
13.



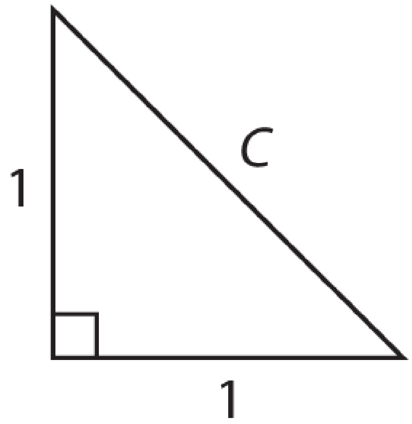
14.



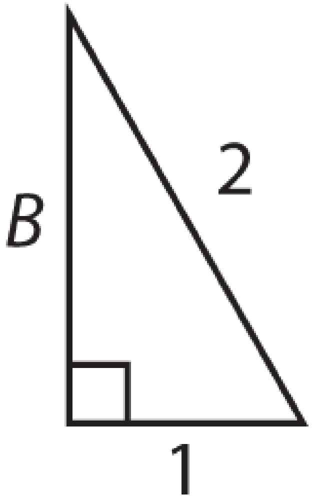
15. Find the area



16. What is the length of hypotenuse *C*?



17. What is the length of leg *B*?



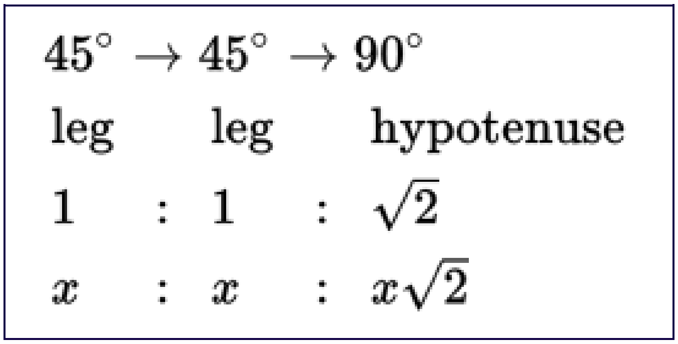
18. Triangle *ABC* is isosceles. If *AB* = 3, and *BC* = 4, what are the possible lengths of *AC*?

Isosceles Triangles and the 45–45–90 Triangle

As previously noted, an isosceles triangle is one in which two sides are equal. The two angles opposite those two sides will also be equal. The most important isosceles triangle on the GRE is the isosceles right triangle.

An isosceles right triangle has one 90° angle (opposite the hypotenuse) and two 45° angles (opposite the two equal legs). This triangle is called the 45–45–90 triangle.

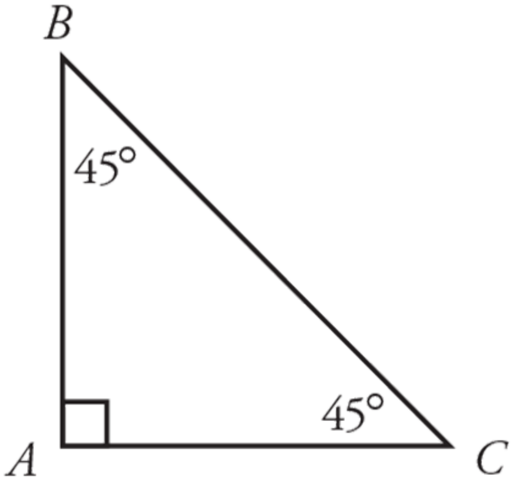
The lengths of the legs of every 45–45–90 triangle have a specific ratio, which you must memorize:



$$\begin{array}{lllll} 45^\circ & \rightarrow & 45^\circ & \rightarrow & 90^\circ \\ \text{leg} & & \text{leg} & & \text{hypotenuse} \\ 1 & : & 1 & : & \sqrt{2} \\ x & : & x & : & x\sqrt{2} \end{array}$$

What does it mean that the sides of a 45–45–90 triangle are in a $1:1:\sqrt{2}$ ratio? It doesn’t mean that they are actually 1, 1, and $\sqrt{9}$ (although that’s a possibility). It means that the sides are some multiple of $1:1:\sqrt{2}$. For instance, they could be 2, 2, and $\sqrt[3]{64}$, or 5.5, 5.5, and

$5.5\sqrt{2}$. In the last two cases, the number you multiplied the ratio by—either 2 or 5.5—is called the **multiplier**. Using a multiplier of 2 has the same effect as doubling a recipe—each of the ingredients gets doubled. Of course, you can also triple a recipe or multiply it by any other number, even a fraction. Try this problem:



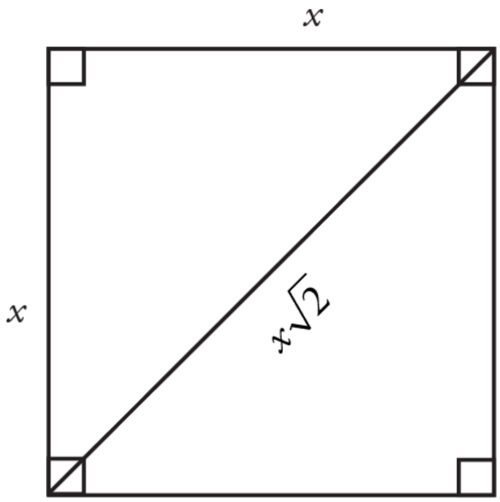
> If the length of side *AB* is 5, what are the lengths of sides *BC* and *AC*?

Because *AB* is 5, use the ratio $1 : 1 : \sqrt{2}$ for sides *AB* : *AC* : *BC* to determine that the multiplier *x* is 5. You then find that the sides of the triangle have lengths $5 : 5 : 5\sqrt{2}$. Therefore, the length of side *AC* = 5, and the length of side $BC = 5\sqrt{2}$. Using the same figure, though without the information from the previous question, review the following problem:

> If the length of side *BC* is $\sqrt[3]{64}$, what are the lengths of sides *AB* and *AC*?

Because the hypotenuse *BC* is $\sqrt{18} = x\sqrt{2}$, solve for x: $\sqrt{18} \div \sqrt{2} = \sqrt{9} = 3$. Thus, the sides *AB* and *AC* are each equal to *x*, which is 3.

One reason that the 45–45–90 triangle is so important is that this triangle is exactly half of a square! That is, two 45–45–90 triangles put together make up a square. Thus, if you are given the diagonal of a square, you can use the 45–45–90 ratio to find the length of a side of the square.



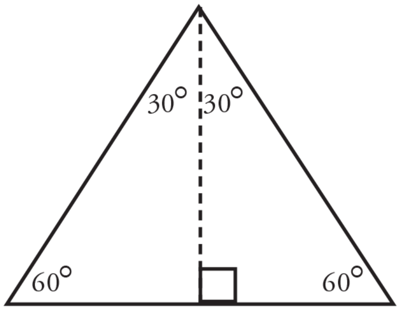
Check Your Skills

19. What is the area of a square with diagonal of 6?

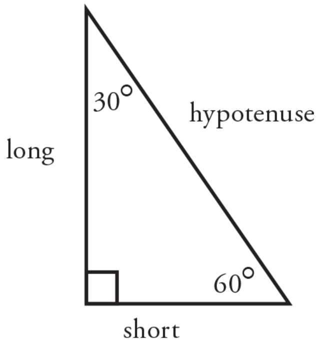
20. What is the diagonal of a square with an area of 25?

Equilateral Triangles and the 30–60–90 Triangle

An equilateral triangle is one in which all three sides (and all three angles) are equal. Each angle of an equilateral triangle is 60° (because all three angles must sum to 180°). A close relative of the equilateral triangle is the 30–60–90 triangle. Notice that two 30–60–90 triangles, when put together, form an equilateral triangle:



Equilateral Triangle



30–60–90 Triangle

The lengths of the legs of every 30–60–90 triangle have the following ratio, which you must memorize:

$$\begin{array}{lllll}
30^\circ & \rightarrow & 60^\circ & \rightarrow & 90^\circ \\
\text{short leg} & & \text{long leg} & & \text{hypotenuse} \\
1 & : & \sqrt{3} & : & 2 \\
x & : & x\sqrt{3} & : & 2x
\end{array}$$

> If the short leg of a 30–60–90 triangle has a length of 6, what are the lengths of the long leg and the hypotenuse?

The short leg, which is opposite the 30° angle, is 6. Use the ratio $1 : \sqrt{3} : 2$ to determine that the multiplier *x* is 6. You then find that the sides of the triangle have lengths $6 : 6\sqrt{3} : 12$. The long leg measures $\sqrt[3]{64}$ and the hypotenuse measures 12. Try another problem:

> If an equilateral triangle has a side of length 10, what is its height?

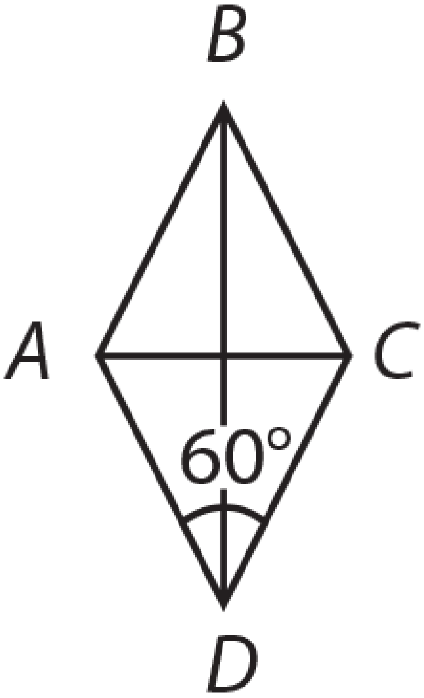
Looking at the equilateral triangle above, you can see that the side of an equilateral triangle is the same as the hypotenuse of a 30–60–90 triangle. Additionally, the height of an equilateral triangle is the same as the long leg of a 30–60–90 triangle.

Because you are told that the hypotenuse is 10, use the ratio $x : x\sqrt{3} : 2x$ to get 2*x* = 10 and determine that the multiplier *x* is 5. You then find that the sides of the 30–60–90 triangle have lengths $5 : 5\sqrt{3} : 10$. Thus, the long leg has a length of $\sqrt[3]{64}$, which is the height of the equilateral triangle.

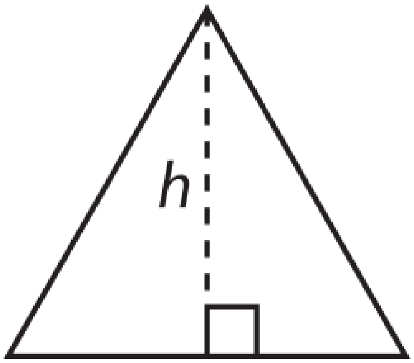
If you get tangled up on a 30–60–90 triangle, try to find the length of the short leg. The other legs will then be easier to figure out.

Check Your Skills

21. Quadrilateral *ABCD* is composed of four 30–60–90 triangles. If $BD = 10\sqrt{3}$, what is the perimeter of *ABCD*?



22. Each side of the equilateral triangle shown is 2. What is the height *h* of the triangle?


h

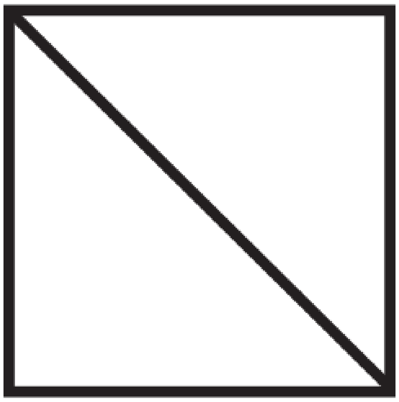
Diagonals of Other Polygons

Right triangles are useful for more than just triangle problems. They are also helpful for finding the diagonals of other polygons, specifically squares, cubes, rectangles, and rectangular solids.

The diagonal of a square can be found using the formula $\sqrt[3]{64} = 4$, where *s* is a side of the square. This is also the diagonal of a face of a cube.

Alternatively, you can recall that any square can be divided into two 45–45–90 triangles, and you can use the ratio $1 : \sqrt{3} : 2$ to find the diagonal. You can also always use the Pythagorean theorem:

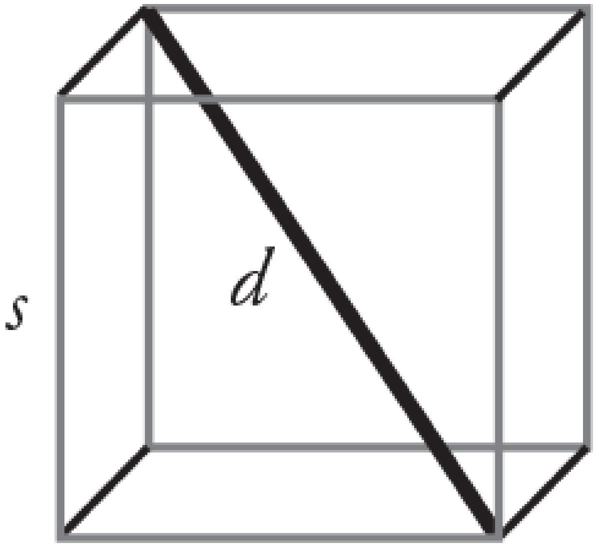
> If a square has a side of length 7, what is the length of the diagonal of the square?



Using the formula $\sqrt[3]{64} = 4$, you find that the length of the diagonal of the square is $\sqrt[3]{64}$.

The main diagonal of a cube can be found using the formula $\sqrt[3]{64} = 4$, where *s* is an edge of the cube. Try an example:

> What is the measure of an edge of a cube with a main diagonal of length $\sqrt[3]{64}$?

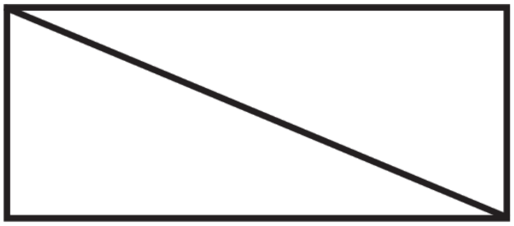


Again, using the formula $\sqrt[3]{64} = 4$, solve as follows:

$$\sqrt{60} = s\sqrt{3} \rightarrow s = \frac{\sqrt{60}}{\sqrt{3}} = \sqrt{20}$$

Thus, the length of the edge of the cube is $\sqrt{20} = 2\sqrt{5}$.

To find the diagonal of a rectangle, you must know EITHER the length and the width OR one dimension and the proportion of one to the other.



Use the rectangle shown here for the next two problems:

> If the rectangle above has a length of 12 and a width of 5, what is the length of the diagonal?

Using the Pythagorean theorem, solve:

$$5^2 + 12^2 = c^2$$
$$25 + 144 = c^2$$
$$c = 13$$

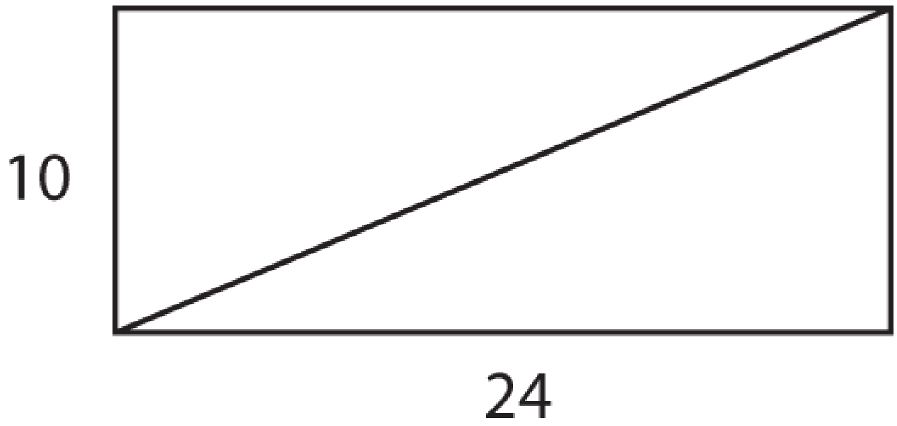
Thus, the diagonal length is 13.

> If the rectangle above has a width of 6, and the ratio of the width to the length is 3 : 4, what is the diagonal?

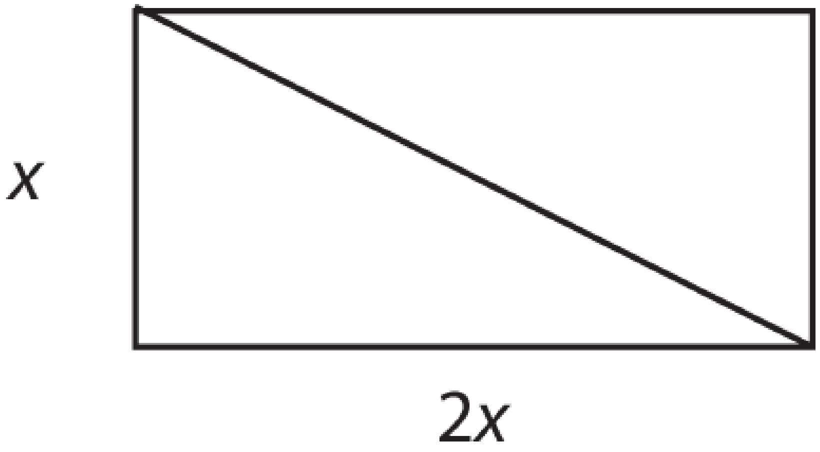
In this problem, you can use the ratio to find the value of the length. Using the ratio of 3 : 4 given in this problem, you find that the length is 8. Now you can use the Pythagorean theorem. Alternatively, you can recognize that this is a 6–8–10 triangle. Either way, the diagonal length is 10.

Check Your Skills

23. What is the diagonal of the rectangle shown?



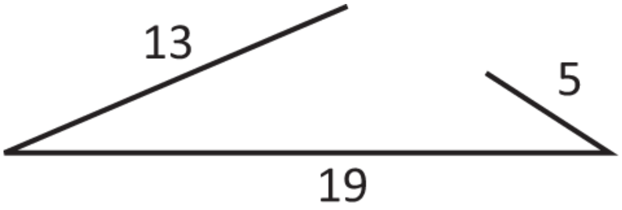
24. If the rectangle shown has a perimeter of 6, what is its diagonal?



Check Your Skills Answer Key

1. **No**

   If the two known sides of the triangle are 5 and 19, then the third side of the triangle cannot have a length of 13, because that would violate the rule that any two sides of a triangle must add up to greater than the third side: $5 + 13 = 18$, and $18 < 19$:



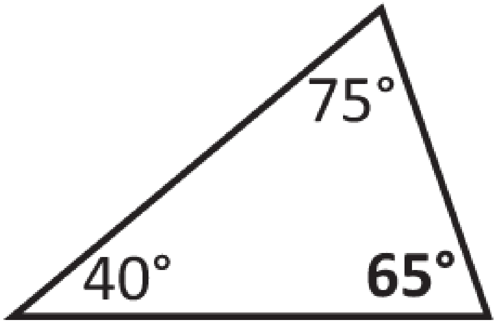
No possible triangle with these lengths.

2. **9 < third side < 25**

   If the two known sides of the triangle are 8 and 17, then the third side must be less than the sum of the other two sides: $8 + 17 = 25$, so the third side must be less than 25. The third side must also be greater than the difference of the other two sides: $17 - 8 = 9$, so the third side must be greater than 9. That means that $9 < \text{third side} < 25$.

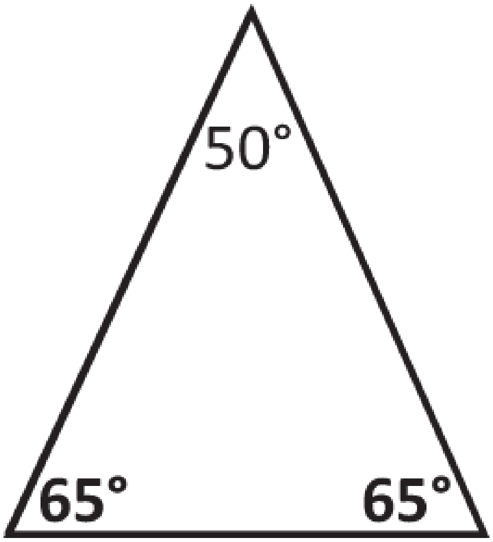
3. **65°**

   The internal angles of a triangle must add up to 180°, so you know that $40 + 75 + x = 180$. Solving for x gives you $x = 65°$:



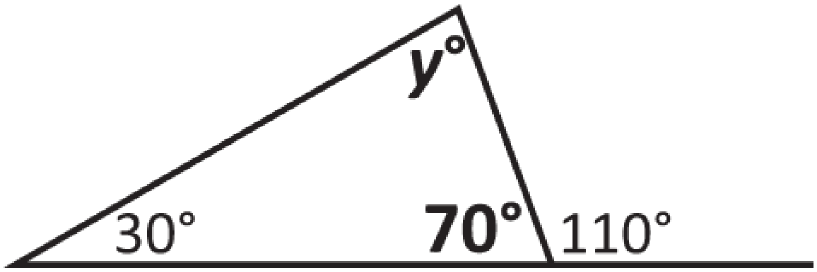
4. **65°**

   The three internal angles of the triangle must add up to 180°, so $50 + x + x = 180$. That means that $2x = 130$, and $x = 65$:

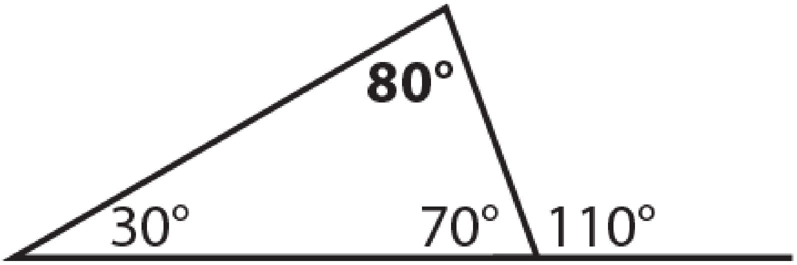


5. ***x* = 70°, *y*=80°**

   To determine the missing angles of the triangle, you need to do a little work with the picture. You can figure out the value of x, because straight lines have a degree measure of 180, so $110 + x = 180$, which means $x = 70$. That means your picture looks like this:

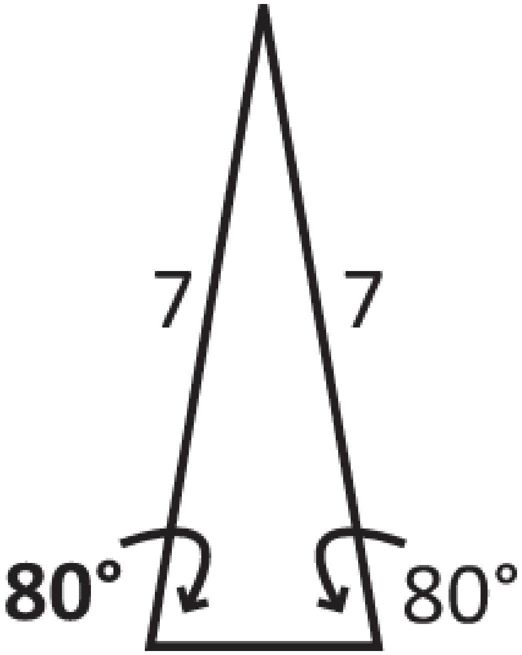


Now you can find *y*, because 30 + 70 + *y* = 180. Solving for *y* gives you *y* = 80:



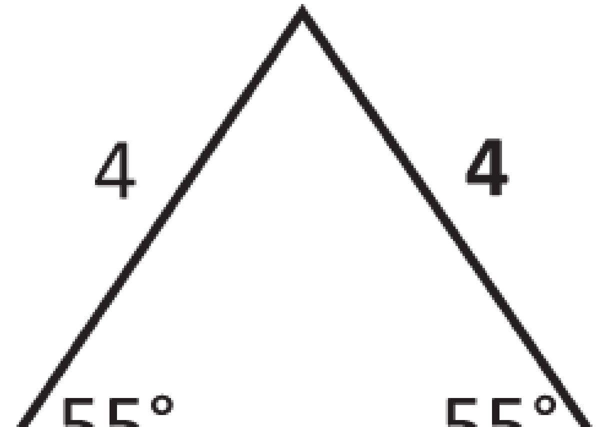
6. **80°**

   In this triangle, two sides have the same length, which means this triangle is isosceles. You also know that the two angles opposite the two equal sides will also be equal. That means that *x* must be 80:



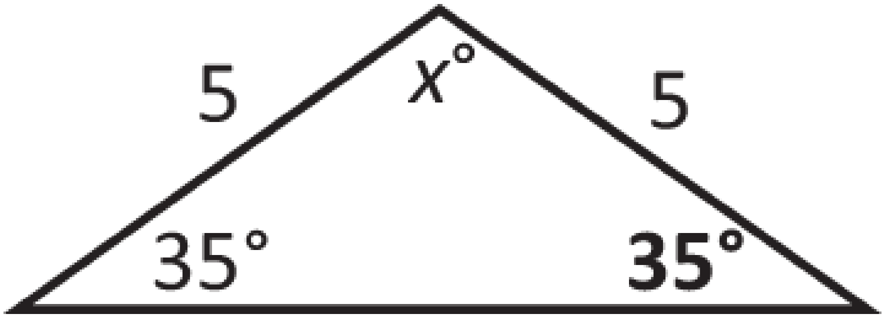
7. **4**

   In this triangle, two angles are equal, which means this triangle is isosceles. Thus, you also know that the two sides opposite the equal angles must also be equal, so *x* must equal 4:

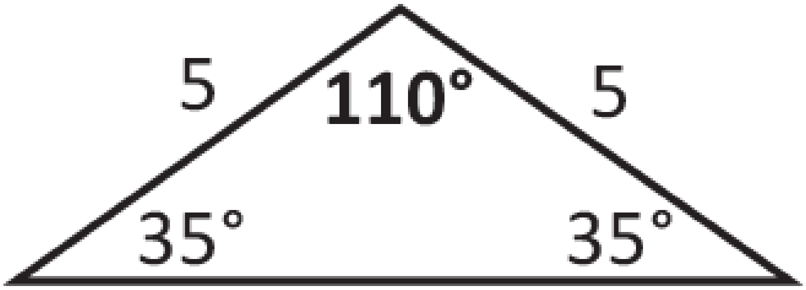


8. **110°**

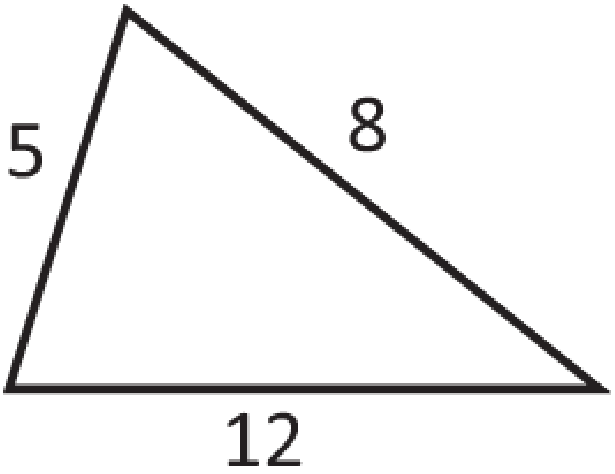
This triangle is isosceles, because two sides have the same length. That means the angles opposite the equal sides must also be equal. That means the triangle really looks like this:



Now you can find *x*, because you know 35 + 35 + *x* = 180. Solving for *x* gives you *x* = 110:



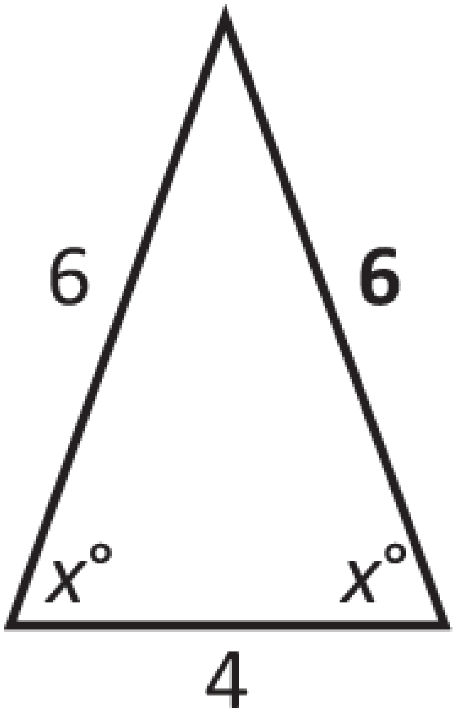
9. **25**



To find the perimeter of the triangle, add up all three sides: 5 + 8 + 12 = 25. Thus, the perimeter is 25.

10. **16**

To find the perimeter of the triangle, you need the lengths of all three sides. This is an isosceles triangle, because two angles are equal. That means that the sides opposite the equal angles must also be equal. So your triangle looks like this:



The perimeter is 6 + 6 + 4, which equals 16.

11. **15**

The area of a triangle is $\frac{1}{2}b \times h$. In the triangle shown, the base is 6 and the height is 5, so the area is $\frac{5}{16} < \frac{5}{15}$, which equals 15.

12. **35**

In this triangle, the base is 10 and the height is 7. Remember that the height must be perpendicular to the base—it doesn't need to lie within the triangle. Thus, the area is $\frac{3}{10} \times \frac{6}{7} =$, which equals 35.

13. **6**

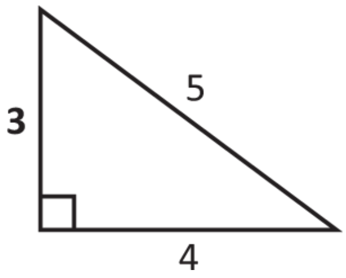
This is a right triangle, so you can use the Pythagorean theorem to solve for the length of the third side. The hypotenuse is the side with length 10, so the formula is $(8)^2 + b^2 = (10)^2$. Thus, $64 + b^2 = 100$, so b^2 is 36, which means $b = 6$. The third side of the triangle has a length of 6. Alternatively, you could recognize that this triangle is one of the Pythagorean triples—a 6–8–10 triangle, which is just a doubled 3–4–5 triangle.

14. **13**

    This is a right triangle, so you can use the Pythagorean theorem to solve for the length of the third side. The hypotenuse is the unknown side, so the formula is $(5)^2 + (12)^2 = c^2$. Thus, $25 + 144 = c^2$, so $c^2 = 169$, which means c is 13. The third side of the triangle has a length of 13. Alternatively, you could recognize that this triangle is one of the Pythagorean triples—a 5–12–13 triangle.

15. **6**

    This is a right triangle, so you can use the Pythagorean theorem to solve for the third side, or recognize that this is a 3–4–5 triangle. Either way, the result is the same: the length of the third side is 3:



Now you can find the area of the triangle. Area of a triangle is $\frac{1}{2}b \times h$, so the area of this triangle is $\frac{1}{2}(3) \times (4)$, which equals 6.

16. $\sqrt{9}$

    Apply the Pythagorean theorem directly, substituting 1 for *a* and *b*, and *C* for *c*:

$$\begin{aligned} 1^2 + 1^2 &= C^2 \\ 2 &= C^2 \\ C &= \sqrt{2} \end{aligned}$$

17. $\sqrt{9}$

    Apply the Pythagorean theorem directly, substituting 1 for *a* and 2 for *c*, and *B* for *b*:

$$\begin{aligned} 1^2 + B^2 &= 2^2 \\ 1 + B^2 &= 4 \\ B^2 &= 3 \\ B &= \sqrt{3} \end{aligned}$$

18. **3 or 4**

    Because an isosceles triangle has two equal sides, the third side must be equal to one of the two named sides.

19. **18**

Call the side length of the square *x*. Thus, the diagonal would be $x\sqrt{2}$.

You know the diagonal is 6, so $x\sqrt{2} = 6$. This means $x = \frac{6}{\sqrt{2}}$.

The area is *x* × *x*, or $\frac{6}{\sqrt{2}} \times \frac{6}{\sqrt{2}} = \frac{36}{2} = 18$.

20. $\sqrt[3]{64}$

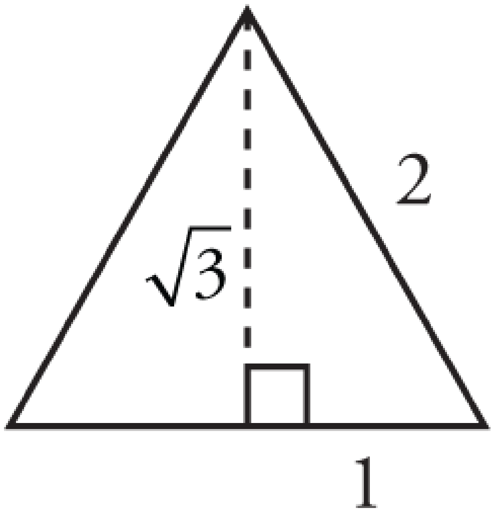
If the area is 25, the side length *x* is 5. Because the diagonal is $x\sqrt{2}$, the diagonal is $\sqrt[3]{64}$.

21. **40**

The long diagonal *BD* is the sum of two long legs of the 30–60–90 triangle, so each long leg is $\sqrt[3]{64}$. The leg:leg:hypotenuse ratio of a 30–60–90 triangle is $x : x\sqrt{3} : 2x$, which means that $5\sqrt{3} = x\sqrt{3}$. Therefore, *x* = 5, so the length of the short leg is 5 and the length of the hypotenuse is 10. Because the perimeter of the figure is the sum of four hypotenuses, the perimeter of this figure is 40.

22. $\sqrt{9}$

The line along which the height is measured in the figure bisects the equilateral triangle, creating two identical 30–60–90 triangles, each with a base of 1. The base of each of these triangles is the short leg of a 30–60–90 triangle. Because the leg:leg:hypotenuse ratio of a 30–60–90 triangle is $1 : 1 : \sqrt{2}$, the long leg of each 30–60–90 triangle, also the height of the equilateral triangle, is $\sqrt{9}$:



23. **26**

The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. In this case, that means that the legs of the right triangle are 10 and 24. Plug these leg lengths into the Pythagorean theorem:

$$
\begin{aligned}
&a^2 + b^2 = c^2 \\
&10^2 + 24^2 = c^2 \\
&c^2 = 100 + 576 = 676 \\
&c = \sqrt{676} = 26
\end{aligned}
$$

You could use the calculator to take this big square root.

Alternatively, you could recognize the 10 : 24 : 26 triangle (a multiple of the more common 5 : 12 : 13 triangle) and save yourself the trouble.

24. $\sqrt{9}$

The perimeter of a rectangle is 2(length + width). In this case, that means 2(x + 2x), which equals 6x. You are told the perimeter equals 6, so 6x = 6, and x is 1. Therefore, the length (2x) is 2 and the width (x) is 1. The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. Plug the leg lengths into the Pythagorean theorem:

$$a^2 + b^2 = c^2$$

$$1^2 + 2^2 = c^2$$

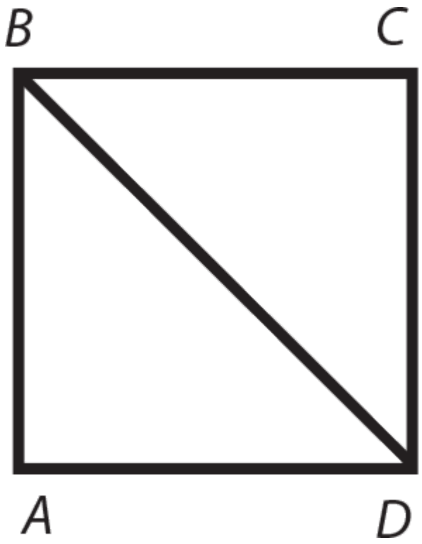
$$c^2 = 1 + 4 = 5$$

$$c = \sqrt{5}$$

Problem Set

(Note: Figures are not drawn to scale.)

1. A square is bisected into two equal triangles (see figure). If the length of *BD* is $16\sqrt{2}$ inches, what is the area of the square?



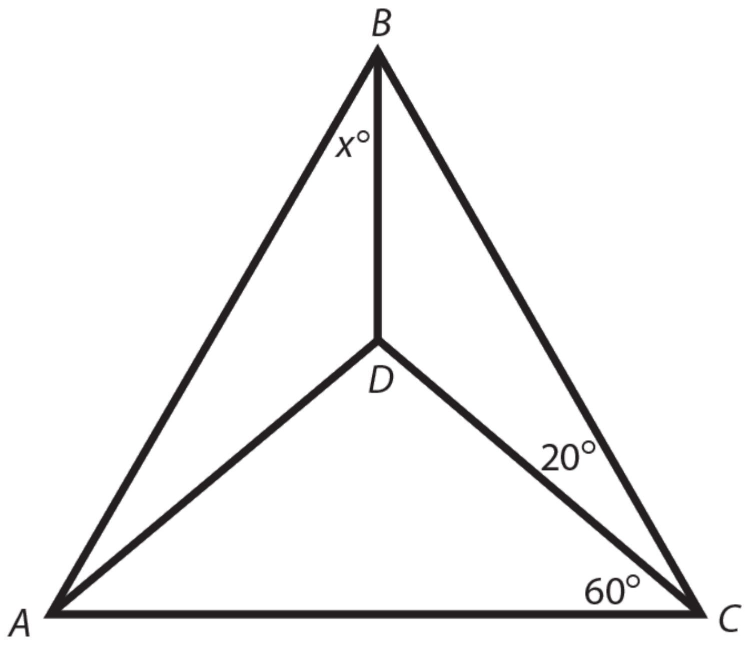
2. Beginning in Town A, Biker Bob rode his bike 10 miles west, 3 miles north, 5 miles east, and then 9 miles north to Town B. How far apart are Town A and Town B? (Assume perfectly flat terrain.)

3. Now in Town B, Biker Bob walked 10 miles due west, and then straight north to Town C. If Town B and Town C are 26 miles apart, how many miles north did he go? (Again, assume perfectly flat terrain.)

4. The longest side of an isosceles right triangle measures $16\sqrt{2}$. What is the area of the triangle?

5. A square field has an area of 400 square meters. Posts are set at all corners of the field. What is the longest distance between any two posts?

6. In triangle *ABC*, *AD* = *BD* = *DC* (see figure). What is *x*?

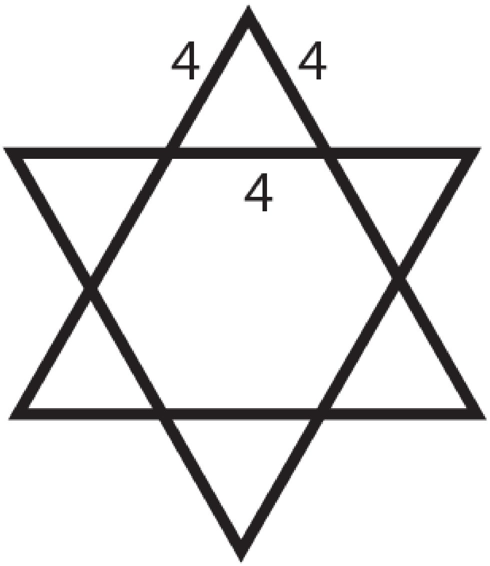


7. Two sides of a triangle are 4 and 10. If the third side is an integer *x*, how many possible values are there for *x*?

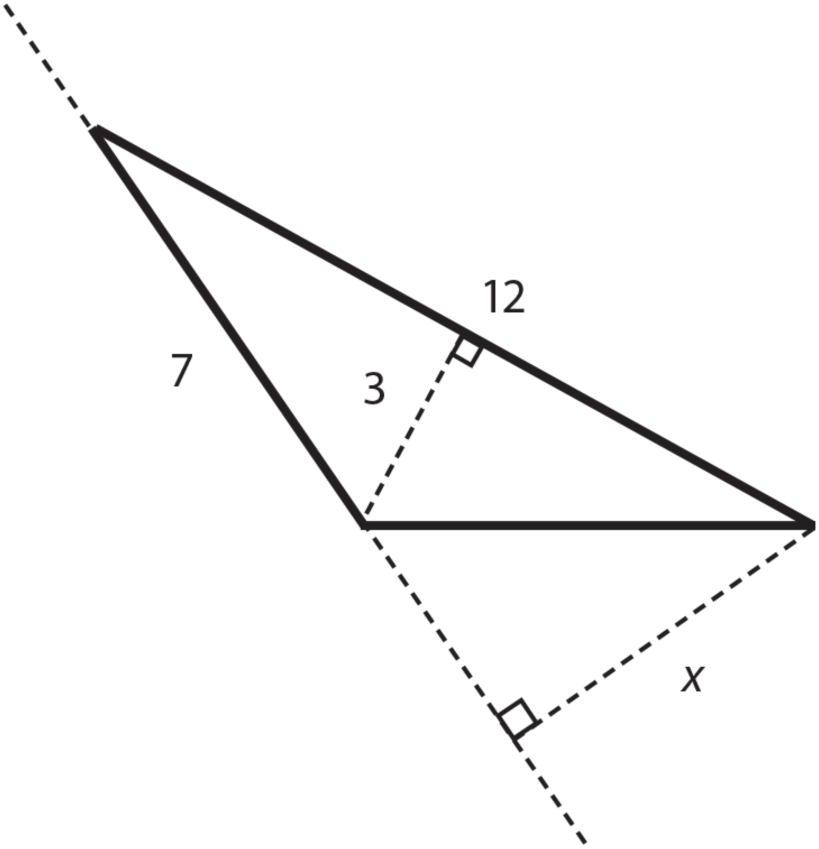
8. Jack has a box in the shape of a cube, the inside edges of which are 4 inches long. What is the longest object he could fit inside the box (i.e., what is the diagonal of the cube)?

9. What is the area of an equilateral triangle whose sides measure 8 cm long?

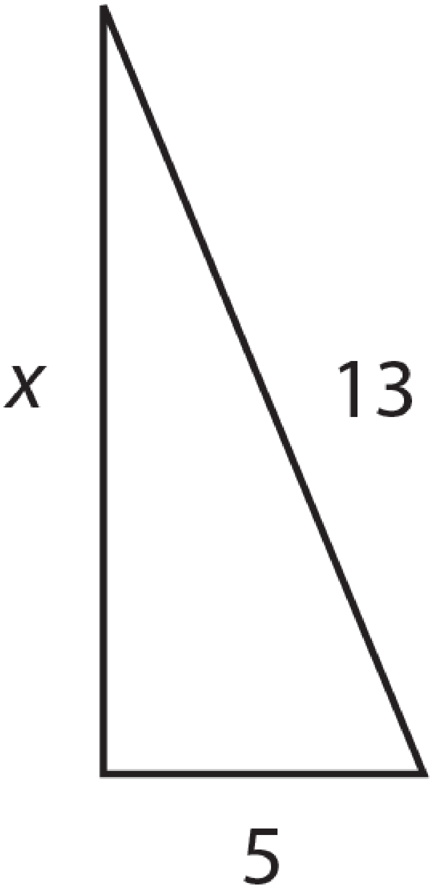
10. The points of a six-pointed star consist of six identical equilateral triangles, with each side 4 cm (see figure). What is the area of the entire star, including the center?



11. What is *x* in the following figure?

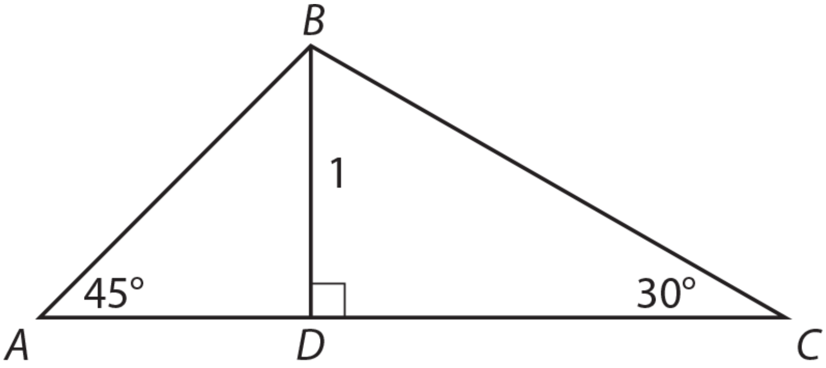

12
7
3
x

12.



Quantity A	**Quantity B**
x	12

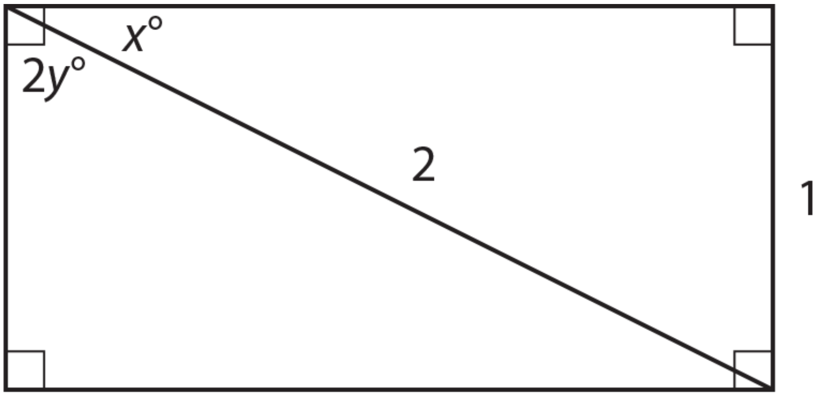
13.



Quantity A | **Quantity B**

The perimeter of triangle ABC | 5

14.



Quantity A | **Quantity B**

x | y

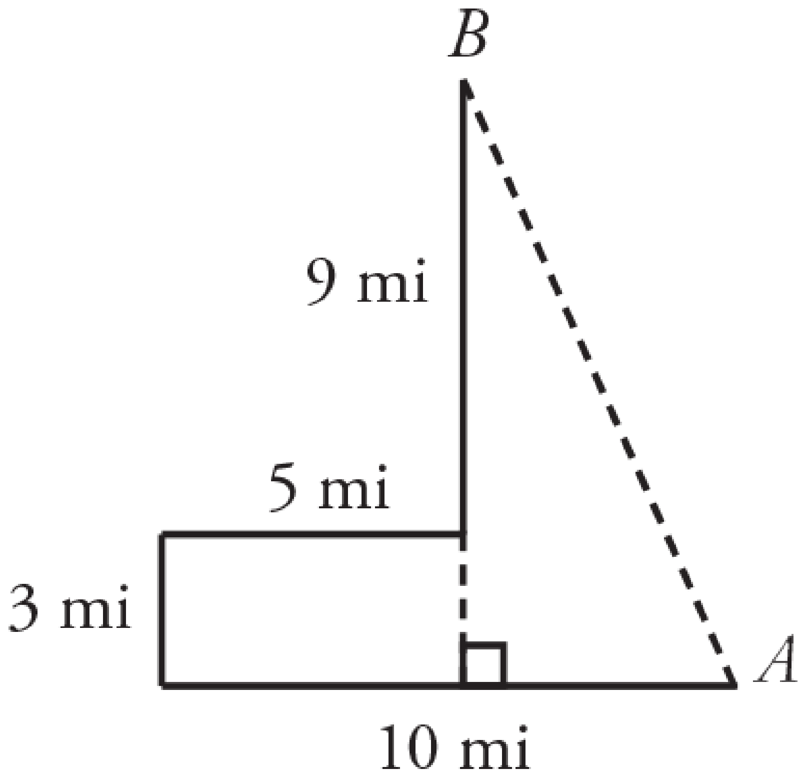
Solutions

1. **256 inches^2**

   The diagonal of a square is $s\sqrt{2}$; therefore, the side length of square *ABCD* is 16. The area of the square is s^2, or 16^2, which is 256.

2. **13 miles**

   If you draw a rough sketch of the path Biker Bob takes, as shown to the right, you can see that the direct distance from *A* to *B* forms the hypotenuse of a right triangle. The short leg (horizontal) is 10 − 5 = 5 miles, and the long leg (vertical) is 9 + 3 = 12 miles. Therefore, you can use the Pythagorean theorem to find the direct distance from *A* to *B*:

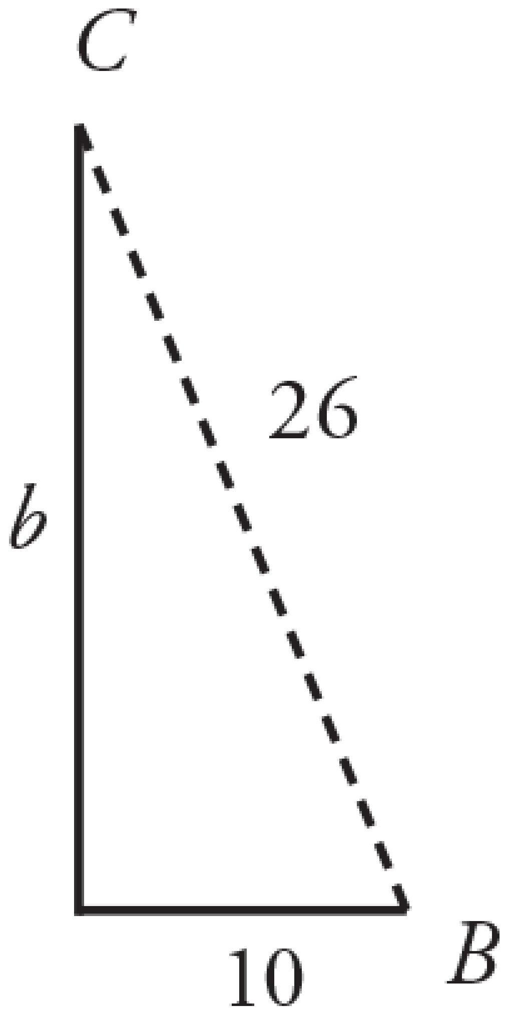


$$\begin{aligned} 5^2 + 12^2 &= c^2 \\ 25 + 144 &= c^2 \\ c^2 &= 169 \\ c &= 13 \end{aligned}$$

You might recognize the common right triangle: 5–12–13.

3. **24 miles**

   If you draw a rough sketch of the path Biker Bob takes, as shown, you can see that the direct distance from *B* to *C* forms the hypotenuse of a right triangle:



$$\begin{aligned}10^2 + b^2 &= 26^2\\100 + b^2 &= 676\\b^2 &= 576\\b &= 24\end{aligned}$$

You might also recognize this as a multiple of the common 5–12–13 triangle.

4. **200**

An isosceles right triangle is a 45–45–90 triangle, with sides in the ratio of $1 : 1 : \sqrt{2}$. If the longest side, the hypotenuse, measures $16\sqrt{2}$, the two other sides each measure 20. Therefore, the area of the triangle is:

$$A = \frac{b \times h}{2} = \frac{20 \times 20}{2} = 200$$

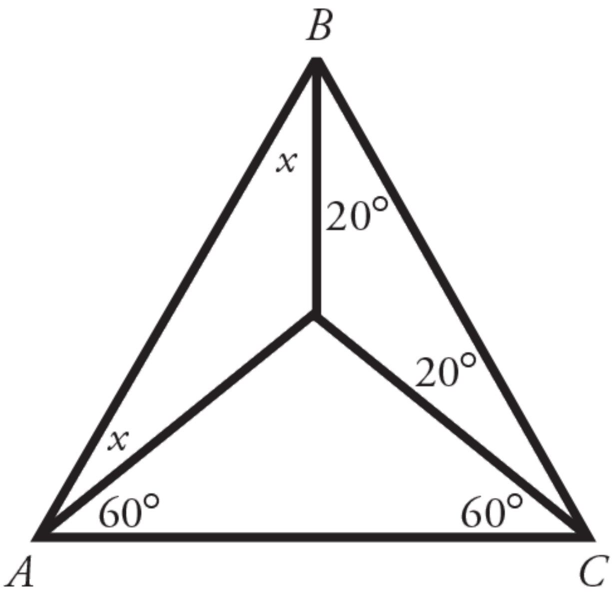
5. $20\sqrt{2}\ \text{meters}$

   The longest distance between any two posts is the diagonal of the field. If the area of the square field is 400 square meters, then each side must measure 20 meters. Diagonal is $\sqrt[3]{64} = 4$, so *d* is $16\sqrt{2}$.

6. **10**

   If *AD* = *BD* = *DC*, then the three triangular regions in this figure are all isosceles triangles. Therefore, you can fill in some of the missing angle measurements as shown here. Because you know that there are 180° in the large triangle *ABC*, you can write the following equation:

$$\begin{aligned} x + x + 20 + 20 + 60 + 60 &= 180 \\ 2x + 160 &= 180 \\ x &= 10 \end{aligned}$$



7. **7**

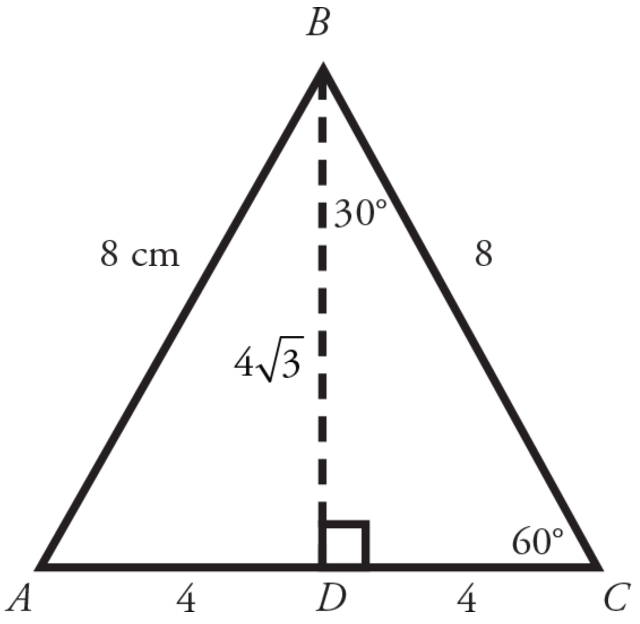
   If two sides of a triangle are 4 and 10, the third side must be greater than 10 – 4 and smaller than 10 + 4. Therefore, the possible values for *x* are {7, 8, 9, 10, 11, 12, and 13}. You can draw a sketch to convince yourself of this result:

8. $\sqrt[3]{64}$ **inches**

   The diagonal of a cube with side *s* is $s\sqrt{2}$. Therefore, the longest object Jack could fit inside the box would be $\sqrt[3]{64}$ inches long.

9. $x = \sqrt{200}$

Draw in the height of the triangle (see figure). If triangle *ABC* is an equilateral triangle, and *ABD* is a right triangle, then *ABD* is a 30–60–90 triangle. Therefore, its sides are in the ratio of $\mathbf{1 : \sqrt{3} : 2}$. If the hypotenuse is 8, then the short leg is 4, and the long leg is $\mathbf{\sqrt[3]{64}}$. This is the height of equilateral triangle *ABC*. Find the area of triangle *ABC* with the formula for area of a triangle:



$$A = \frac{b \times h}{2} = \frac{8 \times 4\sqrt{3}}{2} = 16\sqrt{3}$$

10. $\mathbf{48\sqrt{3}\ cm^2}$
    Think of this star as a large equilateral triangle with sides 12 centimeters long, and three additional smaller equilateral triangles (shaded in the figure below) with sides 4 inches long. Using the same 30–60–90 logic you applied in problem 9, you can see that the height

of the larger equilateral triangle is $\sqrt[3]{64}$, and the height of the smaller equilateral triangle is $\sqrt[3]{64}$.

Therefore, the areas of the triangles are as follows:

Large triangle:

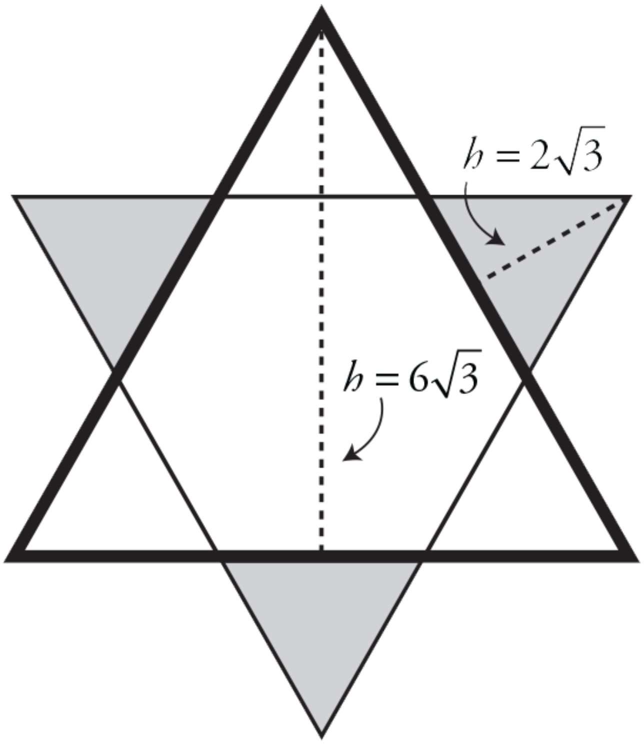
$$A = \frac{b \times h}{2} = \frac{12 \times 6\sqrt{3}}{2} = 36\sqrt{3}$$

Small triangles:

$$A = \frac{b \times h}{2} = \frac{4 \times 2\sqrt{3}}{2} = 4\sqrt{3}$$

The total area of three smaller triangles and one large triangle is:

$$36\sqrt{3} + 3(4\sqrt{3}) = 48\sqrt{3}\,\mathrm{cm}^2$$



11. $\frac{23}{7}$

You can calculate the area of the triangle using the side of length 12 as the base:

$$\frac{1}{2}(12)(3) = 18$$

Next, use the side of length 7 as the base (remember, any side can function as the base, provided that you can find the corresponding height) and write the equation for the area:

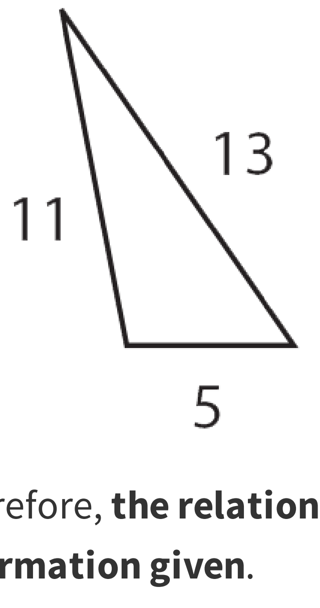
$$\frac{1}{2}(7)(x) = 18$$

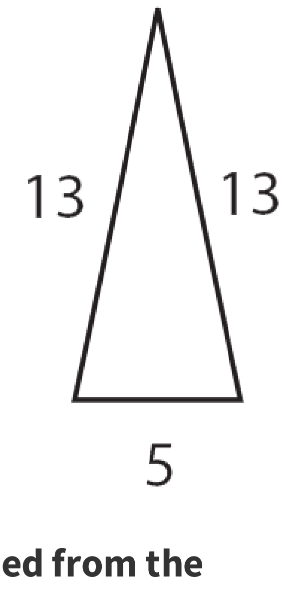
Now solve for *x*, the unknown height:

$$7x = 36$$
$$x = \frac{36}{7}$$

You could also solve this problem using the Pythagorean theorem, but the process is *much* harder.

12. **(D)**
Although this appears to be a 5 : 12 : 13 triangle, you do not know that it is a right triangle. There is no right angle symbol in the diagram. Remember, don’t trust the picture! Here are a couple of possible triangles:

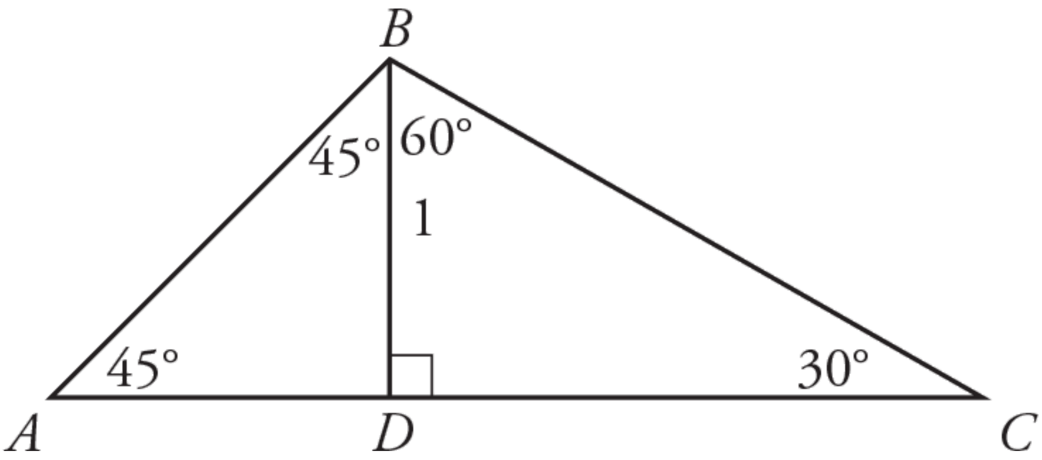




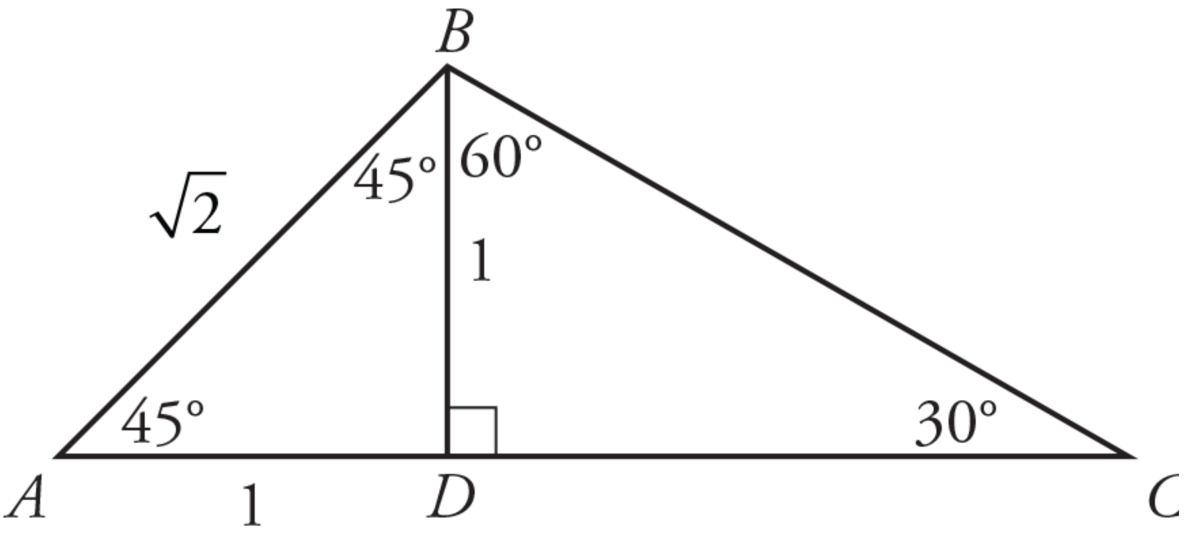
Therefore, **the relationship cannot be determined from the information given**.

13. **(A)**

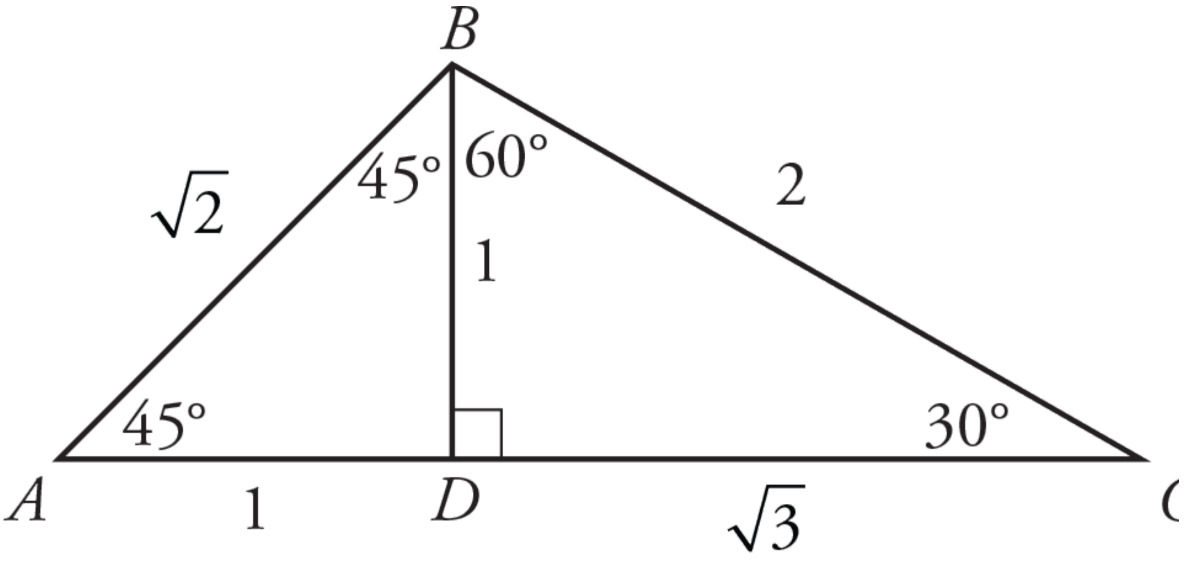
Although there seems to be very little information here, the two small triangles that comprise triangle *ABC* may seem familiar. First, fill in the additional angles in the diagram:



With the additional angles filled in, it is clear that the two smaller triangles are special right triangles: a 45–45–90 triangle and a 30–60–90 triangle. You know the ratios of the side lengths for each of these triangles. For a 45–45–90 triangle, the ratio is $x : x : x\sqrt{2}$. In this diagram, the value of *x* is 1 (side *BD*), so *AD* is 1 and *AB* is $\sqrt{9}$:



For a 30–60–90 triangle, the ratio is $x : x\sqrt{3} : 2x$. In this diagram, *x* is 1 (side *BD*), so *DC* is $\sqrt{9}$ and *BC* is 2:



Now calculate the perimeter of triangle *ABC*:

Quantity A	**Quantity B**

The perimeter of triangle *ABC* $= 1 + 2 + \sqrt{2} + \sqrt{3}$	5

Now you need to compare this sum to 5. A good approximation of $\sqrt{9}$ is 1.4 and a good approximation of $\sqrt{9}$ is 1.7:

Quantity A	**Quantity B**
$1 + 2 + \sqrt{2} + \sqrt{3} \approx$ $1 + 2 + 1.4 + 1.7 = 6.1$	5

In fact, simply knowing that each square root is greater than 1 would let you conclude that, **Quantity A is greater.**

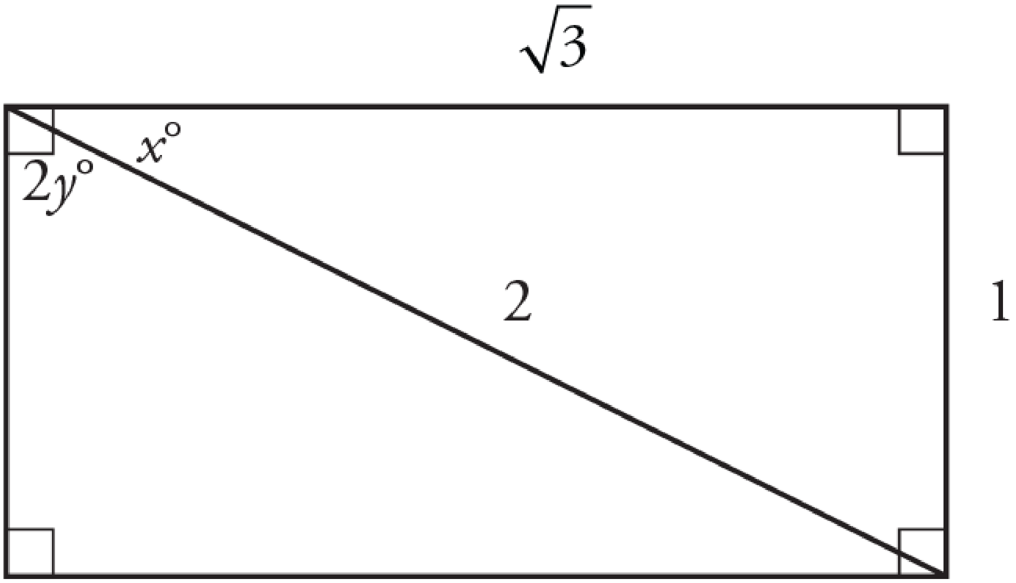
Alternatively, you could use the calculator to compute Quantity A.

14. **(C)**

The diagonal of the rectangle is the hypotenuse of a right triangle whose legs are the length and width of the rectangle. In this case, you are given the width and the diagonal. Plug these into the Pythagorean theorem to determine the length:

$$\begin{aligned} a^2 + b^2 &= b^2 \\ 1^2 + b^2 &= 2^2 \\ 1 + b^2 &= 4 \\ b^2 &= 3 \\ b &= \sqrt{3} \end{aligned}$$

Label this value on the diagram:



The key to this question is recognizing that each of the triangles is a 30–60–90 triangle. Any time you see a right triangle and one of the sides has a length of $\sqrt{9}$ or a multiple of $\sqrt{9}$, you should check to see whether it is a 30–60–90 triangle. Another clue is a right triangle in which the hypotenuse is twice the length of one of the other sides.

Now, in addition to the side lengths, you can fill in the values of the angles in this diagram. Angle x is opposite the short leg, which means it has a degree measure of 30. Similarly, $2y$ is opposite the long leg, which means it has a degree measure of 60:

$$2y = 60$$
$$y = 30$$

Quantity A	**Quantity B**

$x = \mathbf{30}$ $y = \mathbf{30}$

Therefore, **the two quantities are equal**.

Chapter 11

POLYGONS

In This Chapter...

Quadrilaterals: An Overview
Polygons and Interior Angles
Polygons and Perimeter
Polygons and Area
3 Dimensions: Surface Area
3 Dimensions: Volume
Quadrilaterals
Maximum Area of Polygons

Chapter 11

Polygons

Polygons are a very familiar sight on the GRE. As you saw in the last chapter, many questions about triangles will often involve other polygons, most notably quadrilaterals. Mastery of polygons will ultimately involve understanding the basic properties, such as perimeter and area, and will also involve the ability to distinguish certain polygons from other polygons or circles within the context of a larger diagram.

A polygon is defined as a closed shape formed entirely by line segments. The polygons tested on the GRE include the following:

- Three-sided shapes (triangles)
- Four-sided shapes (quadrilaterals)
- Other polygons with *n* sides (where *n* is five or more)

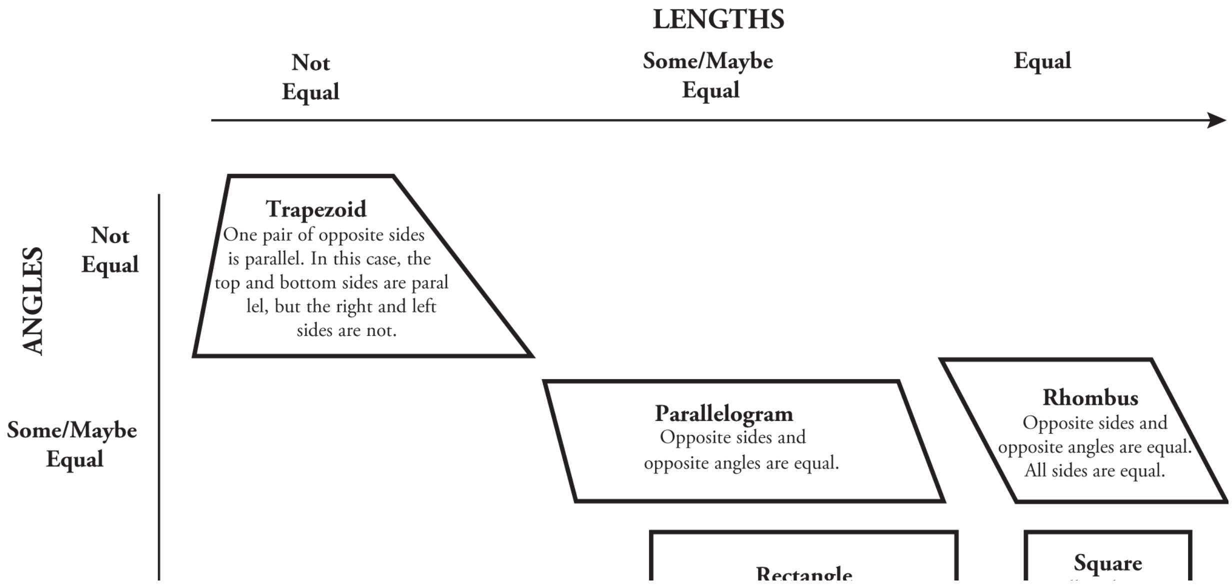
This section will focus on polygons of four or more sides. In particular, the GRE emphasizes quadrilaterals—four-sided polygons—especially squares, rectangles, parallelograms, and trapezoids.

Polygons are two-dimensional shapes—they lie in a plane. The GRE tests your ability to work with different measurements associated with polygons. The measurements you must be adept with are 1) interior angles, 2) perimeter, and 3) area.

The GRE also tests your knowledge of three-dimensional shapes formed from polygons, particularly rectangular solids and cubes. The measurements you must be adept with are 1) surface area, and 2) volume.

Quadrilaterals: An Overview

The most common polygon tested on the GRE, aside from the triangle, is the quadrilateral (any four-sided polygon). Almost all GRE polygon problems involve the special types of quadrilaterals shown below:



Trapezoid One pair of opposite sides is parallel. In this case, the top and bottom sides are parallel, but the right and left sides are not.

Parallelogram Opposite sides and opposite angles are equal.

Rhombus Opposite sides and opposite angles are equal. All sides are equal.


Equal
All angles are 90°, and opposite sides are equal.
All angles are 90°. All sides are equal.

Polygons and Interior Angles

The sum of the interior angles of a given polygon depends on the **number of sides in the polygon**. The following chart displays the relationship between the type of polygon and the sum of its interior angles:

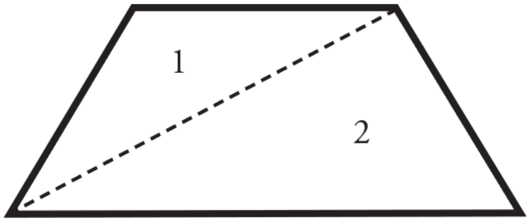
Polygon	**# of Sides**	**Sum of Interior Angles**
Triangle	3	180°
Quadrilateral	4	360°
Pentagon	5	540°
Hexagon	6	720°

The sum of the interior angles of a polygon follows a specific pattern that depends on n, the number of sides that the polygon has. This sum is always $(n - 2) \times 180$, because the polygon can be cut into $(n - 2)$ triangles, each of which contains 180°.

$(n - 2) \times 180$ = the SUM of interior angles of a polygon

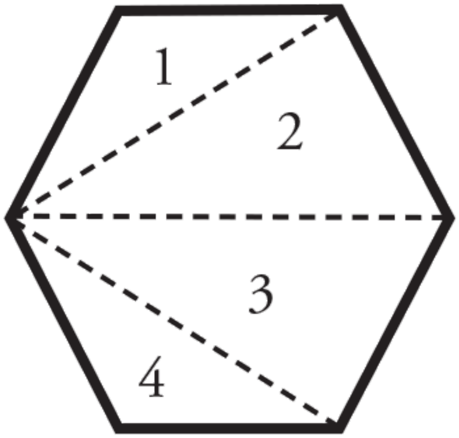
If you forget this formula, you can always say, “Okay, a triangle has 180°, a rectangle has 360°,” and so on. Add 180° for each additional side.

Take a look at the following picture. Because this polygon has four sides, the sum of its interior angles is (4 − 2)180 = 2(180) = 360°. Alternatively, note that a quadrilateral can be cut into two triangles by a line connecting opposite corners. Thus, the sum of the angles is 2(180), which equals 360°.



If a polygon has six sides, as in the figure to the right, the sum of its interior angles is (6 − 2)180 = 4(180) = 720°.

Alternatively, note that a hexagon can be cut into four triangles by three lines connecting corners. Thus, the sum of the angles is 4(180), which is 720°.



By the way, the corners of polygons are also known as vertices (singular: vertex).

Check Your Skills

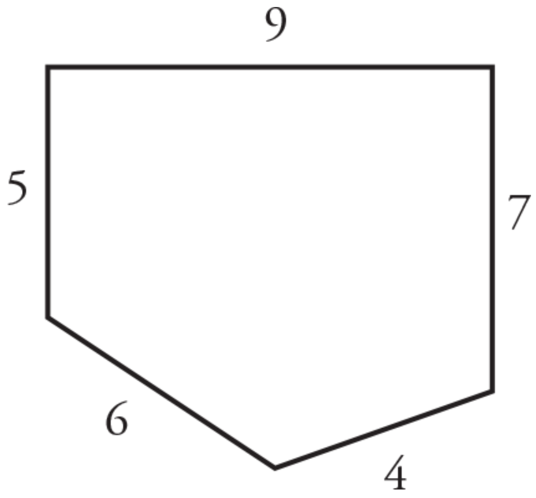
1. What is the sum of the interior angles of an octagon (eight-sided polygon)?

2. A regular polygon is a polygon in which every side is of equal length and all interior angles are equal. What is the degree measure of each interior angle in a regular hexagon (six-sided polygon)?

Polygons and Perimeter

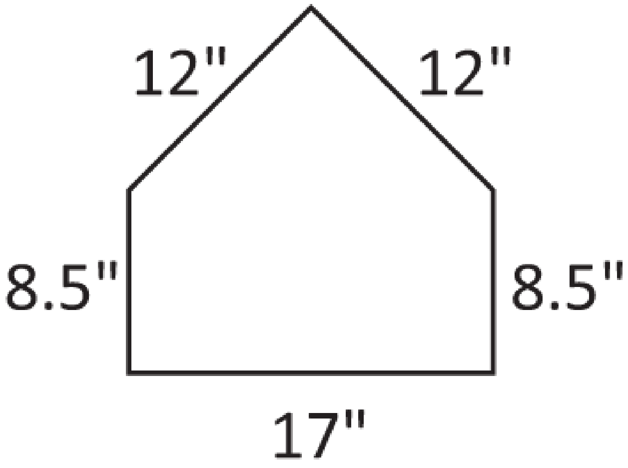
The perimeter refers to the distance around a polygon, or the sum of the lengths of all the sides. The amount of fencing needed to surround a yard would be equivalent to the perimeter of that yard (the sum of all the sides).

The perimeter of the pentagon to the right is:



9 + 7 + 4 + 6 + 5 = **31**

Check Your Skills



3.

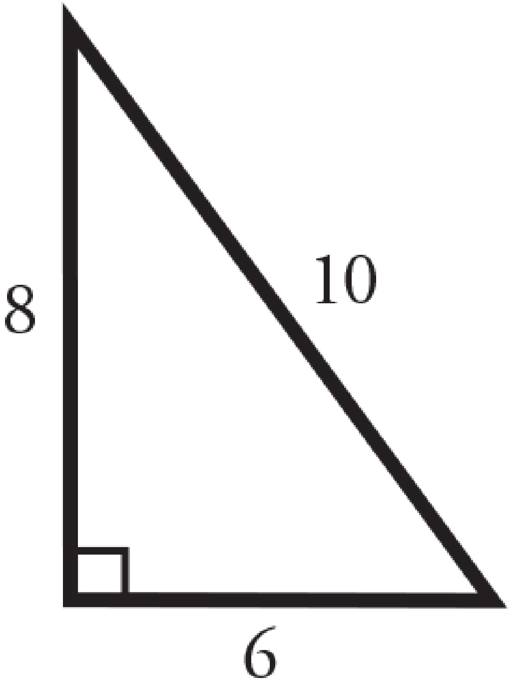
The figure shown represents a standard baseball home plate. What is the perimeter of this figure?

Polygons and Area

The area of a polygon refers to the space inside the polygon. For example, the amount of space that a garden occupies is the area of that garden. Area is measured in square units, such as cm^2 (square centimeters), m^2 (square meters), or ft^2 (square feet).

On the GRE, there are two polygon area formulas you **must** know:

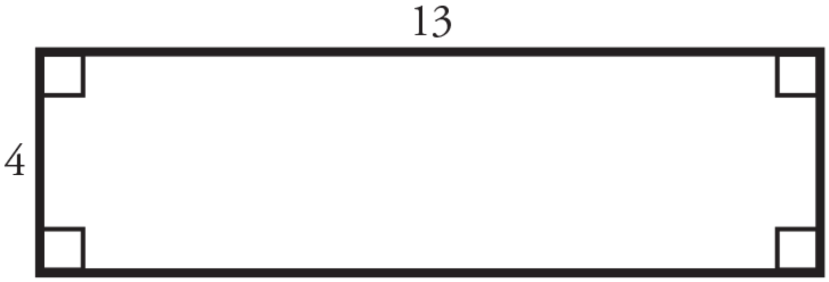
1. $\textbf{Area of a Triangle} = \dfrac{\textbf{Base} \times \textbf{Height}}{\textbf{2}}$



The height *always* refers to a line that is perpendicular (at a 90° angle) to the base.

In this triangle, the base is 6 and the height (perpendicular to the base) is 8. Thus, the area is (6 × 8) ÷ 2 = 48 ÷ 2 = 24.

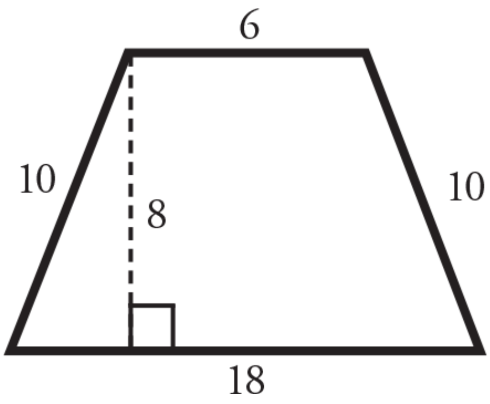
2. Area of a Rectangle = **Length** × **Width**



The length of this rectangle is 13, and the width is 4. Therefore, the area is 13 × 4 = 52.

The GRE will occasionally ask you to find the area of a polygon more complex than a simple triangle or rectangle. The following formulas can be used to find the areas of other types of quadrilaterals:

3. $\text{Area of a Trapezoid} = \frac{(\text{Base}_1 + \text{Base}_2)}{2} \times \textbf{Height}$

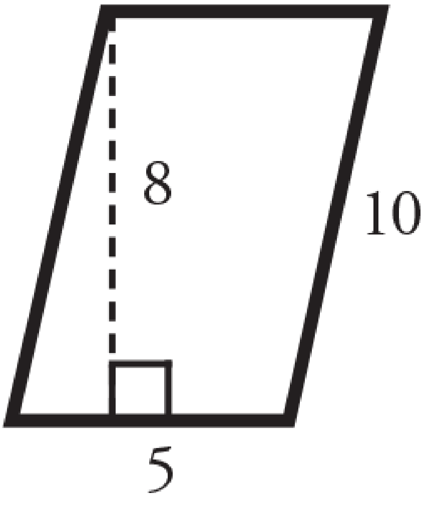


Note that the height refers to a line perpendicular to the two bases, which are parallel. (You often have to draw in the height, as in this case.)

In the trapezoid shown, $\text{base}_1 = 18$, $\text{base}_2 = 6$, and the height = 8.

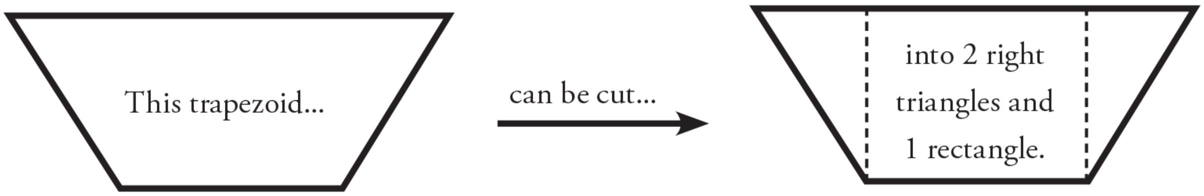
Thus, the area is $\frac{(18+6)}{2} \times 8 = 96$. Another way to think about this is to take the *average* of the two bases and multiply it by the height.

4. Area of any Parallelogram = **Base** × **Height**



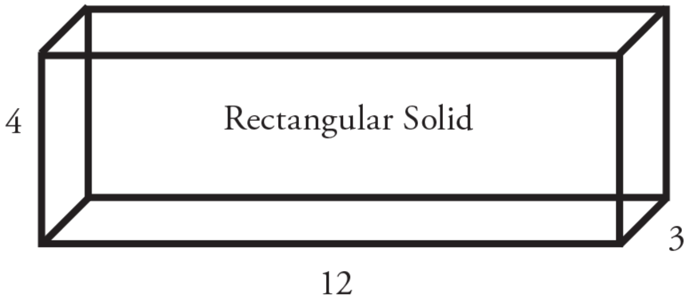
Note that the height refers to the line perpendicular to the base. (As with the trapezoid, you often have to draw in the height.) In the parallelogram shown, the base = 5 and the height = 8. Therefore, the area is 5 × 8, which is 40.

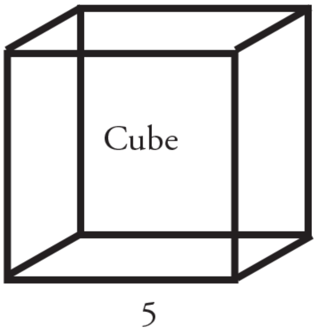
Strange polygons can generally be divided into some combination of rectangles and right triangles. If you need the area of a strange polygon, or if you forget the trapezoid formula, simply cut the shape into rectangles and right triangles, and then find the areas of these individual pieces. For example:



3 Dimensions: Surface Area

The GRE tests two particular three-dimensional shapes formed from polygons: the rectangular solid and the cube. Note that a cube is just a special type of rectangular solid:





The surface area of a three-dimensional shape is the amount of space on the surface of that particular object. For example, the amount of paint that it would take to fully cover a rectangular box could be determined by finding the surface area of that box. As with simple area, surface area is measured in square units such as in^2 (square inches) or ft^2 (square feet).

Surface Area = **the SUM of the areas of ALL of the faces**

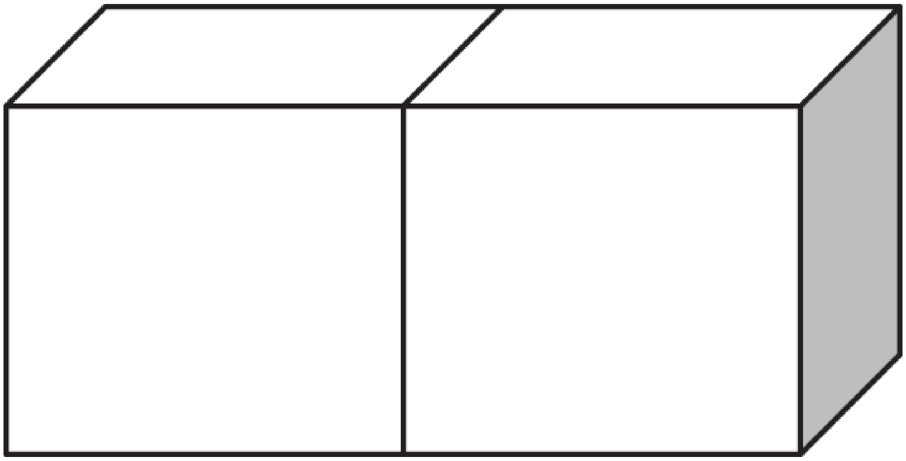
Both a rectangular solid and a cube have **six faces**.

To determine the surface area of a rectangular solid, you must find the area of each face. Notice, however, that in a rectangular solid, the front and back faces have the same area, the top and bottom faces have the same area, and the two side faces have the same area. In the rectangular solid, the area of the front face is equal to 12 × 4 = 48. Thus, the back face also has an area of 48. The area of the bottom face is equal to 12 × 3 = 36. Thus, the top face also has an area of 36. Finally, each side face has an area of 3 × 4 = 12. Therefore, the surface area, or the sum of the areas of all six faces, is: 48(2) + 36(2) + 12(2) = 192.

To determine the surface area of a cube, you only need the length of one side. You can see from the preceding cube that a cube is made of six identical square surfaces. First, find the area of one face: 5 × 5 = 25. Then, multiply by 6 to account for all of the faces: 25 × 6 = 150.

Check Your Skills

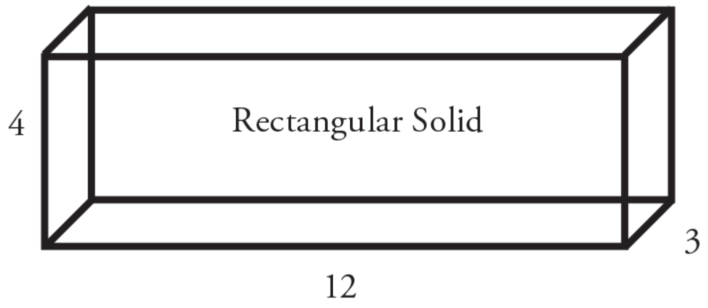
4.

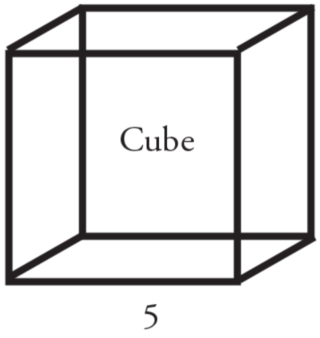


The figure here shows two wooden cubes joined to form a rectangular solid. If each cube has a surface area of 24, what is the surface area of the resulting rectangular solid?

3 Dimensions: Volume

The volume of a three-dimensional shape is the amount of “stuff” it can hold. For example, the amount of liquid that a rectangular milk carton holds can be determined by finding the volume of the carton. Volume is measured in cubic units such as in^3 (cubic inches) or ft^3 (cubic feet).





Volume = **Length** × **Width** × **Height**

By looking at the rectangular solid above, you can see that the length is 12, the width is 3, and the height is 4. Therefore, the volume is 12 × 3 × 4, which is 144.

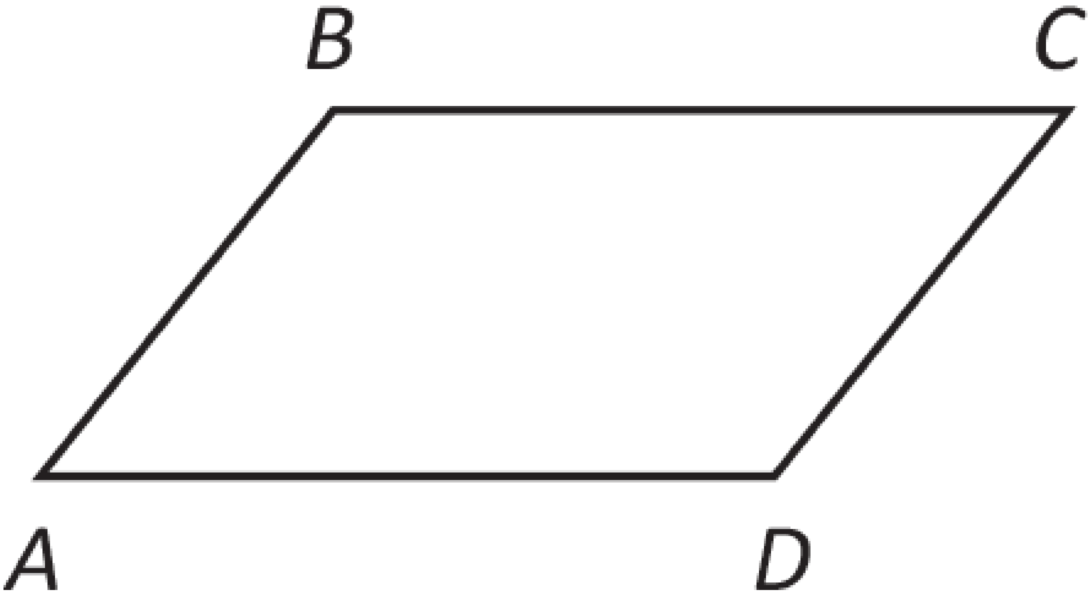
In a cube, all three of the dimensions—length, width, and height—are identical. Therefore, knowing the measurement of just one side of the cube is sufficient to find the volume. In the cube shown, the volume is 5 × 5 × 5, which equals 125.

Check Your Skills

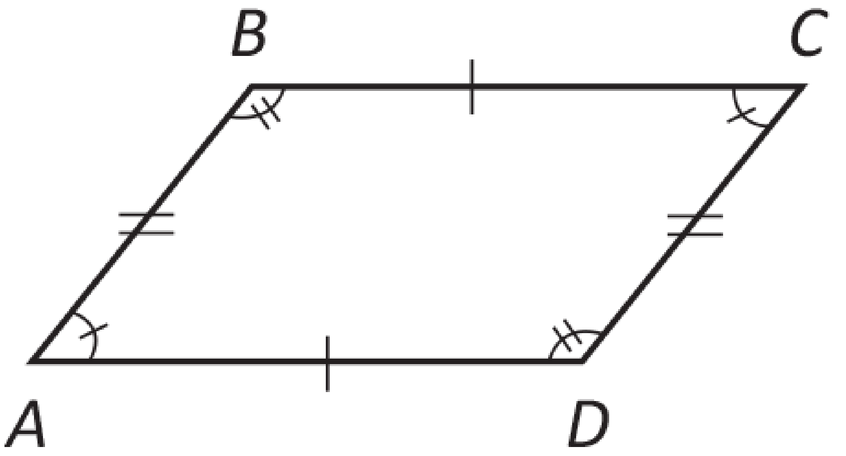
5. The volume of a rectangular solid with length 8, width 6, and height 4 is how many times the volume of a rectangular solid with length 4, width 3, and height 2?

Quadrilaterals

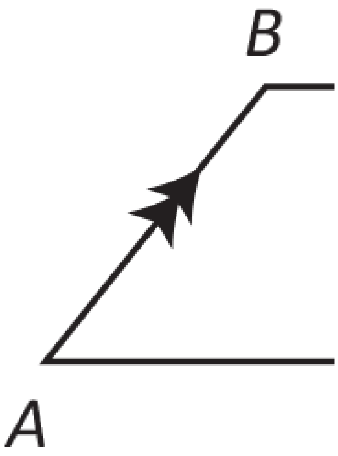
A quadrilateral is any figure with four sides. The GRE largely deals with one class of quadrilaterals known as **parallelograms**. A parallelogram is any four-sided figure in which the opposite sides are parallel and equal and in which opposite angles are equal. This is an example of a parallelogram:



In this figure, sides *AB* and *CD* are parallel and have equal lengths, sides *AD* and *BC* are parallel and have equal lengths, angles *ADC* and *ABC* are equal, and angles *BAD* and *BCD* are equal:



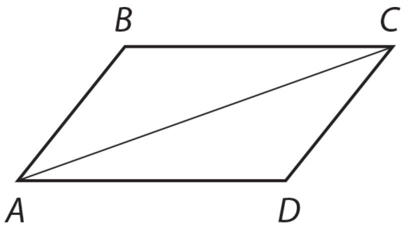
Use hash marks to indicate equal lengths or equal angles.



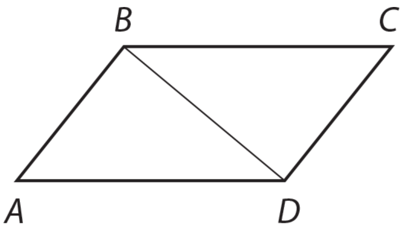
Use arrows to indicate para

Any quadrilateral with two sets of opposite and equal sides is a parallelogram, as is any quadrilateral with two sets of opposite and equal angles.

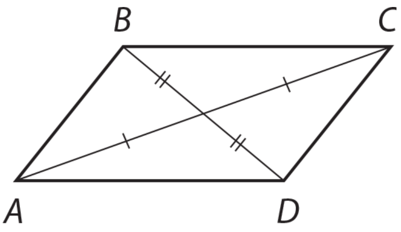
An additional property of any parallelogram is that the diagonal will divide the parallelogram into two identical triangles:



Triangle *ABC* = Triangle *ACD*

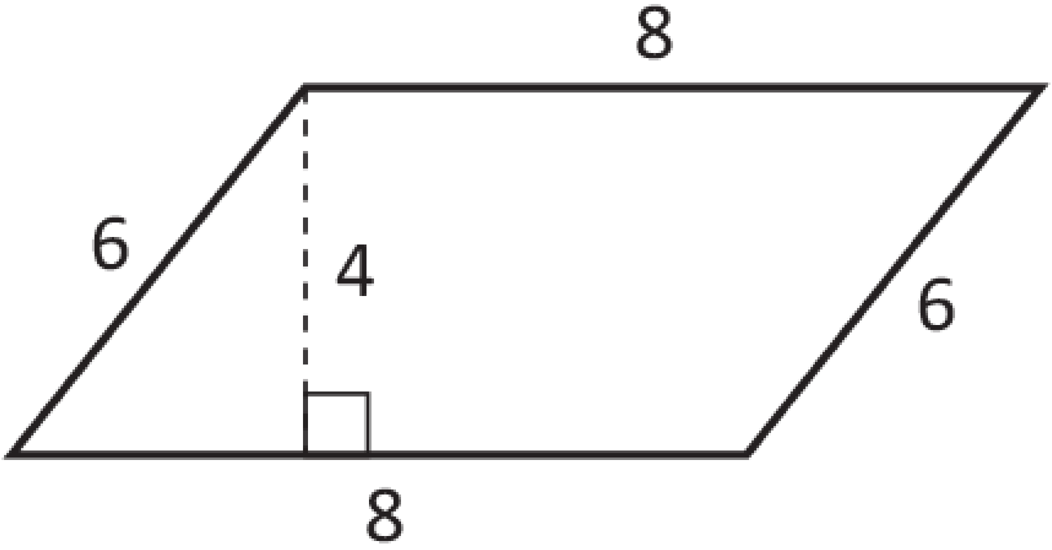


Triangle *ABD* = Triangle *BCD*



The diagonals also cut each other in half (bisect each other)

For any parallelogram, the perimeter is the sum of the lengths of all the sides and the area is equal to (base) × (height). With parallelograms, as with triangles, it is important to remember that the base and the height *must* be perpendicular to one another.

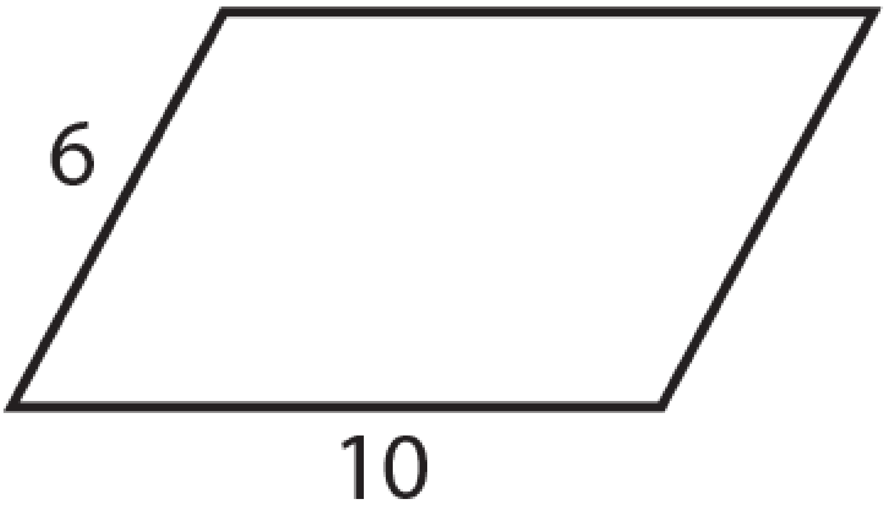


In this parallelogram, what is the perimeter, and what is the area? The perimeter is the sum of the sides, so it is equal to 6 + 8 + 6 + 8, which is 28. Alternatively, you can use one of the properties of parallelograms to calculate the perimeter in a different way. You know that parallelograms have two sets of equal sides. In this parallelogram, two of the sides have a length of 6, and two of the sides have a length of 8. So the perimeter equals 2 × 6 + 2 × 8. You can factor out a 2, and say that perimeter is 2 × (6 + 8), which equals 28.

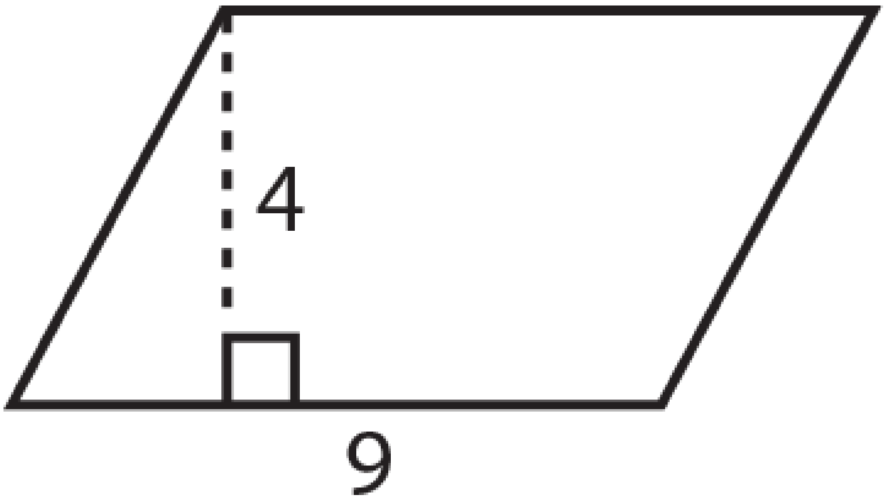
To calculate the area, you need a base and a height. It might be tempting to say that the area is 8 × 6 = 48. But the two sides of this parallelogram are not perpendicular to each other. The dotted line drawn into the figure, however, is perpendicular to the base. The area of the parallelogram is 8 × 4 = 32.

Check Your Skills

6. What is the perimeter of the parallelogram?

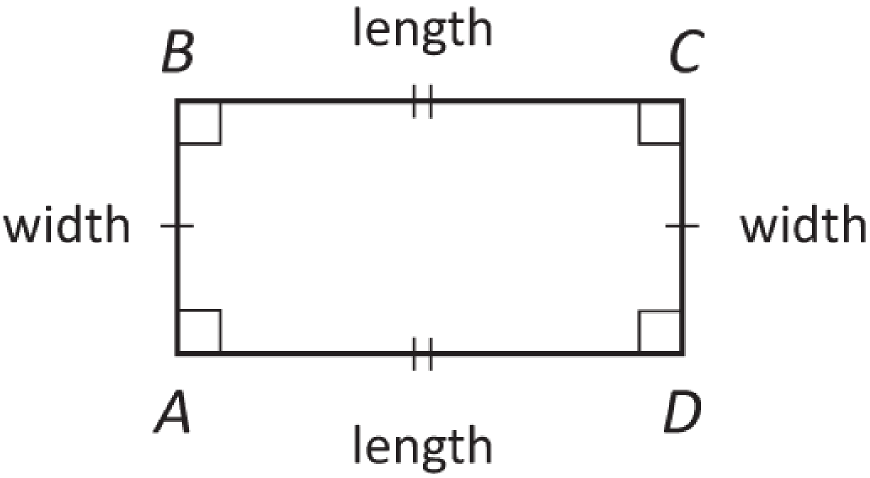


7. What is the area of the parallelogram?



RECTANGLES

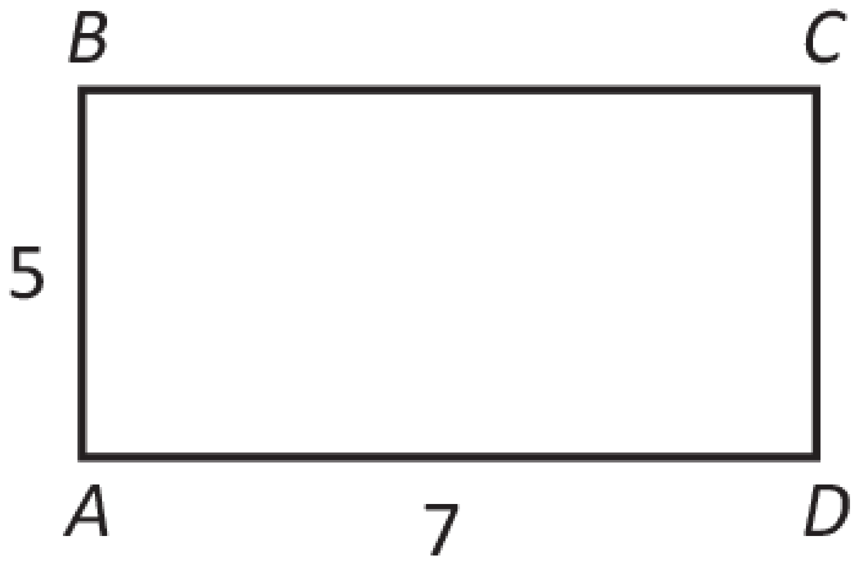
Rectangles are a specific type of parallelogram. Rectangles have all the same properties as parallelograms, with one additional property—all four internal angles of a rectangle are right angles. Additionally, with rectangles, one pair of sides is referred to as the length and one pair of sides as the width.



The formula for the perimeter of a rectangle is the same as for the perimeter of a parallelogram—either sum the lengths of the four sides or add the length and the width then multiply by 2.

The formula for the area of a rectangle is also the same as for the area of a parallelogram, but for any rectangle, the length and width are by definition perpendicular to each other, so you don't need a separate height. For this reason, the area of a rectangle is commonly expressed as (length) × (width).

Here's practice. For the following rectangle, find the perimeter and the area:

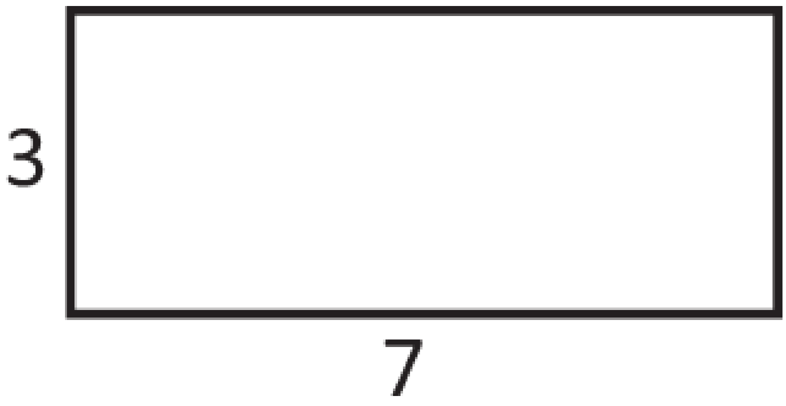


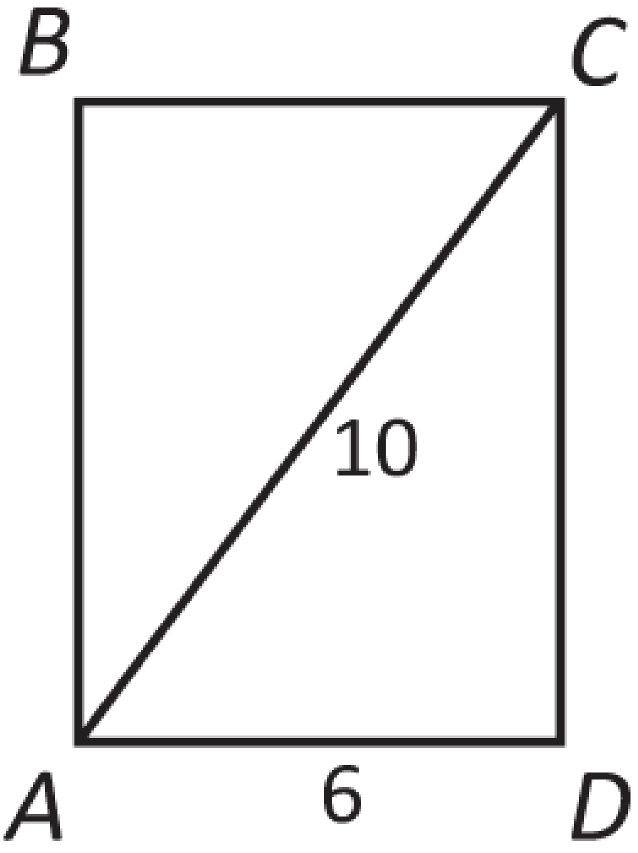
Start with the perimeter. Again, recognize that you have two sides with a length of 5 and two sides with a length of 7. Therefore, the perimeter is 2 × (5 + 7), which equals 24. Or, just add the sides up; 5 + 5 + 7 + 7 also equals 24.

Now to find the area. The formula for area is (length) × (width). For the purposes of finding the area, it is irrelevant which side is the length and which side is the width. If you make side *AB* the length and side *AD* the width, then the area = (5) × (7) = 35. If, instead, you make side *AD* the length and side *AB* the width, then you have area = (7) × (5) = 35. The only thing that matters is that you choose two sides that are perpendicular to each other.

Check Your Skills

Find the area and perimeter of each rectangle.

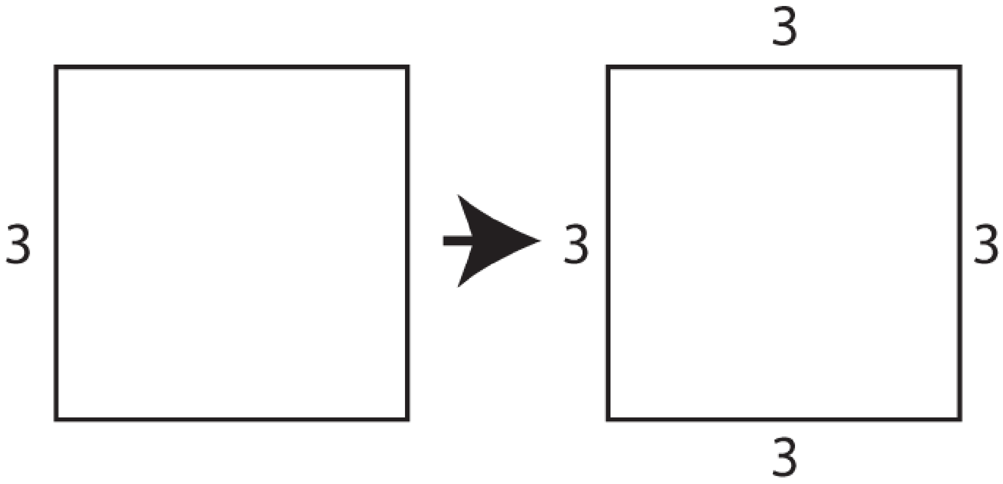
8. 

9. 

SQUARES

One particular type of rectangle warrants mention—the square. Everything that is true of rectangles is true of squares as well. However, a square is a rectangle in which the lengths of all four sides are equal. Thus, knowing only one side of a square is enough to determine the perimeter and area of a square.

For instance, if you have a square, and you know that the length of one of its sides is 3, you know that all four sides have a length of 3:



The perimeter of the square is 3 + 3 + 3 + 3, which equals 12. Alternatively, once you know the length of one side of a square, you can multiply that length by 4 to find the perimeter: 3 × 4 = 12.

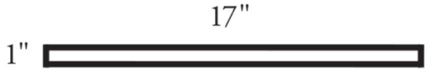
To find the area, use the same formula as for a rectangle: Area = (length) × (width). But, because the shape is a square, you know that the length and the width are equal. Therefore, you can say that the area of a square is Area = $(\text{side})^2$. In this case, Area = $(3)^2 = 9$.

Maximum Area of Polygons

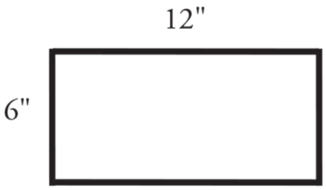
In some problems, the GRE may require you to determine the maximum or minimum area of a given figure. Following a few simple shortcuts can help you solve certain problems quickly.

MAXIMUM AREA OF A QUADRILATERAL

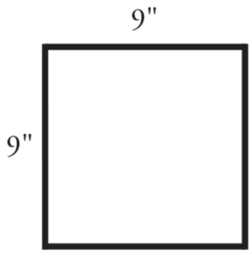
Perhaps the best-known maximum area problem is one that asks you to maximize the area of a *quadrilateral* (usually a rectangle) with a *fixed perimeter*. If a quadrilateral has a fixed perimeter, say, 36 inches, it can take a variety of shapes:



Area = 17 square inches



Area = 72 square inches



Area = 81 square inches

Of these figures, the one with the largest area is the square. This is a general rule: **Of all quadrilaterals with a given perimeter, the *square* has the largest area.** This is true even in cases involving non-integer lengths. For instance, of all quadrilaterals with a perimeter of 25 feet, the one with the largest area is a square with 25 ÷ 4 = 6.25 feet per side.

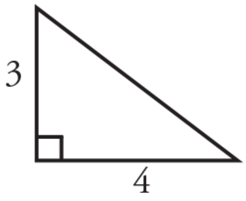
This principle can also be turned around to yield the following corollary: **Of all quadrilaterals with a given area, the *square* has the minimum perimeter.**

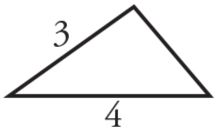
Both of these principles can be generalized for polygons with *n* sides: **A regular polygon with all sides equal and all angles equal will maximize area for a given perimeter and minimize perimeter for a given area.**

MAXIMUM AREA OF A PARALLELOGRAM OR TRIANGLE

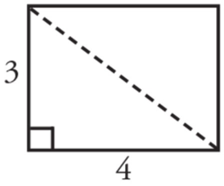
Another common optimization problem involves maximizing the area of a *triangle or parallelogram with given side lengths*.

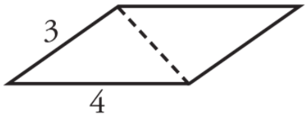
For instance, there are many triangles with two sides 3 and 4 units long. Imagine that the two sides of lengths 3 and 4 are on a hinge. The third side can have various lengths:





There are many corresponding parallelograms with two sides 3 and 4 units long:





The area of a triangle is given by $\frac{6x - 15y}{10}$, and the area of a parallelogram is given by *A* = *bh.* Because both of these formulas involve the perpendicular height *h*, the maximum area of each figure is achieved when the 3-unit side is perpendicular to the 4-unit side, so that the height is 3 units. All the other figures have lesser heights. (Note that in this case, the triangle of maximum area is the famous 3–4–5 right triangle.) If the sides are not perpendicular, then the figure is squished, so to speak.

The general rule is this: **If you are given two sides of a triangle or parallelogram, you can maximize the area by placing those two sides PERPENDICULAR to each other.**

Check Your Skills Answer Key

1. **1,080°**

   One way to calculate the sum of the interior angles of a polygon is by applying the formula $(n - 2)180$ = Sum of the interior angles, where n is the number of sides. Substituting 8 for n yields:

$$\begin{aligned}\text{Sum of the interior angles} &= (8-2)180\\ &= (6)180\\ &= 1{,}080\end{aligned}$$

2. **120°**

   Each interior angle is the same, therefore, you can determine the angle of any one by dividing the sum of the interior angles by 6 (the number of interior angles). Use the formula $(n - 2)180$ = Sum of the interior angles, where n is the number of sides. Substituting 6 for n yields: $(4)180 = 720$. Divide 720 by 6 to get 120.

3. **58"**

   It is simplest to sum the sides in this order: 12 + 12 + 17 + (8½ + 8½) = 12 + 12 + 17 + 17 = 58.

4. **40**

   The surface area of a cube is 6 times the area of one face, therefore, each square face of each cube must have an area of 4. One face of each cube is lost when the two cubes are joined, so the total surface area of the figure will be the sum of the surface areas of both cubes minus the surface areas of the covered faces.

Each cube has surface area of 24, so the total surface area is 48. Subtract the surface area of each of the two touching (and thus non-exterior) faces: 48 − 2(4) = 40.

Alternatively, you could find that, because the surface area of one side of each cube is 4, the side length of each cube is 2. Thus, the length of the overall rectangular solid is 4, while its width is 2 and its height is 2. The surface area will now be equal to the sum of all six faces: 2(2 × 4) + 2(2 × 4) + 2(2 × 2) = 40.

5. **8**

   The volume of a rectangular solid is the product of its three dimensions: length, width, and height:

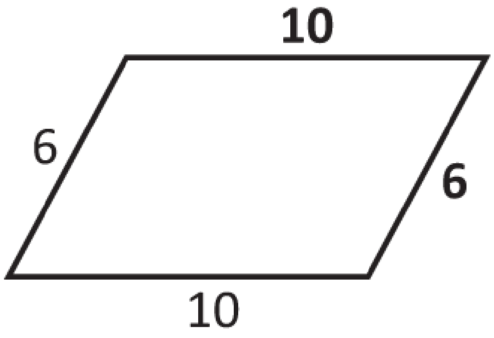
   $8 \times 6 \times 4 = 192$ and $4 \times 3 \times 2 = 24$

   This yields $\frac{192}{24} = 8$, so the volume of the larger solid is 8 times the volume of the smaller solid.

   Alternatively, note that each dimension of the larger solid is 2 times the corresponding dimension of the smaller solid. The volume will be 2 times × 2 times × 2 times = 8 times greater.

6. **32**

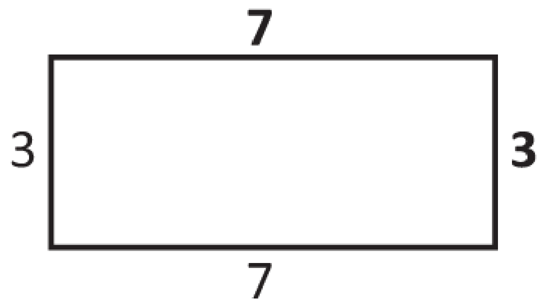
In parallelograms, opposite sides have equal lengths, so you know that two of the sides of the parallelogram have a length of 6 and two of the sides have a length of 10.



Thus, the perimeter is 6 + 10 + 6 + 10, which equals 32. Alternatively, 2(6 + 10) = 32.

7. **36**
   Area of a parallelogram is *b* × *h*. In this parallelogram, the base is 9 and the height is 4, so the area is 9 × 4, which equals 36. The area of the parallelogram is 36.

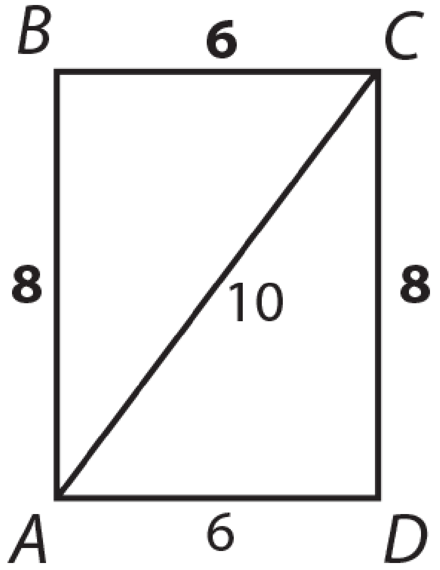
8. **Area = 21, Perimeter = 20**
   In rectangles, opposite sides have equal lengths, so your rectangle looks like this:



The perimeter is 3 + 7 + 3 + 7, which equals 20. The area of a rectangle is $l \times w$, so the area is 7 × 3, which equals 21. The area is 21, and the perimeter is 20.

9. **Area = 48, Perimeter = 28**

   To find the area and perimeter of the rectangle, you need to know the length of either side *AB* or side *CD*. The diagonal of the rectangle creates a right triangle, so you can use the Pythagorean theorem to find the length of side *CD*. Alternatively, you can recognize that triangle *ACD* is a 6–8–10 triangle, and thus the length of side *CD* is 8. Either way, your rectangle now looks like this:



Thus, the perimeter of the rectangle is 6 + 8 + 6 + 8, which equals 28, and the area is 6 × 8, which equals 48.

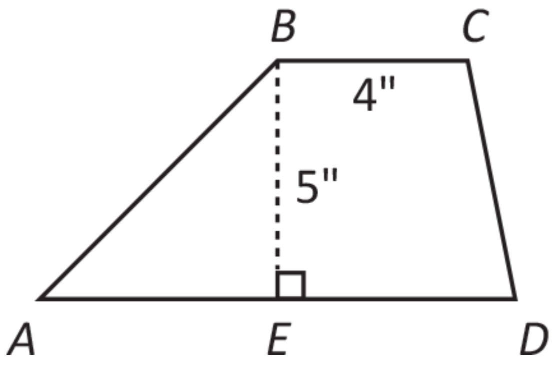
Problem Set

(Note: Figures are not drawn to scale.)

1. Frank the Fencemaker needs to fence in a rectangular yard. He fences in the entire yard, except for one 40-foot side of the yard. The yard has an area of 280 square feet. How many feet of fence does Frank use?

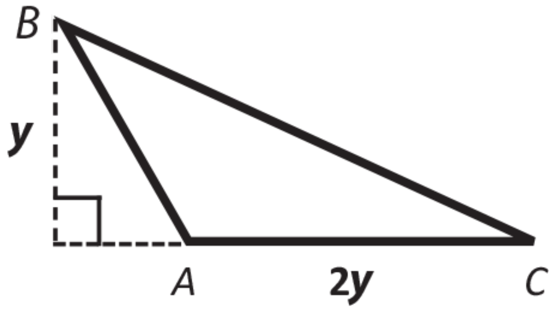
2. A pentagon has three sides with length x, and two sides with the length $3x$. If x is $\frac{1}{2}$ of an inch, what is the perimeter of the pentagon?

3. $ABCD$ is a quadrilateral, with AD parallel to BC (see figure). E is a point between A and D such that BE represents the height of $ABCD$ and E is the midpoint of AD. If the area of triangle ABE is 12.5 square inches, what is the area of $ABCD$?



4. A rectangular tank needs to be coated with insulation. The tank has dimensions of 4 feet, 5 feet, and 2.5 feet. Each square foot of insulation costs $20. How much will it cost to cover the surface of the tank with insulation?

5. Triangle *ABC* (see figure) has a base of $2y$, a height of y, and an area of 49. What is y?



6. Forty percent of Andrea's living room floor is covered by a carpet that is 4 feet by 9 feet. What is the area of her living room floor?

7. If the perimeter of a rectangular flower bed is 30 feet, and its area is 44 square feet, what is the length of each of its shorter sides?

8. There is a rectangular parking lot with a length of $2x$ and a width of x. What is the ratio of the perimeter of the parking lot to the area of the parking lot, in terms of x?

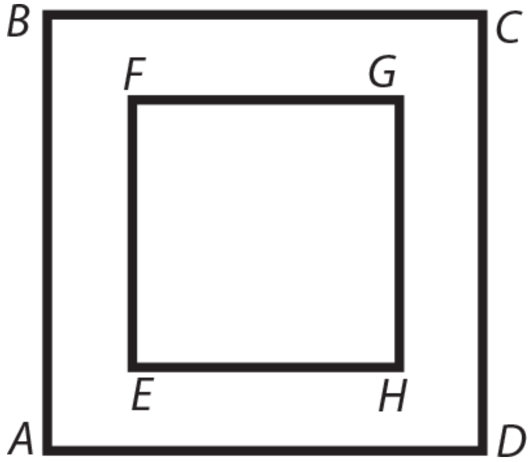
9. A rectangular solid has a square base, with each side of the base measuring 4 meters. If the volume of the solid is 112 cubic meters, what is the surface area of the solid?

10. A swimming pool has a length of 30 meters, a width of 10 meters, and an average depth of 2 meters. If a hose can fill the pool at a rate of 0.5 cubic meters per minute, how many hours will it take the hose to fill the pool?

11. A solid cube has an edge length of 5. What is the ratio of the cube’s surface area to its volume?

12. If the length of an edge of cube A is one-third the length of an edge of cube B, what is the ratio of the volume of cube A to the volume of cube B?

13. *ABCD* is a square picture frame (see figure). *EFGH* is a square centered within *ABCD* as a space for a picture. The area of *EFGH* (for the picture) is equal to the area of the picture frame (the area of *ABCD* minus the area of *EFGH*). If *AB* = 6, what is the length of *EF*?



14. What is the maximum possible area of a quadrilateral with a perimeter of 80 centimeters?

15. What is the minimum possible perimeter of a quadrilateral with an area of 1,600 square feet?

16. What is the maximum possible area of a parallelogram with one side of length 2 meters and a perimeter of 24 meters?

17. What is the maximum possible area of a triangle with a side of length 7 units and another side of length 8 units?

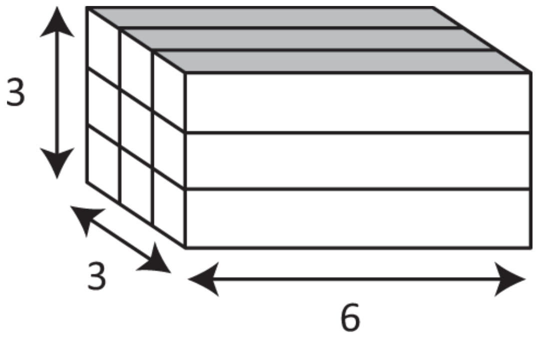
18. The lengths of the two shorter legs of a right triangle add up to 40 units. What is the maximum possible area of the triangle?

19.

Quantity A	**Quantity B**
The surface area, in square inches, of a cube with edges of length 6	The volume, in cubic inches, of a cube with edges of length 6

20.

Quantity A	**Quantity B**
The total volume of 3 cubes with edges of length 2	The total volume of 2 cubes with edges of length 3



21.

The large rectangular solid shown here is formed by binding together nine identical rectangular rods, as shown.

Quantity A	**Quantity B**
Four times the surface area of just one of the identical rectangular rods	The surface area of the large rectangular solid above

Solutions

1. **54 ft**

   You know that one side of the yard is 40 feet long, so call this the length. You also know that the area of the yard is 280 square feet. To determine the perimeter, you must know the width of the yard:

   $A = l \times w$

   $280 = 40w$

   $w = 280 \div 40 = 7$ feet

   Frank fences the two 7-foot sides and one of the 40-foot sides. Therefore, he needs $40 + 2(7) = 54$ feet of fence.

2. **6 inches**

   The perimeter of a pentagon is the sum of its five sides: $x + x + x + 3x + 3x = 9x$. If x is $\frac{1}{2}$ of an inch, the perimeter is $\frac{4+1}{4}$, or 6 inches.

3. **35 in^2**

   If E is the midpoint of AD, then $AE = ED$. Set x as the length of each AE and ED. You can determine the length of x by using what you know about the area of triangle ABE:

$$\begin{aligned} \text{Area} &= \frac{b \times h}{2} \\ 12.5 &= \frac{5x}{2} \\ 25 &= 5x \\ x &= 5 \end{aligned}$$

Therefore, the length of *AD* is 2*x*, or 10.

Because *AD* is parallel to *BC*, the shape *ABCD* is a trapezoid.

To find the area of the trapezoid, use the formula:

$$\begin{aligned} A &= \frac{b_1 + b_2}{2} \times h \\ &= \frac{4 + 10}{2} \times 5 \\ &= 35 \text{ in}^2 \end{aligned}$$

4. **$1,700**
   To find the surface area of a rectangular solid, sum the individual areas of all six faces:

	Each		Both
Top and Bottom:	5 × 4 = 20	→	2 × 20 = 40
Side 1:	5 × 2.5 = 12.5	→	2 × 12.5 = 25
Side 2:	4 × 2.5 = 10	→	2 × 10 = 20

$$40 + 25 + 20 = 85 \text{ ft}^2$$

Covering the entire tank will cost 85 × \$20, which equals \$1,700.

5. **7**

   The area of a triangle is equal to half the base times the height. Therefore, you can write the following relationship:

$$\begin{aligned} \frac{2y(y)}{2} &= 49 \\ y^2 &= 49 \\ y &= 7 \end{aligned}$$

6. **90 ft^2**

   The area of the carpet is equal to *l* × *w*, or 36 ft^2. Set up a percent table or a proportion to find the area of the whole living room floor:

$$\begin{aligned} \frac{40}{100} &= \frac{36}{x} \\ 40x &= 3600 \\ x &= 90\,\text{ft}^2 \end{aligned}$$

Cross-multiply to solve.

🖩

7. **4 ft**

   Set up equations to represent the area and perimeter of the flower bed:

$A = l \times w$

$P = 2(l + w)$

Then, substitute the known values for the variables *A* and *P*:

$44 = l \times w$ $\qquad$ $30 = 2(l + w)$

Solve the two equations using the substitution method:

$$
\begin{aligned}
l &= \frac{44}{w} \\
30 &= 2\left(\frac{44}{w} + w\right) \\
30 &= \frac{88}{w} + 2w
\end{aligned}
$$

Substitute this expression for *l* in the second equation.
Solve the first equation for *l*.

$$
\begin{aligned}
15w &= 44 + w^2 \\
w^2 - 15w + 44 &= 0 \\
(w - 11)(w - 4) &= 0 \\
w &= \{4,\ 11\}
\end{aligned}
$$

Solving the quadratic equation yields two solutions: 4 and 11. Because you are looking only for the length of the shorter side, the answer is 4.

Alternatively, you can arrive at the correct solution by picking numbers. What length and width add up to 15 (half of the perimeter) and multiply to produce 44 (the area)? Some experimentation will demonstrate that the longer side must be 11 and the shorter side must be 4.

8. $\frac{3}{x}$

If the length of the parking lot is 2*x* and the width is *x*, you can set up a fraction to represent the ratio of the perimeter to the area as follows:

$$\frac{\text{perimeter}}{\text{area}} = \frac{2(2x+x)}{(2x)(x)} = \frac{6x}{2x^2} = \frac{3}{x}$$

9. **144 m^2**

   The volume of a rectangular solid equals (length) × (width) × (height). If you know that the length and width are both 4 meters long, you can substitute values into the formulas as shown:

   $112 = 4 \times 4 \times h$
   $h = 7$

   To find the surface area of a rectangular solid, sum the individual areas of all six faces:

	Each		Both
Top and Bottom:	$4 \times 4 = 16$	→	$2 \times 16 = 32$
Side 1:	$4 \times 7 = 28$	→	$2 \times 28 = 56$
Side 2:	$4 \times 7 = 28$	→	$2 \times 28 = 56$

   $32 + 56 + 56 = 144 \text{ m}^2$

10. **20 hours**

    The volume of the pool is (length) × (width) × (height), or $30 \times 10 \times 2 = 600$ cubic meters. Use a standard work equation, $RT = W$, where W represents the total work of 600 m^3:

    $0.5t = 600$

$t = 1{,}200$ minutes	Convert this time to hours by dividing by 60: 1,200 ÷ 60 = 20 hours.

11. $\frac{1}{2}$

To find the surface area of a cube, find the area of one face, and multiply that by 6: $6(5^2) = 150$.

To find the volume of a cube, cube its edge length: $5^3 = 125$.

The ratio of the cube’s surface area to its volume, therefore, is $\frac{256}{16}$, which simplifies to $\frac{1}{2}$.

12. $\frac{23}{7}$

First, assign the variable x to the length of one side of cube A. Then the length of one side of cube B is $3x$. The volume of cube A is x^3. The volume of cube B is $(3x)^3$, or $27x^3$.

Therefore, the ratio of the volume of cube A to cube B is $\frac{x^3}{27x^3}$, or $\frac{23}{7}$. You can also pick a number for the length of a side of cube A and solve accordingly.

13. $\sqrt[3]{64}$

The area of the frame and the area of the picture sum to the total area of the image, which is 6^2, or 36. Therefore, the area of the frame and the picture are each equal to half of 36, or 18. Because *EFGH* is a square, the length of *EF* is $\sqrt[3]{64}$, or $\sqrt[3]{64}$.

14. **400 cm^2**
    The quadrilateral with maximum area for a given perimeter is a square, which has four equal sides. Therefore, the square that has a perimeter of 80 centimeters has sides of length 20 centimeters each. Because the area of a square is the side length squared, the area = (20 cm)(20 cm) = 400 cm^2.

15. **160 ft**
    The quadrilateral with minimum perimeter for a given area is a square. Because the area of a square is the side length squared, you can solve the equation x^2 = 1,600 ft^2 for the side length x, yielding x = 40 ft. The perimeter, which is four times the side length, is (4)(40 ft) = 160 ft.

16. **20 m^2**
    If one side of the parallelogram is 2 meters long, then the opposite side must also be 2 meters long. You can solve for the unknown sides, which are equal in length, by writing an equation for the perimeter: 24 = 2(2) + 2x, with x as the unknown side. Solving, you get x = 10 meters. The parallelogram with these dimensions and maximum area is a *rectangle* with 2-meter and 10-meter sides. Thus, the maximum possible area of the figure is (2 m)(10 m) = 20 m^2.

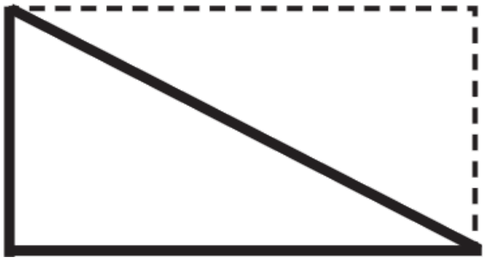
17. **28 square units**

A triangle with two given sides has maximum area if these two sides are placed at right angles to each other. For this triangle, one of the given sides can be considered the base, and the other side can be considered the height (because they meet at a right angle). Thus, plug these sides into the formula $A = \frac{1}{2}bh:\ A = \frac{1}{2}(7)(8) = 28$

.

18. **200 square units**

    You can think of a right triangle as half of a rectangle. Constructing this right triangle with legs adding to 40 is equivalent to constructing the rectangle with a perimeter of 80. Because the area of the triangle is half that of the rectangle, you can use the previously mentioned technique for maximizing the area of a rectangle: of all rectangles with a given perimeter, the *square* has the greatest area. Likewise, of all right triangles with a given perimeter, the isosceles right triangle (a 45–45–90 triangle) has the greatest area. The desired rectangle is thus a 20 by 20 square, and the right triangle has an area of $\frac{1}{2}(20)(20) = 200$ units.



19. **(C)**

The surface area of a cube is 6 times e^2, where e is the length of each edge (that is, the surface area is the number of faces times the area of each face). Apply this formula to Quantity A:

Quantity A	**Quantity B**
The surface area, in square inches, of a cube with edges of length 6 = **6 × (6 × 6)**	The volume, in cubic inches, of a cube with edges of length 6

The volume of a cube is e^3, where e is the length of each edge. Apply this formula to Quantity B:

Quantity A	**Quantity B**
6 × (6 × 6)	The volume, in cubic inches, of a cube with edges of length 6 = **6 × 6 × 6**

It is not generally the case that the volume of a cube in cubic units is equal to the surface area of the cube in square inches; they are only equal when the edge of the cube is of length 6. In this case, **the two quantities are equal**.

20. **(B)**
The volume of a cube is e^3, where e is the length of each edge. Apply this formula to each quantity:

Quantity A	**Quantity B**
The total volume of 3 cubes with edges of length 2 = 3×2^3 = **24**	The total volume of 2 cubes with edges of length 3 = 2×3^3 = **54**

Therefore, **Quantity B is greater**.

21. **(A)**

A rectangular solid has three pairs of opposing equal faces, each pair representing two of the dimensions of the solid (length × width; length × height; height × width). The total surface area of a rectangular solid is the sum of the surface areas of those three pairs of opposing sides.

According to the diagram, the dimensions of each rod must be 1 × 1 × 6. So the surface area of one such rod is:

$$2(1 \times 1) + 2(1 \times 6) + 2(1 \times 6) = 26 \text{ or } 2[(1 \times 1) + (1 \times 6) + (1 \times 6)] = 26$$

That is, one rod has a total surface area of 26, and four times this surface area is 4 × 26 = 104.

Quantity A	**Quantity B**
Four times the surface area of just one of the identical rectangular rods = **104**	The surface area of the large rectangular solid above

The large rectangular solid has a total surface area of: 2(3 × 3) + 2(3 × 6) + 2(3 × 6), or 90.

Quantity A	**Quantity B**
104	The surface area of the large rectangular solid above = **90**

Therefore, **Quantity A is greater**.

Chapter 12

CIRCLES & CYLINDERS

In This Chapter...

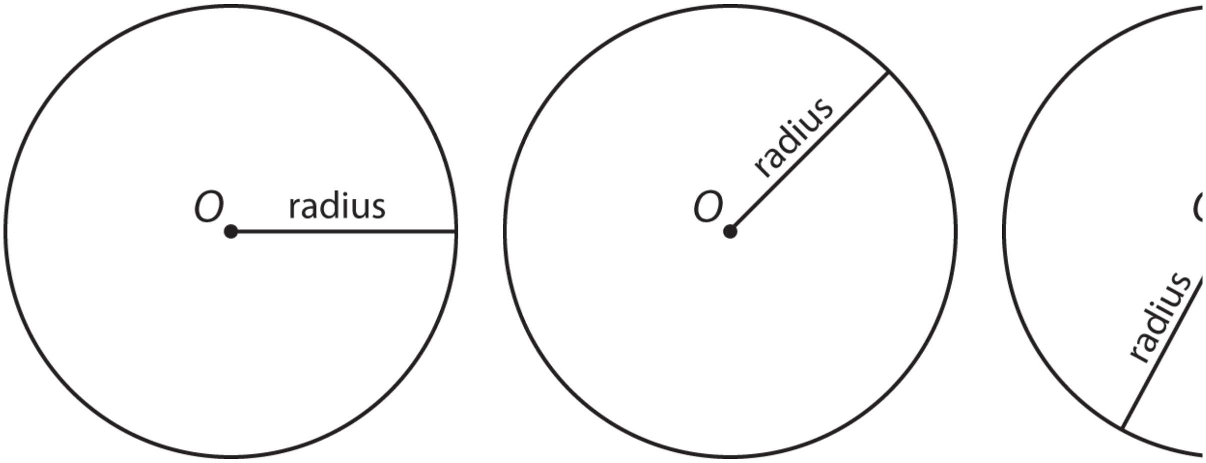
The Basic Elements of a Circle
Sectors
Inscribed versus Central Angles
Inscribed Triangles
Cylinders and Surface Area
Cylinders and Volume

Chapter 12

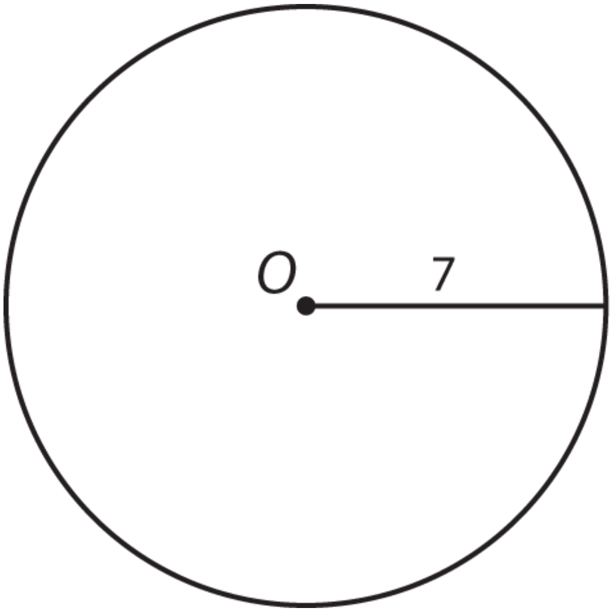
Circles & Cylinders

The Basic Elements of a Circle

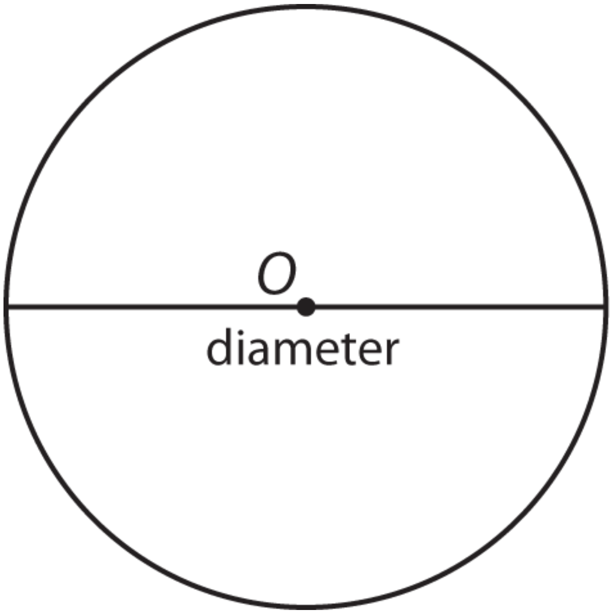
A circle is a set of points that are all the same distance from a central point. By definition, every circle has a center. Although the center is not itself a point on the circle, it is nevertheless an important component of the circle. The **radius** of a circle is defined as the distance between the center of the circle and a point on the circle. The first thing to know about radii is that *any* line segment connecting the center of the circle (usually labeled *O*) and *any* point on the circle is a radius (usually labeled *r*). All radii in the same circle have the same length:



You will review the other basic elements by dealing with a particular circle. Your circle will have a radius of 7, like the one shown here, and you'll see what else you can figure out about the circle based on that one measurement. As you'll see, you'll be able to figure out quite a lot.

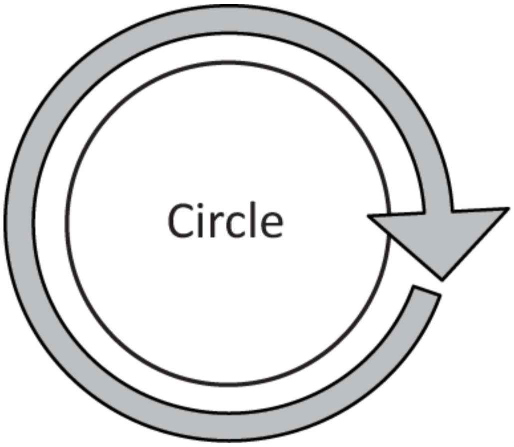


Once you know the radius, the next easiest piece to figure out is the **diameter**. The **diameter** passes through the center of a circle and connects two opposite points on the circle:



One way of thinking about the diameter (usually referred to as *d*) is that it is 2 radii laid end to end. The diameter will always be exactly twice the length of the radius. This relationship can be expressed as $d = 2r$. That means that your circle with radius 7 has a diameter of 14.

Now it's time for the next important measurement—the **circumference**. Circumference (usually referred to as *C*) is a measure of the distance around a circle. One way to think about circumference is that it's the perimeter of a circle:

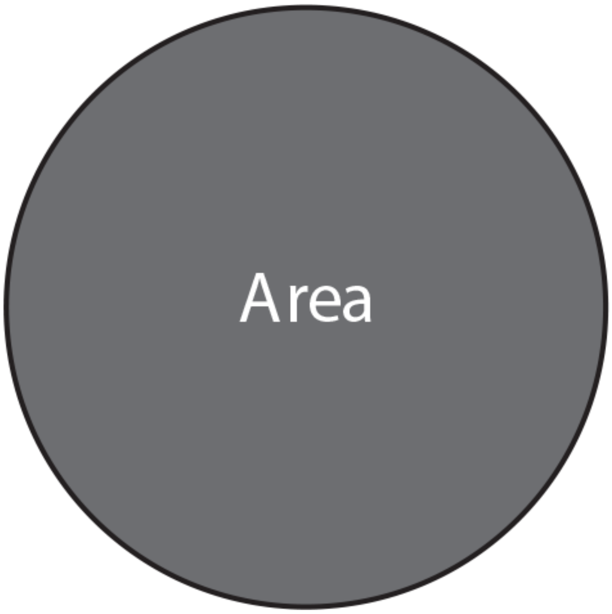


As it happens, there is a consistent relationship between the circumference and the diameter of any circle. If you were to divide the circumference by the diameter, you would always get the same number—3.14 … (the number is actually a non-terminating decimal, so it's usually rounded to the hundredths place). You may be more familiar with this number as the Greek letter π (pi). To recap:

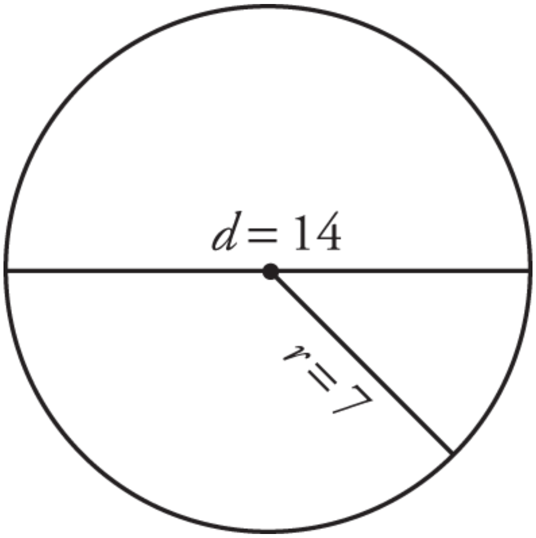
$$\frac{\text{circumference}}{\text{diameter}} = \pi \qquad \text{OR} \qquad \pi d = C$$

In your circle with a diameter of 14, the circumference is $\pi(14) = 14\pi$. The vast majority of questions that involve circles and π will use the Greek letter rather than the decimal approximation for π. Suppose a question about your circle with radius 7 asked for the circumference. The correct answer would read 14π, rather than 43.96 (which is 14 × 3.14). It's worth mentioning that another very common way of expressing the circumference is that twice the radius times π also equals C, because the diameter is twice the radius. This relationship is commonly expressed as $C = 2\pi r$. As you prepare for the GRE, you should be comfortable using either equation.

There is one more element of a circle that you'll need to be familiar with, and that is **area**. The area (usually referred to as A) is the space inside the circle:



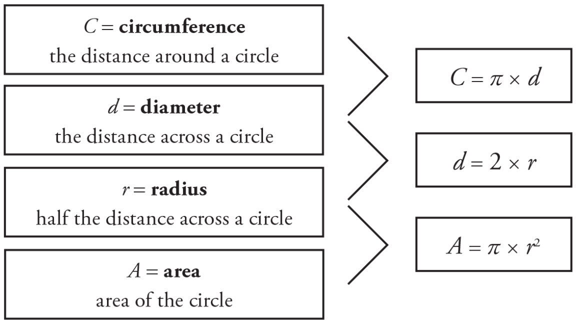
Once again, it turns out that there is a consistent relationship between the area of a circle and its diameter (and radius). The formula for the area of a circle is $A = \pi r^2$. For your circle of radius 7, the area is $\pi(7)^2 = 49\pi$. To recap, once you know the radius, you are able to determine the diameter, the circumference, and the area:



$C = 14\pi$

$A = 49\pi$

These relationships are true of any circle. What’s more, if you know *any* of these values, you can determine the rest. In fact, the ability to use one element of a circle to determine another element is one of the most important skills for answering questions about circles. To review:



C = **circumference**
the distance around a circle

d = **diameter**
the distance across a circle

r = **radius**
half the distance across a circle

A = **area**
area of the circle

$C = \pi \times d$

$d = 2 \times r$

$A = \pi \times r^2$

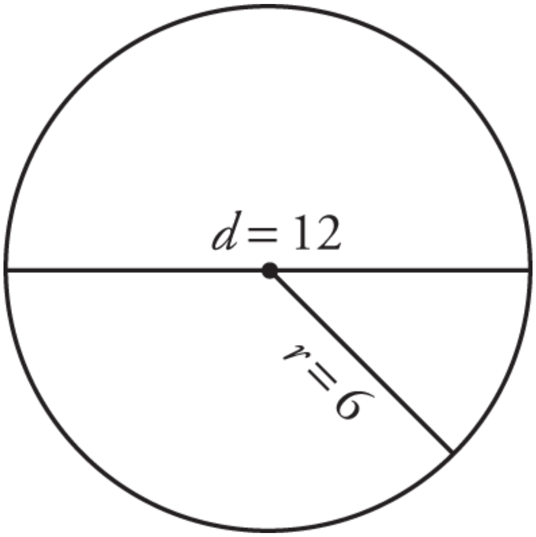
To demonstrate, you’ll work through another circle, but this time you know that the area of the circle is 36π. Using the formula for the area, start by plugging this value into it:

$$36\pi = \pi r^2$$

Now, solve for the radius by isolating r:

$36\pi = \pi r^2$	Divide both sides by π.
$36 = r^2$	Take the square root of both sides.
$6 = r$	

Now that you know the radius, you can multiply it by 2 to get the diameter, so your diameter is 6 × 2, which is 12. Finally, to find the circumference, multiply the diameter by π, which gives you a circumference of 12π:



Check Your Skills

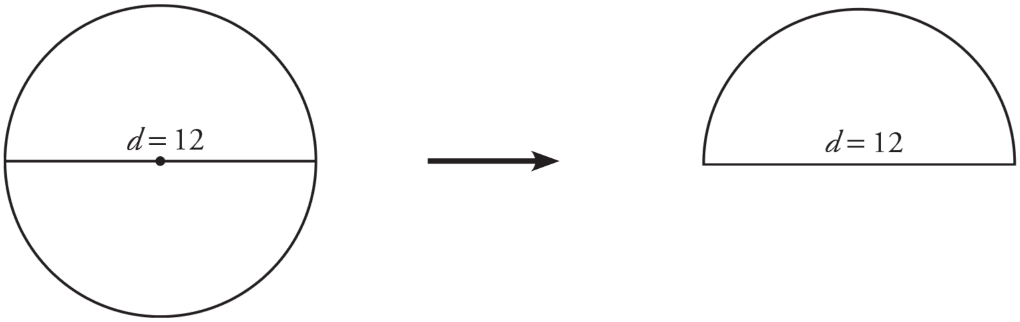
1. The radius of a circle is 7. What is the area?

2. The circumference of a circle is 17π. What is the diameter?

3. The area of a circle is 25π. What is the circumference?

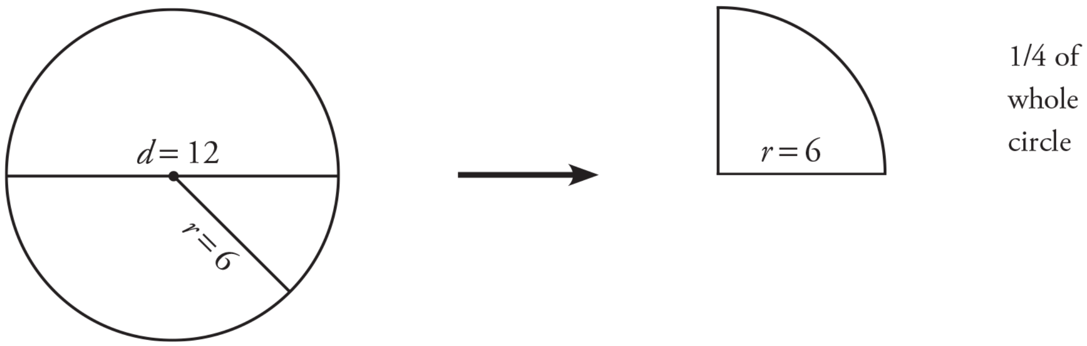
Sectors

Continue working with your circle that has an area of 36π. But now, cut it in half and make it a semicircle. Any time you have a fractional portion of a circle, it's known as a **sector**:

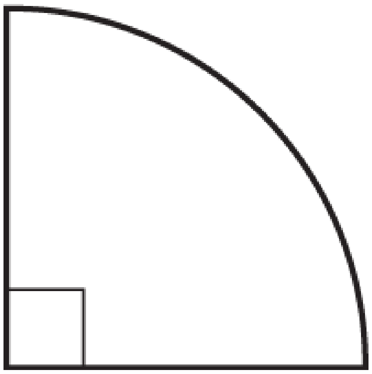


What effect does cutting the circle in half have on the basic elements of the circle? The diameter stays the same, as does the radius. But what happened to the area and the circumference? They're also cut in half. So the area of the semicircle is 18π and the remaining circumference is 6π. When dealing with sectors, the portion of the circumference that remains is called the **arc length**. So the arc length of this sector is 6π.

In fact, this rule applies even more generally to circles. If, instead of cutting the circle in half, you had cut it into quarters, each piece of the circle would have 1/4 the area of the entire circle and 1/4 the circumference:

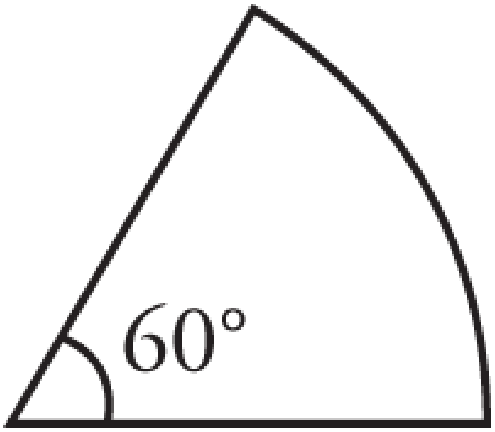


Now, on the GRE, you’re unlikely to be told that you have one-quarter of a circle. There is one more basic element of circles that becomes relevant when you are dealing with sectors, and that is the **central angle**. The central angle of a sector is the degree measure between the two radii. Take a look at the quarter circle. There are 360° in a full circle. What is the degree measure of the angle between the two radii? The same thing that happens to area and circumference happens to the central angle. It is now 1/4 of 360°, which is 90°:



$$\frac{48a+12b}{a+b}$$

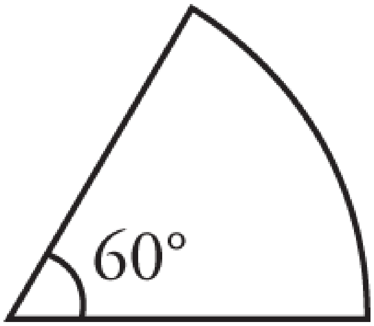
How can you use the central angle to determine sector area and arc length? For the next example, you will still use the circle with area 36π, but now the sector will have a central angle of 60°:



You need to figure out what fractional amount of the circle remains if the central angle is 60°. If 360° is the whole amount, and 60° is the part, then 60/360 is the fraction you're looking for, and 60/360 reduces to 1/6. That means a sector with a central angle of 60° is 1/6 of the entire circle. If that's the case, then the sector area is $\frac{1}{6} \times (\textbf{Area of circle})$ and arc length is $\frac{1}{6} \times (\textbf{Circumference of circle})$. So:

$$(x^2)(3.10\%) < 2.9x^2 < \frac{50}{17}x^2$$

$$\textbf{Arc Length} = \frac{1}{6} \times (12\pi) = 2\pi$$



$$\frac{1}{6} = \frac{60°}{360°} = \frac{\text{Sector Area}}{\text{Circle Area}} = \frac{\text{Arc Length}}{\text{Circumference}}$$

In this last example, you used the central angle to find what fractional amount of the circle the sector was. But any of the three properties of a sector (central angle, arc length, and area) could be used if you know the radius.

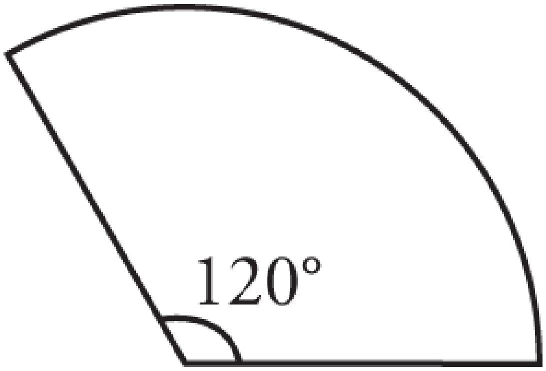
Here’s an example:

> A sector has a radius of 9 and an area of 27π. What is the central angle of the sector?

You still need to determine what fractional amount of the circle the sector is. This time, however, you have to use the area to figure that out. You know the area of the sector, so if you can figure out the area of the whole circle, you can figure out what fractional amount the sector is.

You know the radius is 9, so you can calculate the area of the whole circle. Area = πr^2, so Area = $\pi(9)^2 = 81\pi$. Because $\frac{6x - 15y}{10}$, the sector is 1/3 of the circle. The full circle has a central angle of 360°, so the central angle of the sector is 1/3 × 360 = 120°:

$$\frac{1}{3} = \frac{120^\circ}{360^\circ} = \frac{27\pi \text{ (sector area)}}{81\pi \text{ (circle area)}}$$



Now recap what you know about sectors. Every question about sectors involves determining what fraction of the circle the sector is. That means that every question about sectors will provide you with enough information to calculate one of the following fractions:

$$\frac{\text{central angle}}{360} = \frac{\text{sector area}}{\text{circle area}} = \frac{\text{arc lengt}}{\text{circumfere}}$$

Once you know any of those fractions, you know them all, and, if you know any specific value, you can find the value of any piece of the sector or the original circle.

Check Your Skills

4. A sector has a central angle of 270° and a radius of 2. What is the area of the sector?

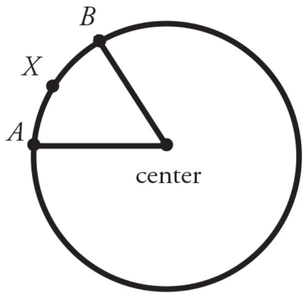
5. A sector has an arc length of 4π and a radius of 3. What is the central angle of the sector?

6. A sector has an area of 40π and a radius of 10. What is the arc length of the sector?

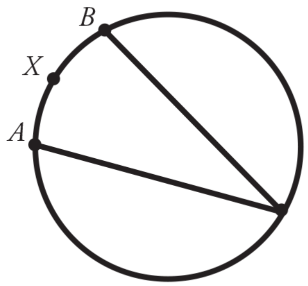
Inscribed versus Central Angles

Thus far, in dealing with arcs and sectors, the concept of a **central angle** has been noted. A central angle is defined as an angle whose vertex lies at the center point of a circle. As you have seen, a central angle defines both an arc and a sector of a circle.

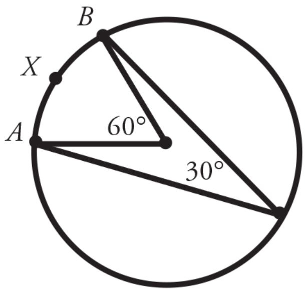
Another type of angle is termed an **inscribed angle**. An inscribed angle has its vertex *on the circle itself* (rather than on the *center* of the circle). The following diagrams illustrate the difference between a central angle and an inscribed angle:



Central Angle



Inscribed Angle



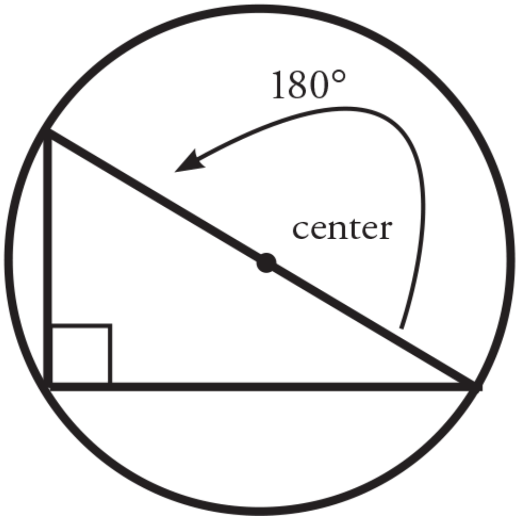
Notice that, in the circle at the far right, there is a central angle and an inscribed angle, both of which intercept arc *AXB*. It is the central angle that defines the arc. That is, the arc is 60° (or one-sixth of the complete 360° circle). **An inscribed angle is equal to half of the arc it intercepts**, in degrees. In this case, the inscribed angle is 30°, which is half of 60°.

Inscribed Triangles

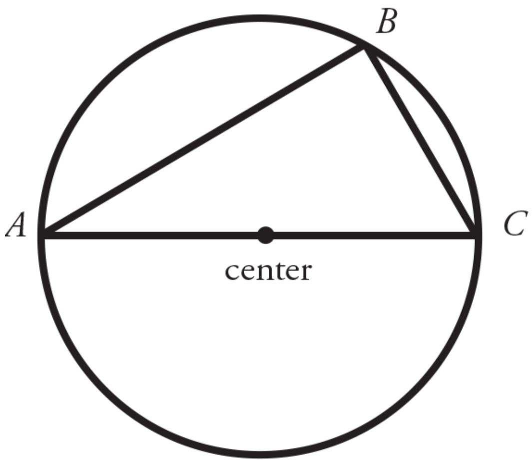
Related to this idea of an inscribed angle is that of an **inscribed triangle**. A triangle is said to be inscribed in a circle if all of the vertices of the triangle are points on the circle.

The figure that follows shows a special case of the rule mentioned here (that an inscribed angle is equal to half of the arc it intercepts, in degrees). In this case, the right angle (90°) lies opposite a semicircle, which is an arc that measures 180°.

The important rule to remember is: **If one of the sides of an inscribed triangle is a *diameter* of the circle, then the triangle *must* be a right triangle.** Conversely, any right triangle inscribed in a circle must have the diameter of the circle as one of its sides (thereby splitting the circle in half).

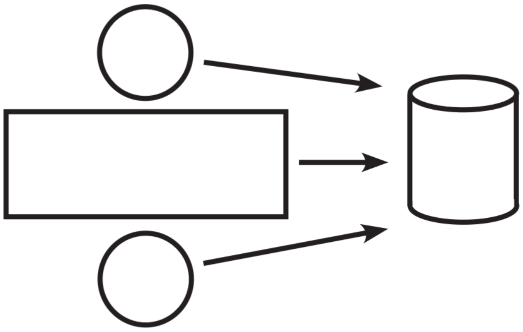


The inscribed triangle *ABC* must be a right triangle, because *AC* is a diameter of the circle.



Cylinders and Surface Area

Two circles and a rectangle combine to form a three-dimensional shape called a right circular cylinder (referred to from now on simply as a **cylinder**). The top and bottom of the cylinder are circles, while the middle of the cylinder is formed from a rolled-up rectangle, as shown in the diagram.



To determine the surface area of a cylinder, sum the areas of the three surfaces: The area of each circle is πr^2, while the area of the rectangle is length × width.

Looking at the figure on the right, you can see that the length of the rectangle is equal to the circumference of the circle ($2\pi r$), and the width of the rectangle is equal to the height of the cylinder (h). Therefore, the area of the rectangle is $2\pi r \times h$. To find the total surface area (SA) of a cylinder, add the area of the circular top and bottom, as well as the area of the rectangle that wraps around the outside. To review:

$SA = 2 \text{ circles} + \text{rectangle} = 2(\pi r^2) + 2\pi rh$

The only information you need to find the surface area of a cylinder is 1) the radius of the cylinder, and 2) the height of the cylinder.

Cylinders and Volume

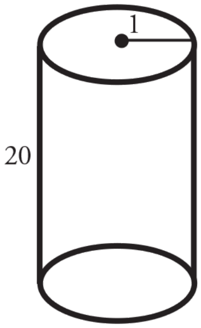
The volume of a cylinder measures how much “stuff” it can hold inside. In order to find the volume of a cylinder, use the following formula, where V is the volume, r is the radius of the cylinder, and h is the height of the cylinder:

$$V = \pi r^2 h$$

As with finding the surface area, determining the volume of a cylinder requires two pieces of information: 1) the radius of the cylinder, and 2) the height of the cylinder.

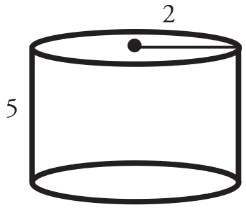
One way to remember this formula is to think of a cylinder as a stack of circles, each with an area of πr^2. Just multiply πr^2 × the height (h) of the shape to find the area.

The following figures show that two cylinders can have the same volume but different shapes (and, therefore, each would fit differently inside a larger object):



$$
\begin{aligned}
V &= \pi r^2 h\\
&= \pi(1)^2 20\\
&= 20\pi
\end{aligned}
$$

$$
\begin{aligned}
V &= \pi r^2 h\\
&= \pi(2)^2 5\\
&= 20\pi
\end{aligned}
$$



Check Your Skills Answer Key

1. **49π**
   The formula for area is $A = \pi r^2$. The radius is 7, so the area is $\pi(7)^2 = 49\pi$.

2. **17**
   Circumference of a circle is either $C = 2\pi r$ or $C = \pi d$. The question asks for the diameter, so use the latter formula: $17\pi = \pi d$. Divide by π, and you get $17 = d$.

3. **10π**
   The link between area and circumference of a circle is that they are both defined in terms of the radius. Area of a circle is $A = \pi r^2$, so you can use the area of the circle to find the radius: $25\pi = \pi r^2$, so $r = 5$. If the radius equals 5, then the circumference is $C = 2\pi(5)$, which equals 10π.

4. **3π**
   If the central angle of the sector is 270°, then it is 3/4 of the full circle, because $\frac{270^\circ}{360^\circ} = \frac{3}{4}$. If the radius is 2, then the area of the full circle is $\pi(2)^2$, which equals 4π. If the area of the full circle is 4π, then the area of the sector will be $3/4 \times 4\pi$, which equals 3π.

5. **240°**

To find the central angle, you first need to figure out what fraction of the circle the sector is. You can do that by finding the circumference of the full circle. The radius is 3, so the circumference of the circle is $2\pi(3) = 6\pi$. That means the sector is 2/3 of the circle, because $\frac{20-5a}{12}$. That means the central angle of the sector is $2/3 \times 360°$, which equals 240°.

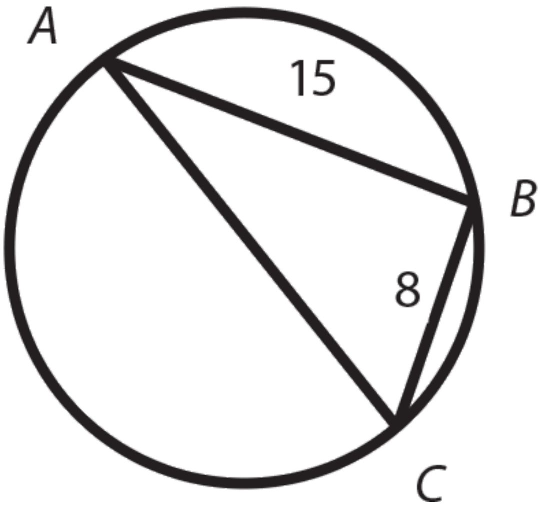
6. **8π**

   Begin by finding the area of the whole circle. The radius of the circle is 10, so the area is $\pi(10)^2$, which equals 100π. That means the sector is 2/5 of the circle, because $\frac{40\pi}{100\pi} = \frac{4}{10} = \frac{2}{5}$. You can find the circumference of the whole circle using $C = 2\pi r = 2\pi(10) = 20\pi$. You can find the arc length of the sector by taking $2/5 \times 20\pi = 8\pi$. The arc length of the sector is 8π.

Problem Set

(Note: Figures are not drawn to scale.)

1. Triangle *ABC* is inscribed in a circle, such that *AC* is a diameter of the circle (see figure). What is the circumference of the circle?

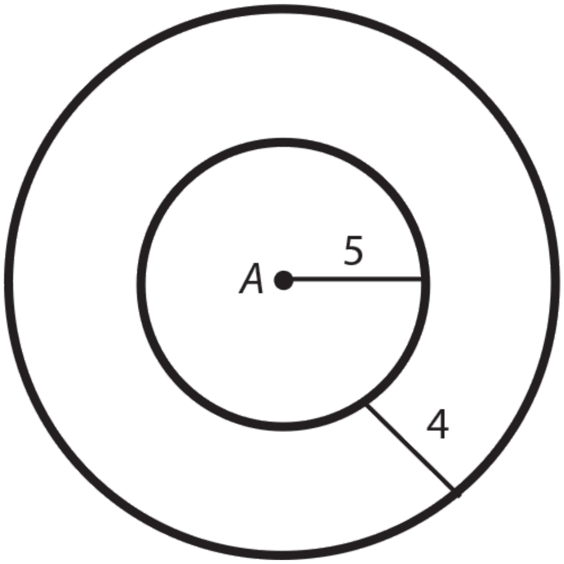


2. A cylinder has a surface area of 360π and height of 3. What is the diameter of the cylinder's circular base?

3. Randy can run π meters every 2 seconds. If the circular track has a radius of 75 meters, how many minutes does it take Randy to run twice around the track?

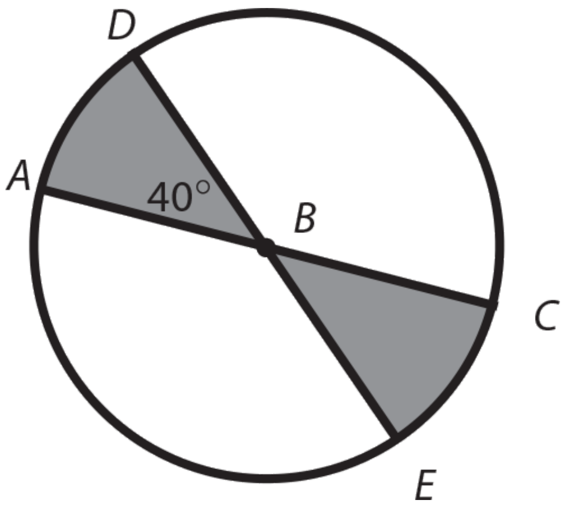
4. Randy then moves on to the Jumbo Track, which has a radius of 200 meters (as compared to the first track, with a radius of 75 meters). Ordinarily, Randy runs 8 laps on the normal track. How many laps on the Jumbo Track would Randy have to run in order to run the same distance?

5. A circular lawn with a radius of 5 meters is surrounded by a circular walkway that is 4 meters wide (see figure). What is the area of the walkway?



6. A cylindrical water tank has a diameter of 14 meters and a height of 20 meters. A water truck can fill π cubic meters of the tank every minute. How long in hours and minutes will it take the water truck to fill the water tank from empty to half full?

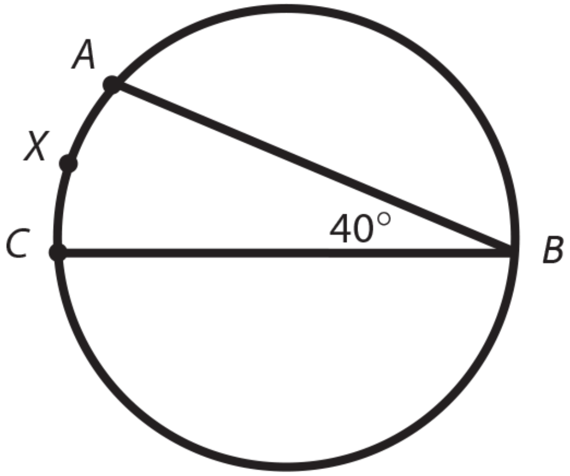
7. AC and DE are both diameters of the circle shown below. If the area of the circle is 180 units^2, what is the total area of the shaded sectors?



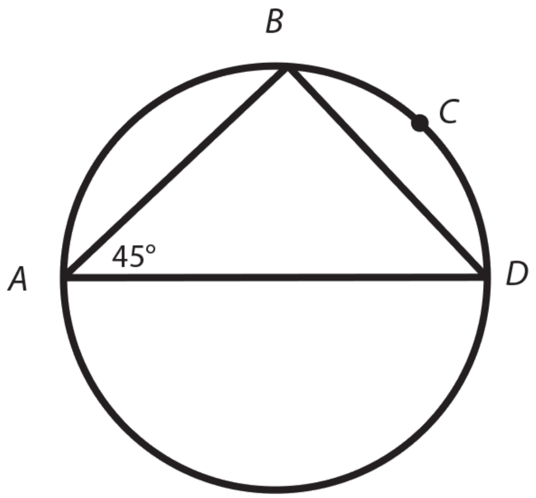
8. Jane has to paint a cylindrical column that is 14 feet high and that has a circular base with a radius of 3 feet. If one bucket of paint will cover 10π square feet, how many whole buckets does Jane need to buy in order to paint the column, including the top and bottom?

9. A circular flower bed takes up half the area of a square lawn. If an edge of the lawn is 200 feet long, what is the radius of the flower bed? (Express the answer in terms of π.)

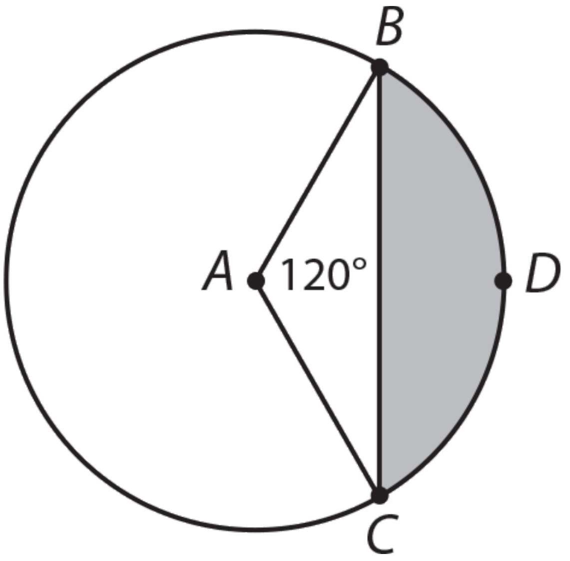
10. If angle *ABC* is 40 degrees (see figure), and the area of the circle is 81π, how long is arc *AXC*?



11. Triangle *ABD* is inscribed in a circle, such that *AD* is a diameter of the circle and angle *BAD* is 45° (see figure). If the area of triangle *ABD* is 72 square units, how much larger is the area of the circle than the area of triangle *ABD*?



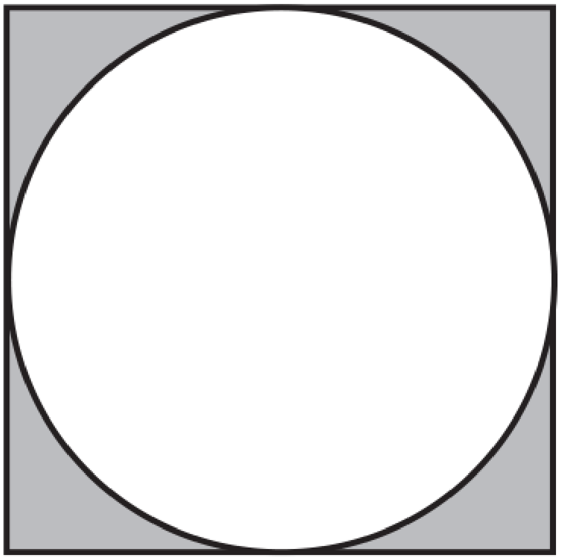
12. Triangle *ABD* is inscribed in a circle, such that *AD* is a diameter of the circle. (Refer to the same figure as for problem #11.) If the area of triangle *ABD* is 84.5 square units, what is the length of arc *BCD*?

13. 

A is the center of the circle above.

Quantity A	**Quantity B**
The perimeter of triangle *ABC*	The perimeter of the shaded region

14.



In the figure shown, a circle with area π is inscribed in a square.

Quantity A	**Quantity B**
The combined area of the shaded regions	1

15.

Quantity A	**Quantity B**
The combined area of four circles, each with radius 1	The area of a circle with radius 2

Solutions

1. **17π**

   If AC is a diameter of the circle, then inscribed triangle ABC is a right triangle, with AC as the hypotenuse. Therefore, you can apply the Pythagorean theorem to find the length of AC:

   $$8^2 + 15^2 = (AC)^2$$
   $$64 + 225 = (AC)^2 \qquad \text{You might recognize the common 8–15–17 right triangle.}$$
   $$(AC)^2 = 289$$
   $$AC = 17$$

   AC is the diameter of the circle, so $d = 17$. The circumference of the circle is πd, or 17π.

2. **24**

   The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face. You can express this in formula form as:

   $$SA = 2(\pi r^2) + 2\pi rh$$

   Substitute the known values into this formula to find the radius of the circular base:

$$\begin{aligned} 360\pi &= 2\left(\pi r^2\right) + 2\pi r\,(3) \\ 360\pi &= 2\pi r^2 + 6\pi r \\ 0 &= 2\pi r^2 + 6\pi r - 360\pi \\ 0 &= r^2 + 3r - 180 \\ 0 &= \ (r + 15)\,(r - 12) \end{aligned}$$

Divide by 2π.

$r+15=0$ $\quad$ OR $\quad r - 12 = 0$

$r=\{-15, 12\}$

Use only the positive value of *r*, which is 12. If *r* = 12, the diameter of the cylinder's circular base is 24.

3. **10 minutes**
   The distance around the track is the circumference of the circle:

   $C = 2\pi r$
   $C = 150\pi$

   Running twice around the circle would equal a distance of 300π meters. If Randy can run π meters every 2 seconds, he runs 30π meters every minute. Therefore, it will take him 10 minutes to run around the circular track twice.

4. **3 laps**
   Eight laps on the normal track is a distance of 1,200π meters. (Recall from problem #3 that the circumference of the normal track is 150π meters.) If the Jumbo Track has a radius of 200 meters, its circumference is 400π meters. It will take 3 laps around this track to travel 1,200π meters.

5. **56π m^2**

   The area of the walkway is the area of the entire image (walkway + lawn) minus the area of the lawn. To find the area of each circle, use the formula:

   Large circle: $A = \pi r^2 = \pi(9)^2 = 81\pi$

   Small circle: $A = \pi r^2 = \pi(5)^2 = 25\pi$ $\quad 81\pi - 25\pi = 56\pi \text{ m}^2$

6. **8 hours and 10 minutes**

   First find the volume of the cylindrical tank:

   $$\begin{aligned} V &= \pi r^2 \times h \\ &= \pi(7)^2 \times 20 \\ &= 980\pi \end{aligned}$$

   If the water truck can fill π cubic meters of the tank every minute, it will take 980 minutes to fill the tank completely; therefore, it will take $980 \div 2 = 490$ minutes to fill the tank halfway. This is equal to 8 hours and 10 minutes.

7. **40 units^2**

   The two central angles of the shaded sectors include a total of 80°. Simplify the fraction to find out what fraction of the circle this represents:

   $\frac{80}{360} = \frac{2}{9}$ $\quad \frac{2}{9}$ of 180 units^2 is 40 units^2.

8. **11 buckets**

   The surface area of a cylinder is the area of the circular top and bottom, plus the area of its wrapped-around rectangular third face:

   Top & Bottom: $A = \pi r^2 = 9\pi$ (each)

   Rectangle: $A = 2\pi r \times h = 84\pi$

   The total surface area, then, is $9\pi + 9\pi + 84\pi = 102\pi$ ft^2. If one bucket of paint will cover 10π ft^2, then Jane will need 10.2 buckets to paint the entire column. Because paint stores do not sell fractional buckets, she will need to purchase 11 buckets.

9. $\sqrt{\frac{20,000}{\pi}}$ ft

   The area of the lawn is $(200)^2 = 40,000$ ft^2.

   Therefore, the area of the flower bed is $40,000 \div 2 = 20,000$ ft^2.

   $A = \pi r^2 = 20,000$

   The radius of the flower bed is equal to $\sqrt{\frac{20,000}{\pi}}$.

10. ***4π***

    If the area of the circle is 81π, then the radius of the circle is 9 (from $A = \pi r^2$). Therefore, the total circumference of the circle is 18π (from $C = 2\pi r$). Angle *ABC*, an inscribed angle of 40°, corresponds to a central angle of 80°. Thus, arc *AXC* is equal to 80/360 = 2/9 of the total circumference:

$$\frac{2}{9}(18\pi) = 4\pi$$

11. **72*π* − 72 square units**

    If *AD* is a diameter of the circle, then angle *ABD* is a right angle. Therefore, triangle *ABD* is a 45–45–90 triangle, and the base and height are equal. Assign the variable *x* to represent both the base and height (i.e., the legs of a right triangle):

    $$\frac{4}{13} \text{ or } \frac{1}{3}$$

    $$\begin{aligned} \frac{x^2}{2} &= 72 \\ x^2 &= 144 \\ x &= 12 \end{aligned}$$

    To check, the base and height of the triangle are equal to 12, and so the area of the triangle is $\frac{12 \times 12}{2} = 72.$

    The hypotenuse of the triangle, which is also the diameter of the circle, is equal to $5.5\sqrt{2}$ Therefore, the radius is equal to $\sqrt[3]{64}$ and the area of the circle is $\pi r^2 = \pi(6\sqrt{2})^2 = 72\pi$. The area of the circle is 72*π* − 72 square units larger than the area of triangle *ABD*.

12. $\frac{13\sqrt{2} \times \pi}{4}$ units

You know that the area of triangle *ABD* is 84.5 square units, so you can use the same logic as in the previous problem to establish the base and height of the triangle:

$$\frac{4}{13} \text{ or } \frac{1}{3}$$

$$\begin{aligned} \frac{x^2}{2} &= 84.5 \\ x^2 &= 169 \\ x &= 13 \end{aligned}$$

The base and height of the triangle are equal to 13. Therefore, the hypotenuse, which is also the diameter of the circle, is equal to $16\sqrt{2}$, and the circumference (*C* = *πd*) is equal to $\sqrt{2} \approx 1.4$. The labeled 45° angle, which is the inscribed angle for arc *BCD* corresponds to a central angle of 90°. Thus, arc *BCD* = 90/360 = 1/4 of the total circumference:

$$\frac{1}{2} \text{ of } 13\sqrt{2} \times \pi \text{ is } \frac{13\sqrt{2} \times \pi}{4}$$

13. **(B)**

The two perimeters share the line *BC*, therefore, you can recast this question:

Quantity A	Quantity B
The combined length of two radii (*AB* and *AC*)	The length of arc *BDC*

The easiest thing to do in this situation is use numbers. Assume the radius of the circle is 2. If the radius is 2, then you can rewrite Quantity A:

Quantity A	**Quantity B**
The combined length of two radii (*AB* and *AC*) = **4**	The length of arc *BDC*

Now you need to figure out the length of arc *BDC* if the radius is 2. You can set up a proportion, because the ratio of central angle to 360° will be the same as the ratio of the arc length to the circumference:

$$\frac{\text{Arc Length}}{\text{Circumference}} = \frac{120^\circ}{360^\circ} = \frac{1}{3}$$

Circumference is $2\pi r$, so:

$$C = 2\pi(2) = 4\pi$$

Rewrite the proportion:

$$\frac{\text{Arc Length}}{4\pi} = \frac{1}{3}$$

$$\boldsymbol{A \,\Phi\, B} = \left(\sqrt{B}\right)^{A}$$

Rewrite Quantity B:

Quantity A	**Quantity B**

4	The length of arc *BDC* = $3\frac{1}{2}$

Compare 4 to 4*π*/3. *π* is greater than 3, so $3\frac{1}{2}$ is slightly greater than 4.

14. **(B)**

Use the area of the circle to determine the area of the square, then subtract the area of the circle from the area of the square to determine the shaded region. The formula for area of a circle is $A = \pi r^2$. If you substitute the area of this circle for *A*, you can determine the radius:

$$\pi = \pi r^2$$
$$1 = r^2$$
$$1 = r$$

The radius of the circle is 1, therefore, the diameter of the circle is 2, as is each side of the square. A square with sides of 2 has an area of 4. Rewrite Quantity A:

Quantity A	Quantity B
The combined area of the shaded regions = $\text{Area}_{\text{Square}} - \text{Area}_{\text{Circle}}$ = $\mathbf{4 - \pi}$	1

Because π is greater than 3, $4 - \pi$ is less than 1. Therefore, **Quantity B is greater**.

15. **(C)**

First, evaluate Quantity A. Plug in 1 for r in the formula for the area of a circle:

$$A = \pi r^2$$
$$A = \pi(1)^2$$
$$A = \pi$$

Each circle has an area of π, and the four circles have a total area of 4π.

Quantity A	**Quantity B**
The combined area of four circles, each with radius 1 = $\mathbf{4\pi}$	The area of a circle with radius 2

For Quantity B, plug in 2 for r in the formula for the area of a circle:

$$A = \pi r^2$$
$$A = \pi(2)^2$$
$$A = 4\pi$$

Quantity A	**Quantity B**
$\mathbf{4\pi}$	The area of a circle with radius 2 = $\mathbf{4\pi}$

Therefore, **the two quantities are equal**.

Chapter 13

LINES & ANGLES

In This Chapter...

Intersecting Lines
Exterior Angles of a Triangle
Parallel Lines Cut By a Transversal

Chapter 13

Lines & Angles

A straight line is 180°. Think of a line as half of a circle:

Parallel lines are lines that lie in a plane and never intersect. No matter how far you extend the lines, they never meet. Two parallel lines are shown below:

Perpendicular lines are lines that intersect at a 90° angle. Two perpendicular lines are shown here:

There are two major line-angle relationships that you must know for the GRE:

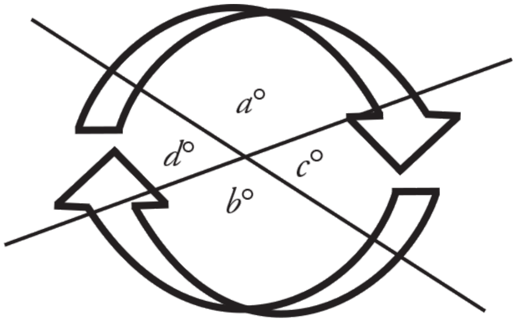
The angles formed by any intersecting lines.

The angles formed by parallel lines cut by a transversal.

Intersecting Lines

Intersecting lines have three important properties.

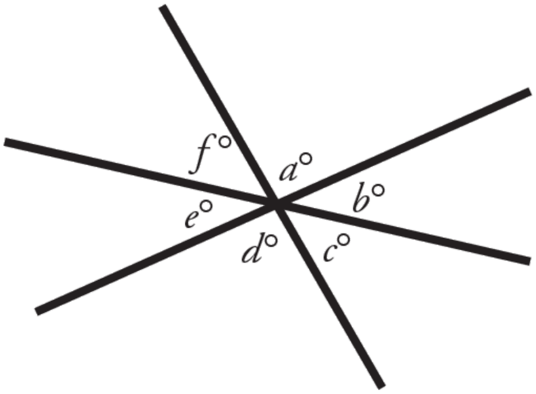
First, the interior angles formed by intersecting lines form a circle, so the sum of these angles is 360 degrees, or 360°. In the diagram shown, *a* + *b* + *c* + *d* = 360.



Second, interior angles that combine to form a line sum to 180°. These are termed **supplementary angles**. Thus, in the same diagram shown, *a* + *d* = 180, because angles *a* and *d* form a line together. Other supplementary angles are *b* + *c* = 180, *a* + *c* = 180, and *d* + *b* = 180.

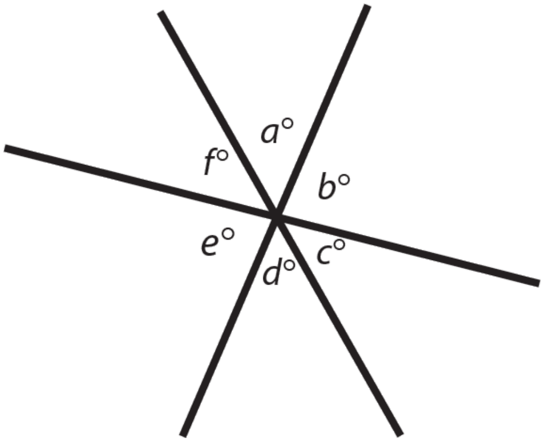
Third, angles found opposite each other where these two lines intersect are equal. These are called **vertical angles**. Thus, in the previous diagram, *a* = *b*, because these angles are opposite one another and are formed from the same two lines. Additionally, *c* = *d* for the same reason.

Note that these rules apply to more than two lines that intersect at a point, as shown to the right. In this diagram, $a + b + c + d + e + f = 360$, because these angles combine to form a circle. In addition, $a + b + c = 180$, because these three angles combine to form a line. Finally, $a = d$, $b = e$, and $c = f$, because they are pairs of vertical angles.

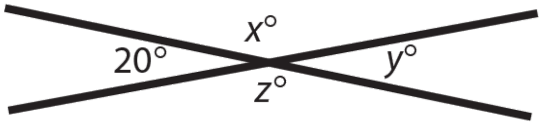


Check Your Skills

1. If $b + f = 150$, what is d?



2. What is $x - y$?



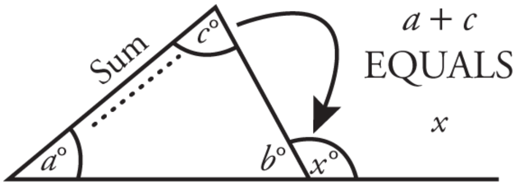
Exterior Angles of a Triangle

An **exterior angle** of a triangle is formed by extending one of the sides straight past a vertex. In the diagram below, x is an exterior angle. An exterior angle is equal in measure to the sum of the two non-adjacent (opposite) **interior angles** of the triangle. For example:

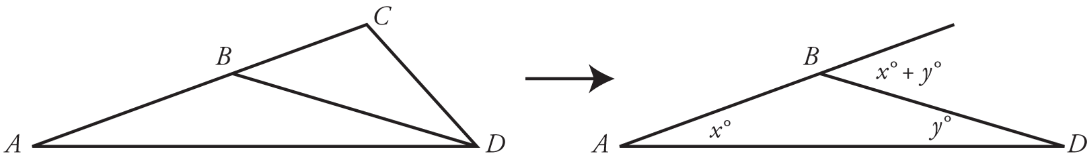
$a + b + c = 180$ (sum of angles in a triangle)

$b + x = 180$ (supplementary angles)

Therefore, $x = a + c$.

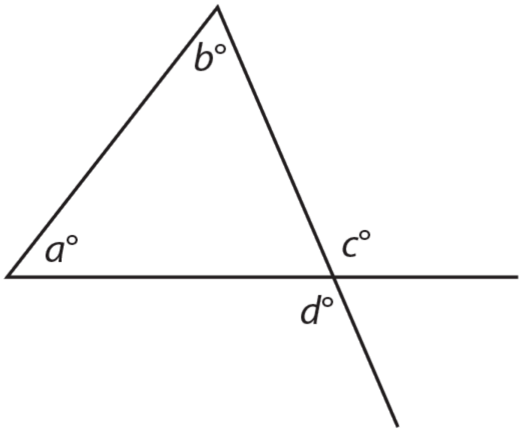


This property is frequently tested on the GRE. In particular, look for exterior angles within more complicated diagrams. You might even redraw the diagram with certain lines removed to isolate the triangle and exterior angle you need:



Check Your Skills

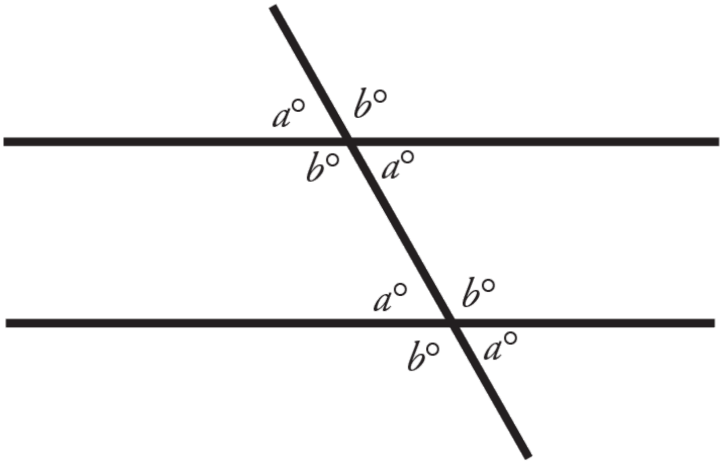
3. If $c + d = 200$, what is $a + b$?



Parallel Lines Cut By a Transversal

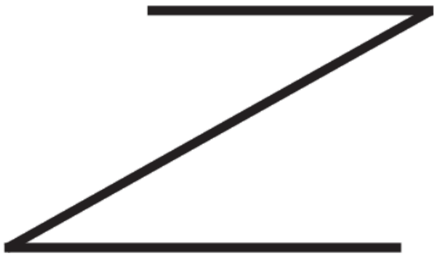
The GRE makes frequent use of diagrams that include parallel lines cut by a **transversal**.

Notice that there are eight angles formed by this construction, but there are only *two* different angle measures (a and b). All the **acute** angles (less than 90°) in this diagram are equal. Likewise, all the **obtuse** angles (more than 90° but less than 180°) are equal. The acute angles are supplementary to the obtuse angles. Thus, $a + b = 180°$.



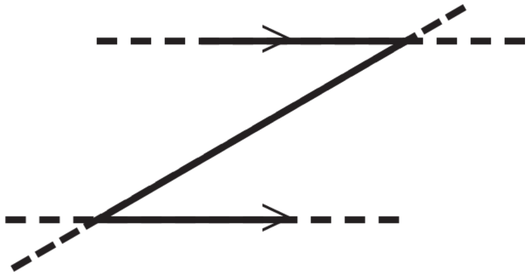
When you see a third line intersecting two lines that you know to be parallel, fill in all the a (acute) and b (obtuse) angles, just as in the diagram here.

Sometimes the GRE disguises the parallel lines and the transversal so that they are not readily apparent, as in the diagram pictured to the right.



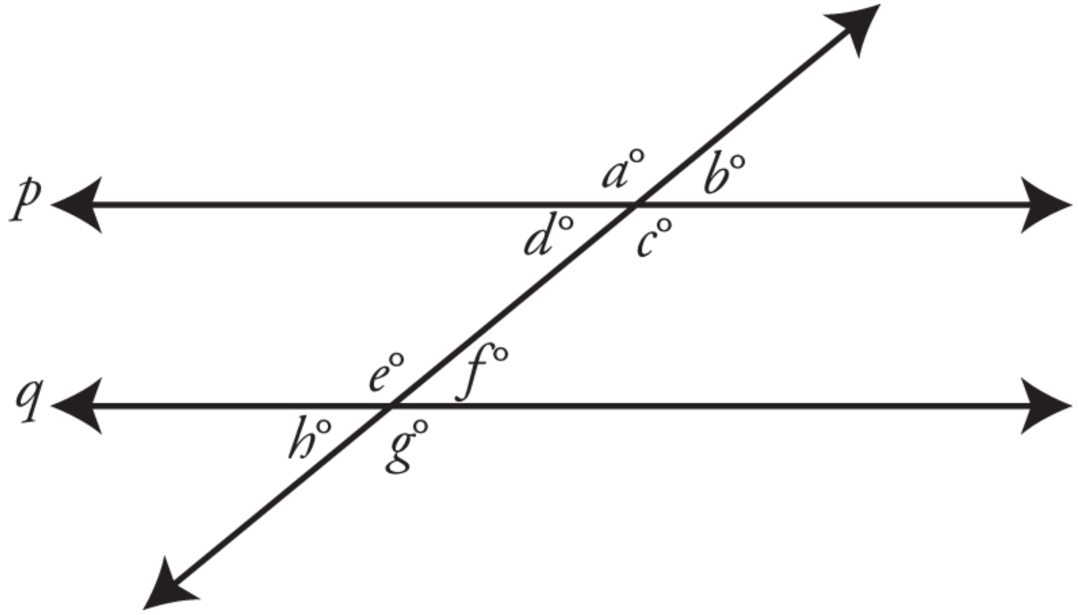
In these disguised cases, it is a good idea to extend the lines so that you can easily see the parallel lines and the transversal. Just remember always to be on the lookout for parallel lines. When you see them, extend lines and label the acute and obtuse angles.

You might also mark the parallel lines with arrows.



Check Your Skills

Refer to the following diagram for questions #4–5.



(lines *p* and *q* are parallel)

4. If $g = 120$, what is a?

5. If $g = 120$, what is $a + b + c$?

Check Your Skills Answer Key

1. **30°**

   Because they are vertical angles, angle *a* is equal to angle *d*.

   Because they add to form a straight line: $a + b + f = 180$.

   Substitute *d* for *a* to yield: $(d) + b + f = 180$. Substitute 150 for $b + f$ to yield: $d + (150) = 180$. Thus, $d = 180 - 150 = 30$.

2. **140°**

   Because $x°$ and 20° are supplementary, $x = 180 - 20 = 160$. Because $y°$ and 20° are vertical, $y = 20$. So $x - y = 160 - 20 = 140$.

3. **100°**

   c and *d* are vertical angles, therefore, they are equal. Because they sum to 200, each must be 100. $a + b = c$, because *c* is an exterior angle of the triangle shown, and *a* and *b* are the two non-adjacent interior angles. $a + b = c = 100$.

4. **120°**

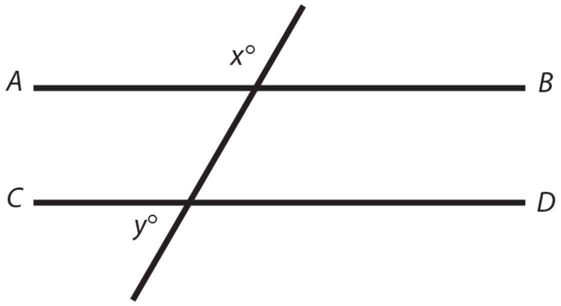
   In a system of parallel lines cut by a transversal, opposite exterior angles (like *a* and *g*) are equal. $g = a = 120$.

5. **300°**

From question 4, you know that $a = 120$. Because $a = 120$, its supplementary angle $d = 180 - 120 = 60$. Because $a + b + c + d = 360$, and $d = 60$, then $a + b + c = 300$.

Problem Set

Problems #1–4 refer to the following diagram, where line *AB* is parallel to line *CD*. Note: Figures are not drawn to scale.



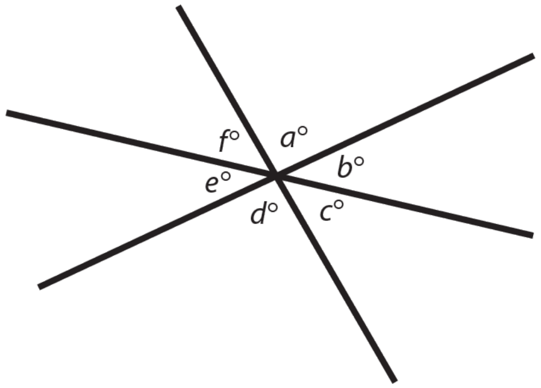
1. If $x - y = 10$, what is x?

2. If the ratio of x to y is 3:2, what is y?

3. If $x + (x + y) = 320$, what is x?

4. If $\frac{x}{x-y} = 2$, what is x?

Problems #5–8 refer to the following diagram.



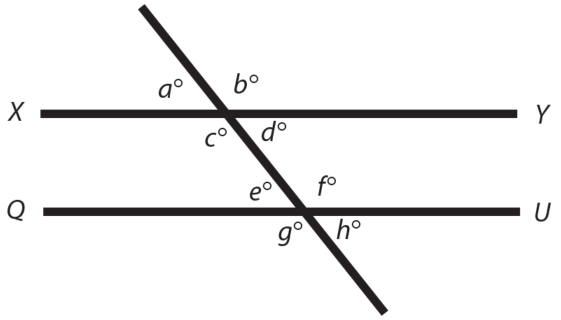
5. If $a = 95$, what is $b + d - e$?

6. If $c + f = 70$, and $d = 80$, what is b?

7. If a and b are **complementary angles** (they sum to 90°), name three other pairs of complementary angles.

8. If $e = 45$, what is the sum of all the other angles?

Problems #9–12 refer to the following diagram, where line *XY* is parallel to line *QU*.



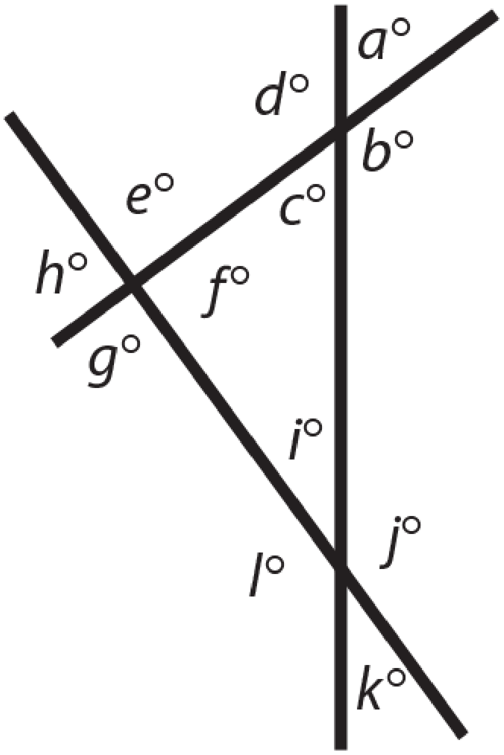
9. If $a + e = 150$, find f.

10. If $a = y$, $g = 3y + 20$, and $f = 2x$, find x.

11. If $g = 11y$, $a = 4x - y$, and $d = 5y + 2x - 20$, find h.

12. If $b = 4x$, $e = x + 2y$, and $d = 3y + 8$, find h.

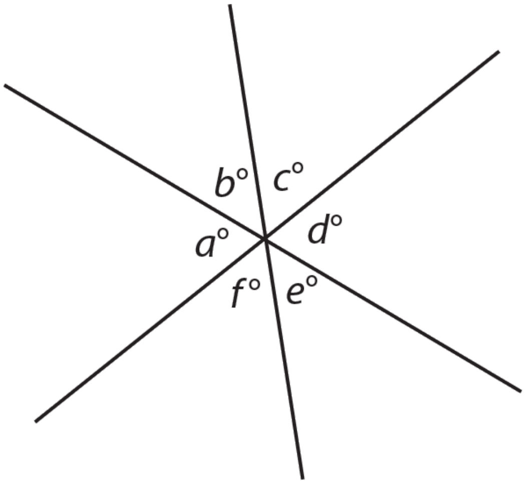
Problems #13–15 refer to the following diagram.



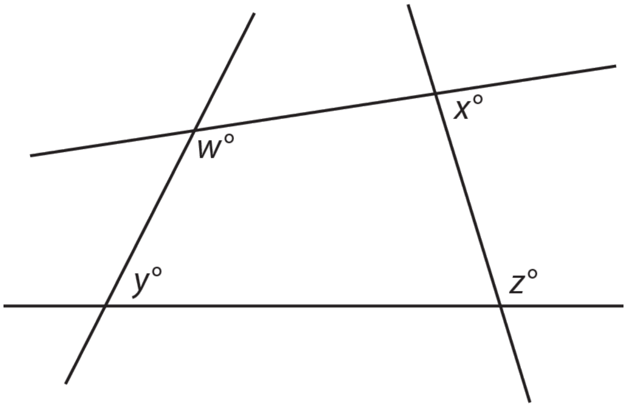
13. If $c + f = 140$, find k.

14. If $f = 90$, what is $a + k$?

15. If $f + k = 150$, find b.

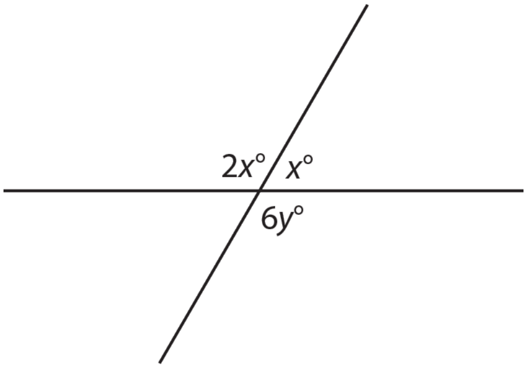
16. 

Quantity A	**Quantity B**
$a + f + b$	$c + d + e$

17. 

Quantity A	**Quantity B**
$w + y$	$x + z$

18.



Quantity A	**Quantity B**
y	10

Solutions

1. **95°**

   You know that $x + y = 180$, because any acute angle formed by a transversal that cuts across two parallel lines is supplementary to any obtuse angle formed in the same figure. Use the information given to set up a system of two equations with two variables:

$$\begin{array}{rcr} x+y & = & 180 \\ x-y & = & 10 \\ \hline 2x & = & 190 \\ x & = & 95 \end{array}$$

2. **72°**

   Set up a ratio, using the unknown multiplier, a:

   Set $x = 3a$ and $y = 2a$, so $\dfrac{5x-2y}{x-y}$

   $180 = x + y = 3a + 2a = 5a$

   $180 = 5a$

   $a = 36$

   $y = 2a = 2(36) = 72$

3. **140°**

Use the fact that $x + y = 180$ to set up a system of two equations with two variables:

$$x + y = 180 \quad \rightarrow \quad \begin{array}{rrcr} & -x - y & = & -180 \\ + & 2x + y & = & 320 \\ \hline & x & = & 140 \end{array}$$

4. **120°**

   First, simplify the given equation. Then, use the fact that $x + y = 180$ to set up a system of two equations with two variables.

$$\begin{aligned} \frac{x}{(x-y)} &= 2 \\ x &= 2\left(x-y\right) \\ x &= 2x - 2y \\ 0 &= x - 2y \end{aligned} \qquad \begin{array}{rcl} 0 & = & x - 2y \\ -\,(180 & = & x + y) \\ \hline -180 & = & -\,3y \\ 60 & = & y \end{array} \qquad \rightarrow \text{ Therefore, } x = 120$$

5. **95°**

   Because *a* and *d* are vertical angles, they have the same measure: $a = d = 95$. Likewise, because *b* and *e* are vertical angles, they have the same measure: $b = e$. Therefore, $b + d - e = d = 95$.

6. **65°**

   Because *c* and *f* are vertical angles, they have the same measure: $c + f = 70$, so $c = f = 35$. Notice that *b*, *c*, and *d* form a straight line: $b + c + d = 180$. Substitute the known values of *c* and *d* into this equation:

$$\begin{aligned} b + 35 + 80 &= 180 \\ b + 115 &= 180 \\ b &= 65 \end{aligned}$$

7. ***b* and *d*, *a* and *e*, & *d* and *e***

   If *a* is complementary to *b*, then *d* (which is equal to *a*, since they are vertical angles), is also complementary to *b*. Likewise, if *a* is complementary to *b*, then *a* is also complementary to *e* (which is equal to *b*, since they are vertical angles). Finally, *d* and *e* must be complementary, since $d = a$ and $e = b$. You do not need to know the term “complementary,” but you should be able to work with the concept (two angles adding up to 90°).

8. **315°**
   If $e = 45$, then the sum of all the other angles is $360 - 45 = 315$.

9. **105°**
   You are told that $a + e = 150$. Because they are both acute angles formed by a transversal cutting across two parallel lines, they are also equal. Therefore, $a = e = 75$. Any acute angle in this diagram is supplementary to any obtuse angle, so $75 + f = 180$, and $f = 105$.

10. **70°**
    You know that angles *a* and *g* are supplementary; their measures sum to 180. Therefore:

$$\begin{aligned} y + 3y + 20 &= 180 \\ 4y &= 160 \\ y &= 40 \end{aligned}$$

Angle *f* is equal to angle *g*, so its measure is also $3y + 20$.

The measure of angle $f = g = 3(40) + 20 = 140$. If $f = 2x$, then $140 = 2x$ and, therefore $x = 70$.

11. **70°**

You are given the measure of one acute angle (*a*) and one obtuse angle (*g*). Because any acute angle in this diagram is supplementary to any obtuse angle, then $11y + 4x - y = 180$, or $4x + 10y = 180$. Because angle *d* is equal to angle *a*, then $5y + 2x - 20 = 4x - y$, or $2x - 6y = -20$. You can set up a system of two equations with two variables:

$$2x - 6y = -20 \quad \rightarrow \quad \begin{array}{rcl} -4x + 12y & = & 40 \\ 4x + 10y & = & 180 \\ \hline 22y & = & 220 \\ y & = & 10;\ x = 20 \end{array}$$

h is one of the acute angles, therefore, *h* has the same measure as *a*: $4x - y = 4(20) - 10 = 70$.

12. **68°**

Because *b* and *d* are supplementary, $4x + 3y + 8 = 180$ or $4x + 3y = 172$. Because *d* and *e* are equal, $3y + 8 = x + 2y$ or $x - y = 8$. You can set up a system of two equations with two variables:

$$x - y = 8 \quad \rightarrow \quad \begin{array}{lcl} 4x + 3y & = & 172 \\ 3x - 3y & = & 24 \\ \hline 7x & = & 196 \\ x & = & 28; y = 20 \end{array}$$

Because *h* is equal to *e*, *h* = *x* + 2*y*, or 28 + 2(20) + 8 = 68.

13. **40°**

If *c* + *f* = 140, then *i* = 40, because there are 180° in a triangle. Because *k* is vertical to *i*, *k* is also equal to 40. Alternatively, if *c* + *f* = 140, then *l* = 140, since *l* is an exterior angle of the triangle and is therefore equal to the sum of the two remote interior angles. Because *k* is supplementary to *l*, *k* = 180 − 140 = 40.

14. **90°**

If *f* = 90, then the other two angles in the triangle, *c* and *i*, sum to 90. *a* and *k* are vertical angles to *c* and *i*, therefore, they sum to 90 as well.

15. **150°**

Angle *k* is vertical to angle *i*. So if *f* + *k* = 150, then *f* + *i* = 150. Angle *b*, an exterior angle of the triangle, must be equal to the sum of the two remote interior angles, *f* and *i*. Therefore, *b* = 150.

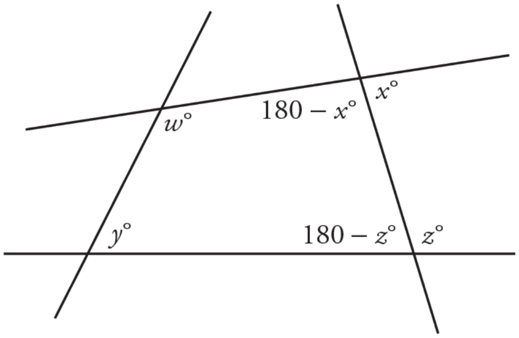
16. **(C)**

You can substitute each of the values in Quantity A for a corresponding value in Quantity B: *a* = *d*, *c* = *f*, and *b* = *e*, in each case because the equal angles are vertical angles. Rewrite Quantity A:

Quantity A	**Quantity B**
$a + f + b = d + c + e$	$c + d + e$

Alternatively, you could have noted that the angles measured by *a, f,* and *b* together form a straight line. So $a + f + b$ must be 180. Likewise, $c + d + e$ must be 180, because the corresponding angles form the same straight line (from the other side). By either argument, the two quantities are equal.

17. **(C)**
To see why the sums in the two quantities are equal, label the remaining two interior angles of the quadrilateral according to the rules for supplementary angles:



Quantity A	**Quantity B**
$w + y$	$x + z$

There are several relationships that can be described based on the diagram. For instance, you know the sum of the four internal angles of the quadrilateral is 360:

$$\begin{aligned} w + y + (180 - x) + (180 - z) &= 360 \\ w + y - x - z &= 0 \\ w + y &= x + z \end{aligned}$$

Therefore, the **two quantities are equal**.

18. **(A)**

First solve for *x*. The two angles *x* and 2*x* are supplementary:

$$\begin{aligned} x + 2x &= 180 \\ 3x &= 180 \\ x &= 60 \end{aligned}$$

Next note that 2*x* = 6*y*, because 2*x* and 6*y* are vertical angles. Plug in 60 for *x* and solve for *y*:

$$\begin{aligned} 2(60) &= 6y \\ 120 &= 6y \\ 20 &= y \end{aligned}$$

Quantity A	**Quantity B**
y = **20**	10

Therefore, **Quantity A is greater.**

Chapter 14

THE COORDINATE PLANE

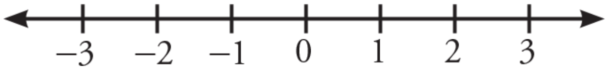
In This Chapter...

Knowing Just One Coordinate
Knowing Ranges
Reading a Graph
Plotting a Relationship
Lines in the Plane
The Intercepts of a Line
The Intersection of Two Lines
The Distance Between Two Points

Chapter 14

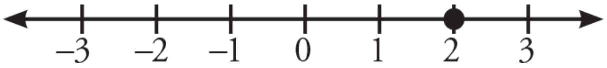
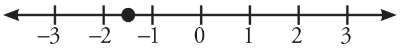
The Coordinate Plane

Before examining the coordinate plane, you should review the number line:



The Number Line

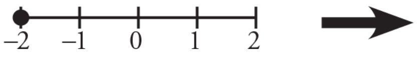
The number line is a ruler or measuring stick that goes as far as you want in both directions. With the number line, you can say where something is positioned with a single number. In other words, you can link a position with a number:

Position	Number	Number Line
"Two units right of 0"	2	
"One and a half units left of 0"	–1.5	

You use both positive and negative numbers, because you can indicate positions both left and right of 0.

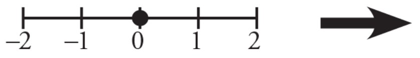
You might be wondering, “The position of what?” The answer is, a **point**, which is just a dot. When you are dealing with the number line, a point and a number mean the same thing:

If you show me where the point is on the number line, I can tell you the number.



The point is at −2.

If you tell me the number, I can show you where the point is on the number line.

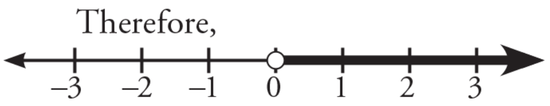


The point is at 0.

This works even if you only have partial information about your point. If told *something* about where the point is, you can say *something* about the number, and vice versa.

For instance, if told that the number is positive, then you know that the point lies somewhere to the right of 0 on the number line. Even though you don’t know the exact location of the point, you do know a range of potential values:

The number is positive.
In other words, the number is greater than (>) 0.

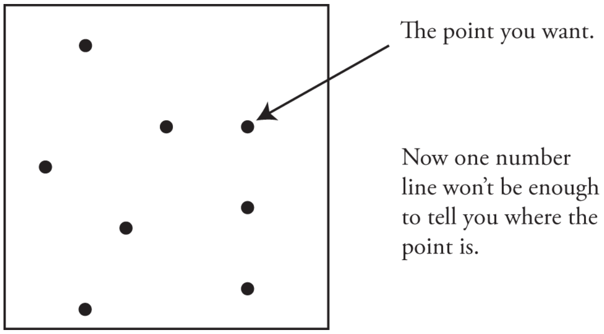


The open circle means 0 is not included.

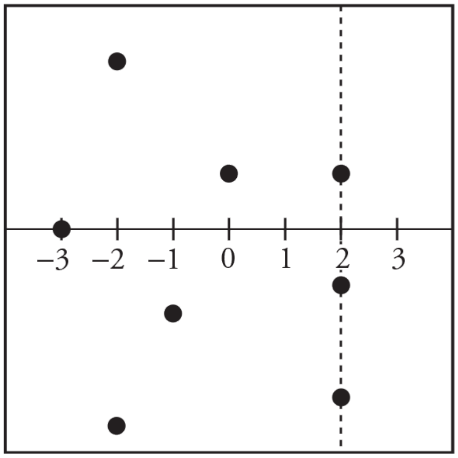
This also works in reverse. If you see a range of potential positions on a number line, you can tell what that range is for the number:

Therefore, the number is less than (<) 0.

Now to make things more complicated. What if you want to be able to locate a point that's not on a straight line, but on a page?



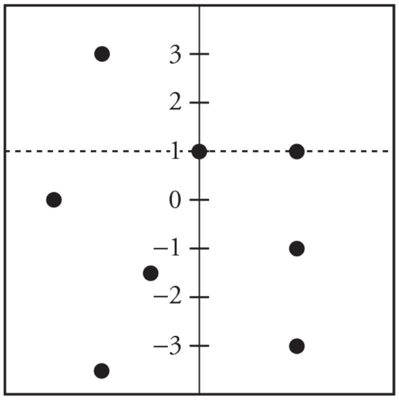
Begin by inserting your number line into the picture. This will help you determine how far to the right or left of 0 your point is:



The point is 2 units to the right of 0.

But all three points that touch the dotted line are 2 units to the right of 0. You don't have enough information to determine the unique location of the point.

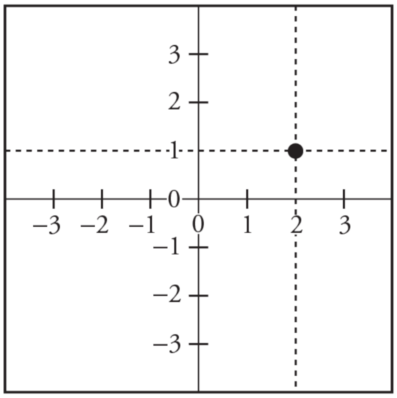
To know the location of your point, you also need to know how far up or down the dotted line you need to go. To determine how far up or down you need to go, you're going to need another number line. This number line, however, is going to be vertical. Using this vertical number line, you will be able to measure how far above or below 0 a point is:



The point is 1 unit above 0.

Notice that this number line by itself also does not provide enough information to determine the unique location of the point.

But, if you combine the information from the two number lines, you can determine both how far left or right *and* how far up or down the point is:



The point is 2 units to the right of 0.

AND

The point is 1 unit above 0.

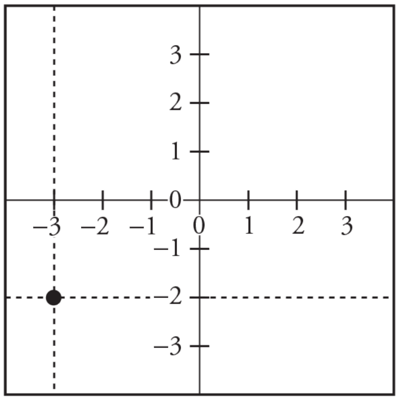
Now you have a unique description of the point's position. There is only one point on the page that is BOTH 2 units to the right of 0 AND 1 unit above 0. So, on a page, you need two numbers to indicate position.

Just as with the number line, information can travel in either direction. If you know the two numbers that give the location, you can place that point on the page:

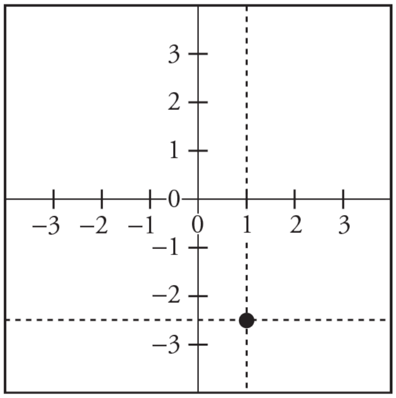
The point is 3 units to the left of 0.

AND

The point is 2 units below 0.



If, on the other hand, you see a point on the page, you can identify its location and determine the two numbers:



The point is 1 unit to the right of 0.

AND

The point is 2.5 units below 0.

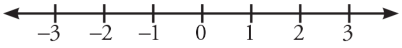
Now that you have two pieces of information for each point, you need to keep straight which number is which. In other words, you need to know which number gives the left-right position and which number gives the up-down position.

To represent the difference, use some technical terms:

The ***x*-coordinate** is the left-right number:

Numbers to the right of 0 are positive.

Numbers to the left of 0 are negative.

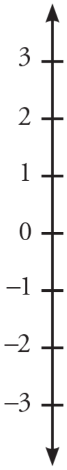


This number line is the ***x*-axis**.

The ***y*-coordinate** is the up-down number:

Numbers above 0 are positive.

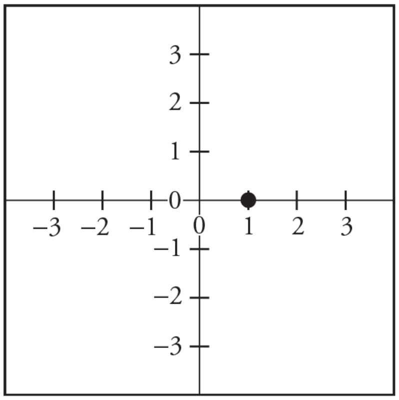
Numbers below 0 are negative.



This number line is the ***y*-axis**.

Now, when describing the location of a point, you can use the technical terms:

The *x*-coordinate of the point is 1 and the *y*-coordinate of the point is 0.

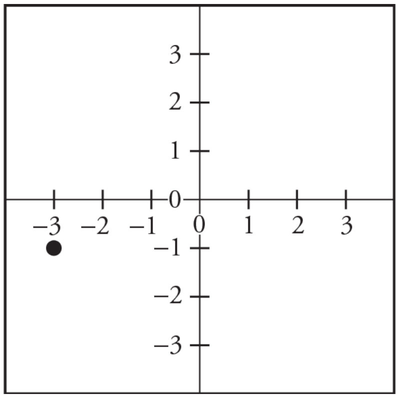


You can condense this and say that, for this point, *x* = 1 and *y* = 0. In fact, you can go even further. You can say that the point is at (1, 0). This shorthand always has the same basic layout. The first number in the parentheses is the *x*-coordinate, and the second number is the *y*-coordinate. One easy way to remember the order in this “ordered pair” is that *x* comes before *y* in the alphabet. For example:

The point is at (−3, −1).

OR

The point has an *x*-coordinate of −3 and a *y*-coordinate of −1.



Now you have a fully functioning **coordinate plane**: an *x*-axis and a *y*-axis drawn on a page. The coordinate plane allows you to determine the unique position of any point on a **plane** (essentially, a really big and flat sheet of paper).

And in case you were ever curious about what one dimensional and two dimensional mean, now you know. A line is one dimensional, because you only need *one* number to identify a point's location. A plane is two-dimensional because you need *two* numbers to identify a point's location.

Check Your Skills

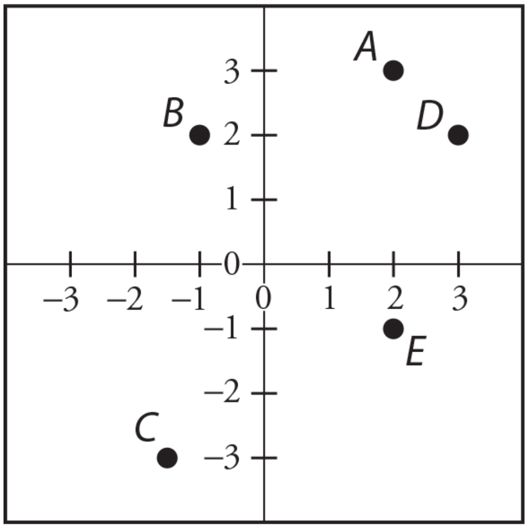
1. Draw a coordinate plane and plot the following points:

   - (3, 1)
   - (−2, 3.5)
   - (0, −4.5)
   - (1, 0)

2. Which point on the coordinate plane shown here is indicated by the following coordinates?

   - (2,−1)
   - (−1.5,−3)
   - (−1,2)
   - (3, 2)

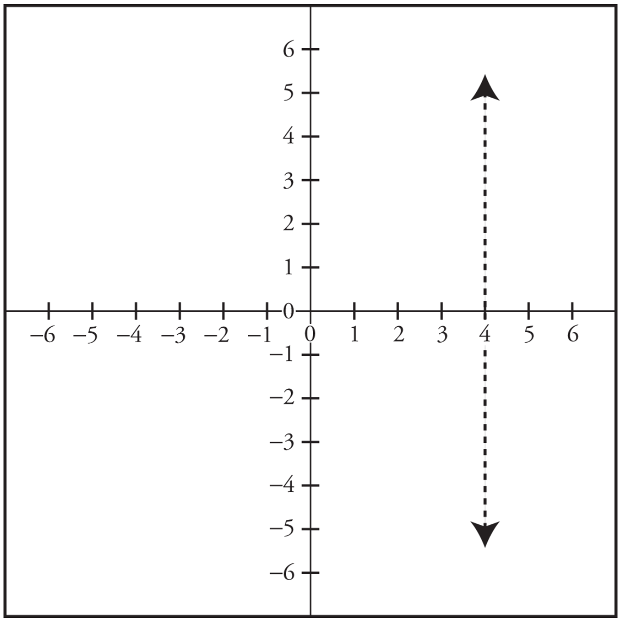
(2, 3)



Knowing Just One Coordinate

As you've just seen, you need to know both the *x*-coordinate and the *y*-coordinate to plot a point exactly on the coordinate plane. If you only know one coordinate, you can't tell precisely where the point is, but you can narrow down the possibilities.

Consider this situation. Say that this is all you know: the point is 4 units to the right of 0:

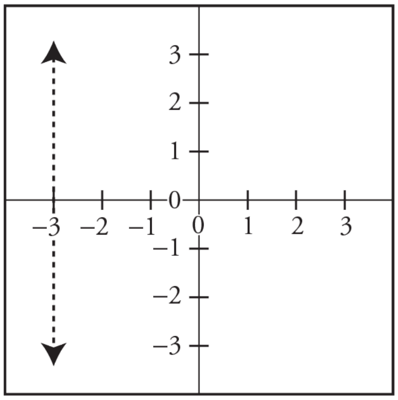


As you saw earlier, any point along the vertical dotted line is 4 units to the right of 0. In other words, every point on the dotted line has an *x*-coordinate of 4. You could shorten that and say $x = 4$. You don't know anything about the *y*-coordinate, which could be any number. All the points along the dotted line have different *y*-coordinates but the same *x*-coordinate, which equals 4.

So, if you know that *x* = 4, then your point can be anywhere along a vertical line that crosses the *x*-axis at (4, 0). Try another example.

If you know that *x* = −3…

Then you know...

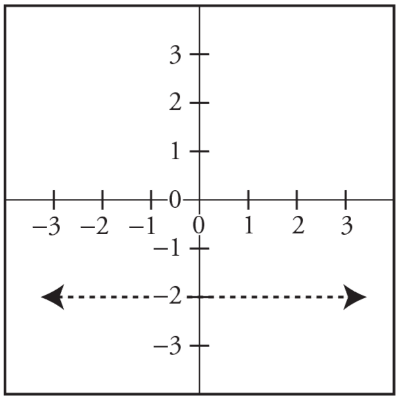


Every point on the dotted line has an *x*-coordinate of −3.

Points on the dotted line include (–3, 1), (–3, –7), (–3, 100), and so on. In general, if you know the *x*-coordinate of a point and not the *y*-coordinate, then all you can say about the point is that it lies somewhere on a vertical line.

The *x*-coordinate still indicates left-right position. If you fix that position but not the up-down position, then the point can only move up and down—forming a vertical line.

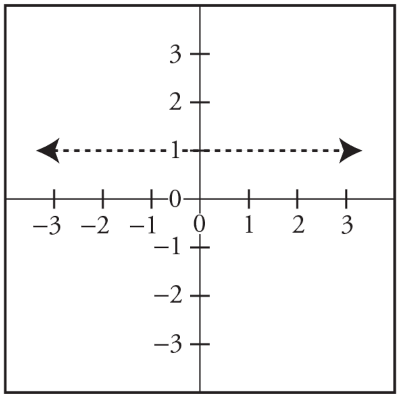
Now imagine that all you know is the *y*-coordinate of a number. Say you know that *y* = −2. How could you represent this on the coordinate plane? In other words, what are all the points for which *y* = –2?



Every point 2 units below 0 fits this condition. These points form a horizontal line. You don't know anything about the *x*-coordinate, which could be any number. All the points along the horizontal dotted line have different *x*-coordinates but the same *y*-coordinate, which equals −2. For instance, (−3, −2), (−2, −2), and (50, −2) are all on the line.

Try another example. If you know that *y* = 1…

Then you know...



Every point on the dotted line has a *y*-coordinate of 1.

If you know the *y*-coordinate but not the *x*-coordinate, then you know the point lies somewhere on a horizontal line.

Check Your Skills

Draw a coordinate plane and plot the following lines.

3. $x = 6$

4. $y = -2$

5. $x = 0$

Knowing Ranges

Now try having even less information. Instead of knowing the actual *x*-coordinate, see what happens if all you know is a range of possible values for *x*. What do you do if all you know is that $x > 0$? To answer that, return to the number line for a moment. As you saw earlier, if $x > 0$, then the target is anywhere to the right of 0:

$x > 0$

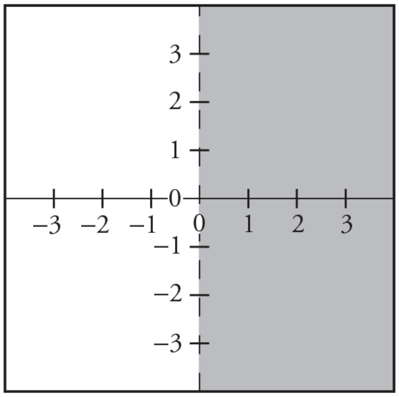
Now look at the coordinate plane. All you know is that *x* is greater than 0. And you don't know *anything* about *y*, which could be any number.

How do you show all the possible points? You can shade in part of the coordinate plane: the part to the right of 0.

If you know that $x > 0$…

Then you know...

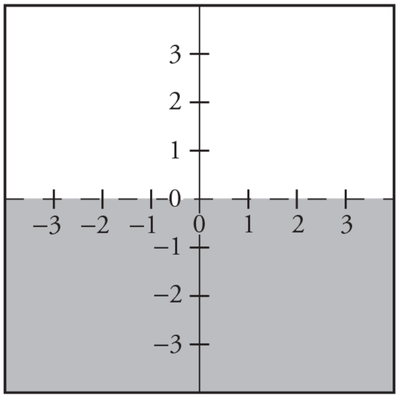
Every point in the shaded region has an *x*-coordinate greater than 0.



Now say that all you know is $y < 0$. Then you can shade in the bottom half of the coordinate plane (see figure that follows)—where the y-coordinate is less than 0. The x-coordinate can be anything. Notice that the dashed line in the plane indicates that y cannot be 0. It must be below the dashed line.

If you know that $y < 0$…

Then you know...

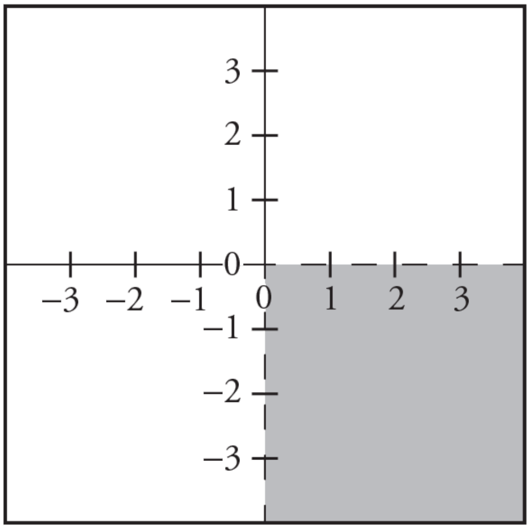


Every point in the shaded region has a y-coordinate less than 0.

Finally, if you know information about both *x* and *y*, then you can narrow down the shaded region.

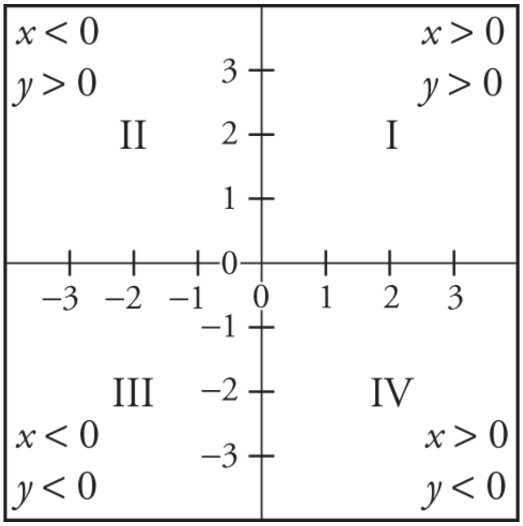
If you know that $x > 0$ AND $y < 0$…

Then you know...



The only place where *x* is greater than 0 AND *y* is less than 0 is the bottom right quarter of the plane. So you know that the point lies somewhere in the bottom right quarter of the coordinate plane.

The four quarters of the coordinate plane are called **quadrants**. Each quadrant corresponds to a different combination of signs of *x* and *y*. The quadrants are always numbered as shown here, starting on the top right and going counter-clockwise:



Check Your Skills

6. In which quadrant do each of the following points lie?

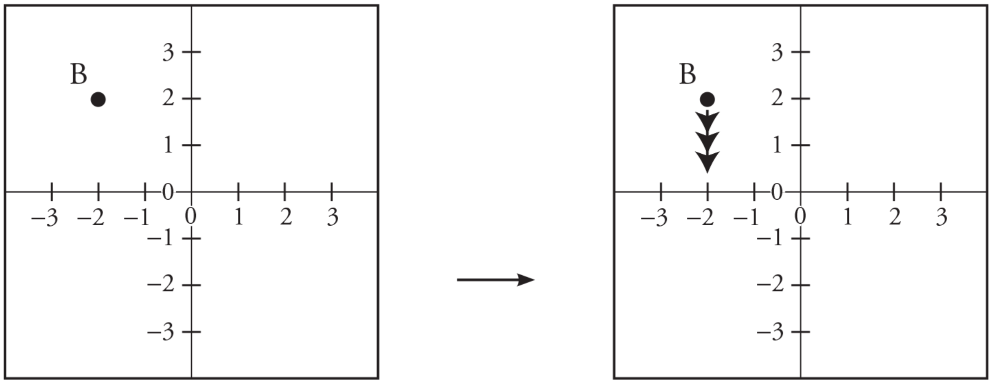
- $(1, -2)$
- $(-4.6, 7)$
- $(-1, -2.5)$
- $(3, 3)$

7. Which quadrant or quadrants are indicated by the following?

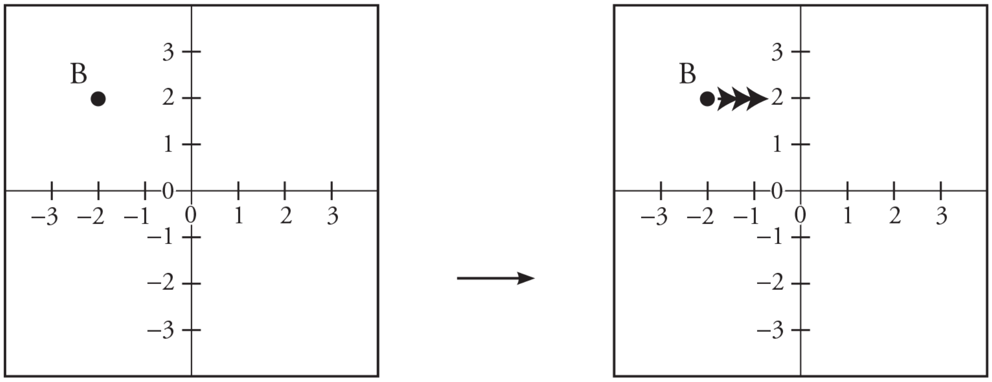
- $x < 0, y > 0$
- $x < 0, y < 0$
- $y > 0$
- $x < 0$

Reading a Graph

If you see a point on a coordinate plane, you can read off its coordinates as follows. To find an *x*-coordinate, drop an imaginary line down to the *x*-axis (if the point is above the *x*-axis) or draw a line up to the *x*-axis (if the point is below the *x*-axis) and read off the number. For example:



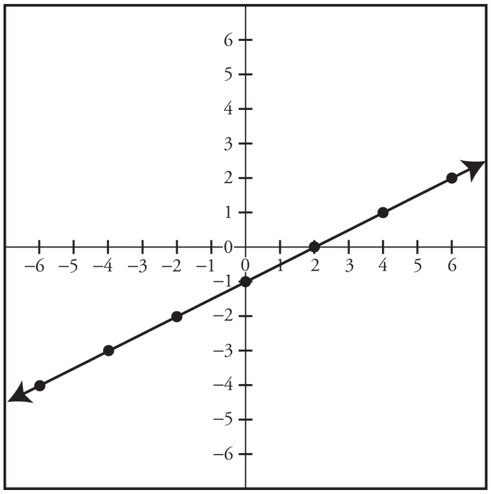
The line hits the *x*-axis at −2, which means the *x*-coordinate of your point is −2. Now, to find the *y*-coordinate, you employ a similar technique, only now you draw a horizontal line instead of a vertical line:



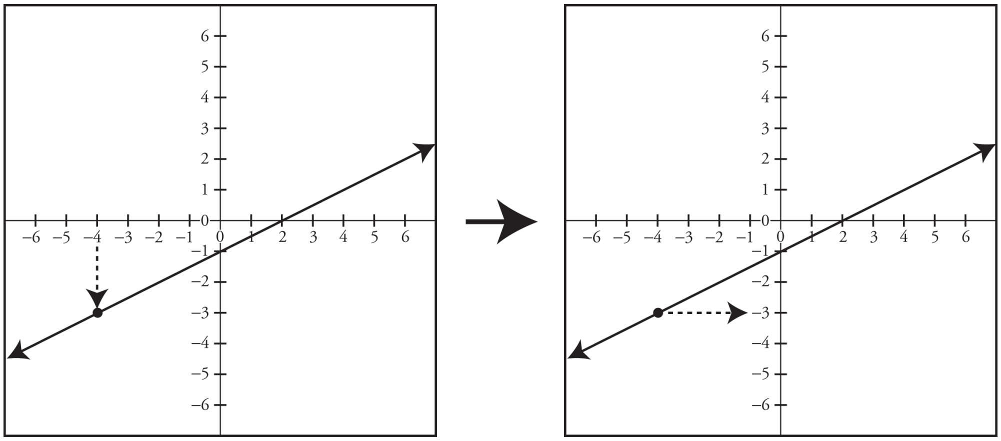
The line touched the *y*-axis at 2, which means the *y*-coordinate of the point is 2. Thus, the coordinates of point B are (−2, 2).

Now suppose that you know the target is on a slanted line in the plane. You can read coordinates off of this slanted line. Try this problem on your own first:

> On the line shown, what is the *y*-coordinate of the point that has an *x*-coordinate of −4?



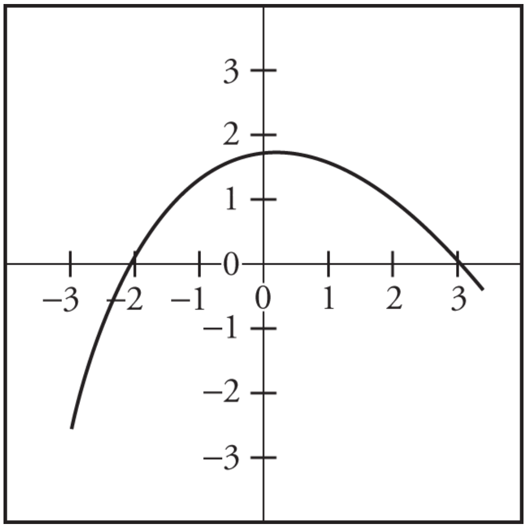
To answer this question, think about reading the coordinates of a point. You went from the point to the axes. Here, you will go from the axis that you know (here, the *x*-axis) to the line that contains the point, and then to the *y*-axis (the axis you don't know):



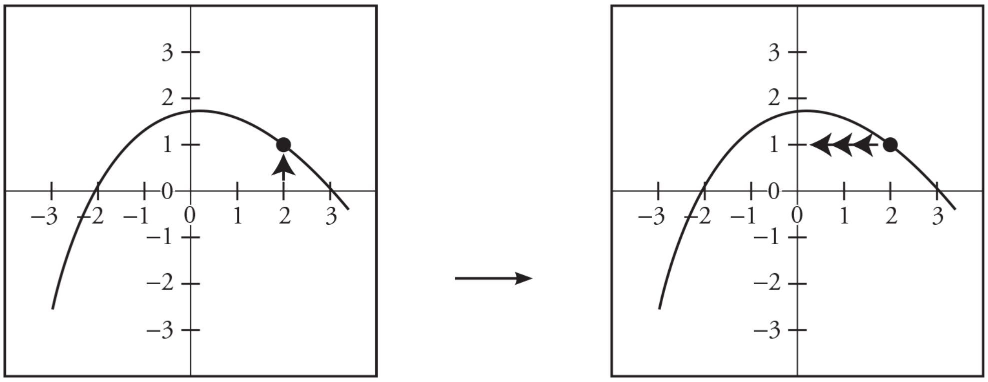
So the point on the line that has an x-coordinate of −4 has a y-coordinate of −3.

This method of locating points applies equally well to any shape or curve you may encounter on a coordinate plane. Try this next problem:

> On the curve shown, what is the value of y when $x = 2$?



Once again, you know the x-coordinate, so draw a line from the x-axis (where you know the coordinate) to the curve, and then draw a line to the *y*-axis:

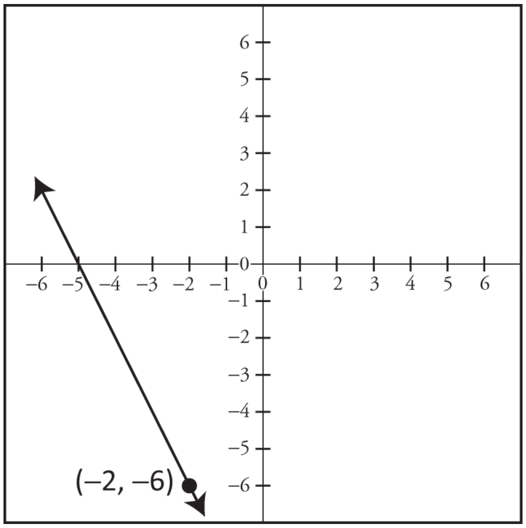


On the curve shown, the point that has an *x*-coordinate of 2 has a *y*-coordinate of 1.

Note that the GRE will mathematically define each line or curve, so you will never be forced to guess visually where a point falls. In fact, if more specific information is *not* given for a coordinate problem on the GRE, you *cannot* infer the location of a point based solely on visual cues. This discussion is *only* meant as an exercise to convey how to use any graphical representation.

Check Your Skills

8. On the following graph, what is the *y*-coordinate of the point on the line that has an *x*-coordinate of −3?


6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−6
−6
−5
−4
−3
−2
−1
0
1
2
3
4
5
6
(−2, −6)

Plotting a Relationship

The most frequent use of the coordinate plane is to display a relationship between *x* and *y*. Often, this relationship is expressed this way: if you tell me *x*, I can tell you *y*.

As an equation, this sort of relationship looks like this:

y = some expression involving x

Another way of saying this is “We have *y* in terms of *x*.”

Examples:

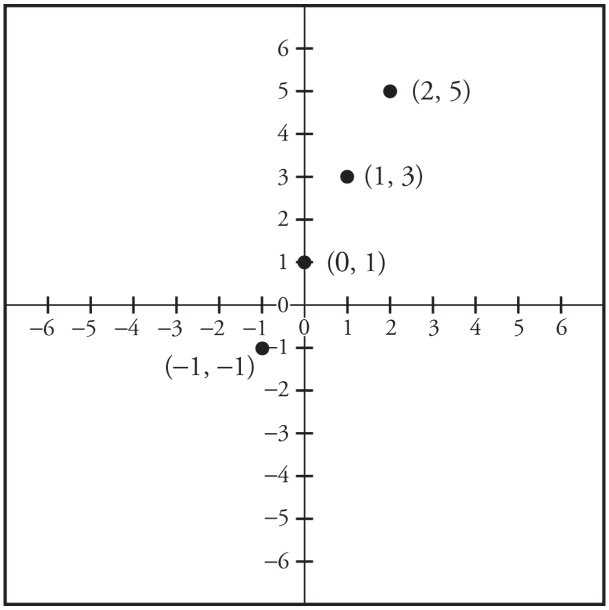
$$\begin{aligned} y &= 2x + 1 \\ y &= x^2 - 3x + 2 \\ y &= \frac{x}{x + 2} \end{aligned}$$

If you plug in a number for *x* in any of these equations, you can calculate a value for *y*.

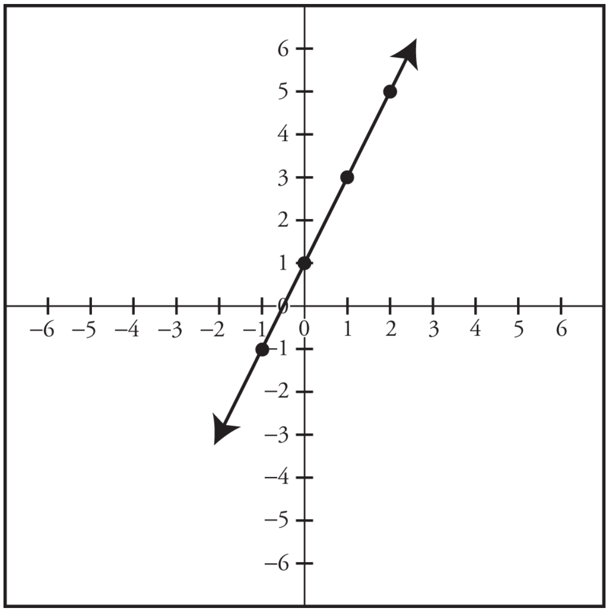
For example, take $y = 2x + 1$. You can generate a set of *y*’s by plugging in various values of *x*. Start by making a table:

x	$y = 2x + 1$
-1	$y = 2(-1) + 1 = -1$
0	$y = 2(0) + 1 = 1$
1	$y = 2(1) + 1 = 3$
2	$y = 2(2) + 1 = 5$

Now that you have some values, see what you can do with them. You can say that when *x* equals 0, *y* equals 1. These two values form a pair. You express this connection by plotting the point (0, 1) on the coordinate plane. Similarly, you can plot all the other points that represent an *x*-*y* pair from your table:

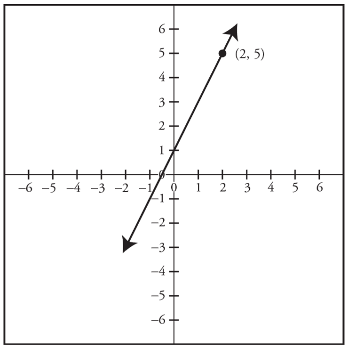


You might notice that these points seem to lie on a straight line. You’re right—they do. In fact, any point that you can generate using the relationship $y = 2x + 1$ will also lie on the line:

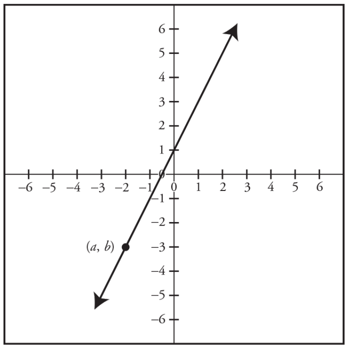


This line is the graphical representation of $y = 2x + 1$.

So now you can talk about equations in visual terms. In fact, that's what lines and curves on the coordinate plane are—they represent all the (x, y) pairs that make an equation true. Take a look at the following example:

	⟷	$y = 2x + 1$ $5 = 2(2) + 1$
The point (2, 5) lies on the line $y = 2x + 1$	⟷	If you plug in 2 for x in $y = 2x + 1$, you get 5 f

You can even speak more generally, using variables:



↔

$y = 2x + 1$

$b = 2(a) + 1$

The point (a, b) lies on the line $y = 2x + 1$ ⟷ If you plug in a for x in $y = 2x + 1$, you get b

Check Your Skills

9. True or False? The point (9, 21) is on the line $y = 2x + 1$.

10. True or False? The point (4, 14) is on the curve $y = x^2 - 2$.

Lines in the Plane

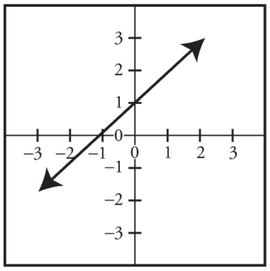
The relationship $y = 2x + 1$ formed a line in the coordinate plane, as you saw. You can actually generalize this relationship. *Any* relationship of the following form represents a line:

$y = mx + b$	m and b represent numbers (positive, negative, or 0)

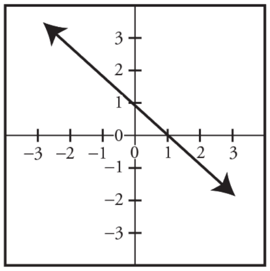
For instance, in the equation $y = 2x + 1$, you can see that $m = 2$ and $b = 1$:

Lines		Not Lines
$y = 3x - 2$	$m = 3, b = -2$	$y = x^2$
$y = -x + 4$	$m = -1, b = 4$	$y = \frac{1}{x}$
These are called linear equations.		These equations are not linear.

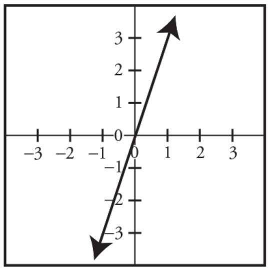
The numbers m and b have special meanings when you are dealing with linear equations. First, m = **slope**. This tells you how steep the line is and whether the line is rising or falling:



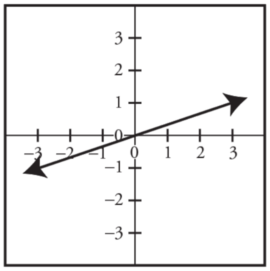
Positive Slope
$m > 0$



Negative Slope
$m < 0$

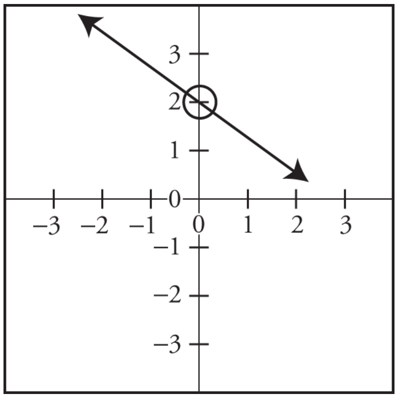


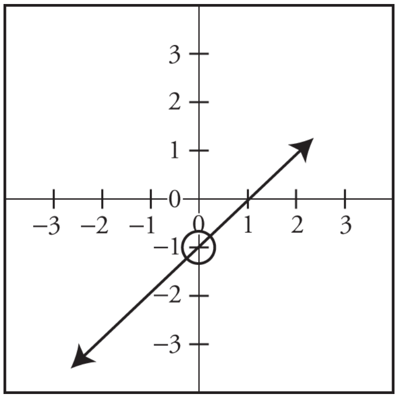
Steep Slope
$m > 1$



Gentle Slope
$0 < m < 1$

Next, b = ***y*-intercept**. This tells you where the line crosses the y-axis. Any line or curve crosses the y-axis when $x = 0$. To find the y-intercept, plug in 0 for x in the equation:





By recognizing linear equations and identifying m and b, you can plot a line more quickly than by plotting several points on the line.

Check Your Skills

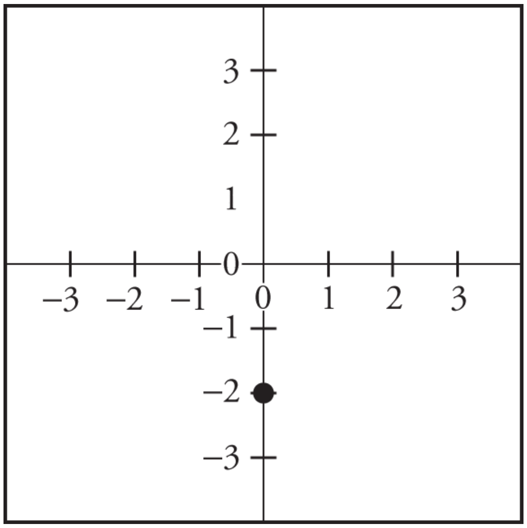
Find the slope and y-intercept of the following lines.

11. $y = 3x + 4$

12. $2y = 5x - 12$

Now the question becomes, how do you use *m* and *b* to sketch a line? Plot the line $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$.

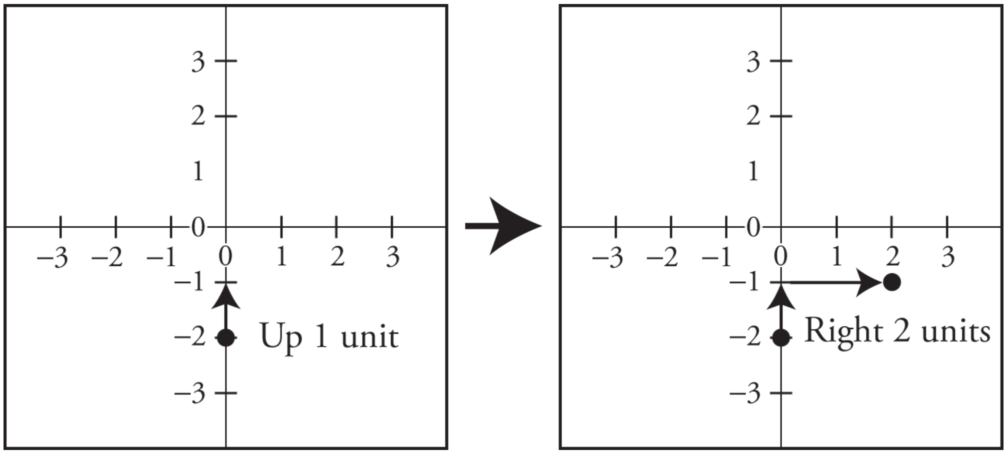
The easiest way to begin graphing a line is to begin with the *y*-intercept. You know that the line crosses the *y*-axis at $y = -2$, so begin by plotting that point on your coordinate plane:



Now you need to figure out how to use slope to finish drawing your line. Every slope, whether an integer or a fraction, should be thought of as a fraction. In this equation, *m* is $\frac{1}{2}$. Look at the parts of the fraction and see what they can tell you about your slope:

$$\frac{1}{2} \begin{matrix}\rightarrow\\\rightarrow\end{matrix} \frac{\text{Numerator}}{\text{Denominator}} \begin{matrix}\rightarrow\\\rightarrow\end{matrix} \frac{\text{Rise}}{\text{Run}} \begin{matrix}\rightarrow\\\rightarrow\end{matrix} \frac{\text{Change in } y}{\text{Change in } x}$$

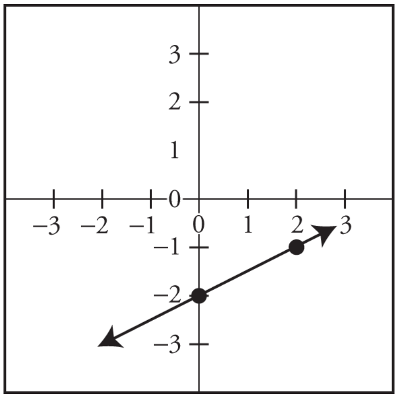
The numerator of your fraction tells you how many units you want to move in the *y* direction—in other words, how far up or down you want to move. The denominator tells you how many units you want to move in the *x* direction—in other words, how far left or right you want to move. For this particular equation, the slope is $\frac{1}{2}$, which means you want to move up 1 unit and right 2 units:



After you went up 1 unit and right 2 units, you ended up at the point (2, −1). What that means is that the point (2, −1) is also a solution to the equation $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$. In fact, you can plug in the *x* value and solve for *y* to check that you did this correctly:

$$\frac{1}{4} + \frac{1}{3} = \frac{1 \times 3}{4 \times 3} + \frac{1 \times 4}{3 \times 4} = \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$$

What this means is that you can use the slope to generate points and draw your line. If you go up another 1 unit and right another 2 units, you will end up with another point that appears on the line. Although you could keep doing this indefinitely, in reality, with only two points you can figure out what your line looks like. Now all you need to do is draw the line that connects the two points you have, and you’re done:

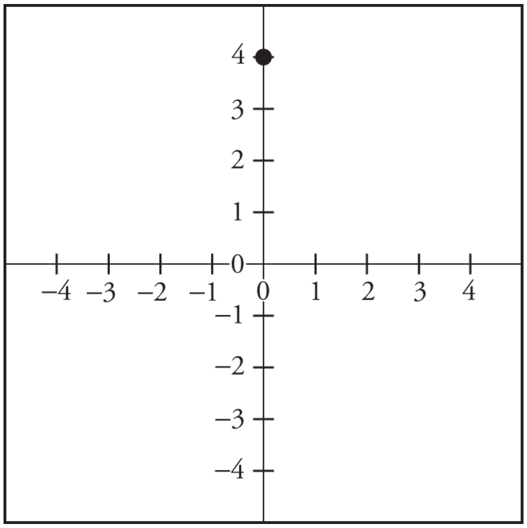


$$\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$$

This line is the graphical representation of $\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$.

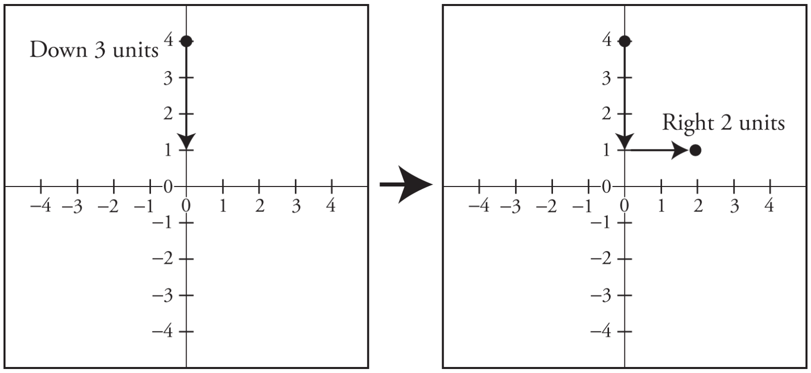
Try another one. Graph the equation $y=\left(-\frac{3}{2}\right)x+4$.

Once again, the best way to start is to plot the *y*-intercept. In this equation, *b* = 4, so you know the line crosses the *y*-axis at the point (0, 4):

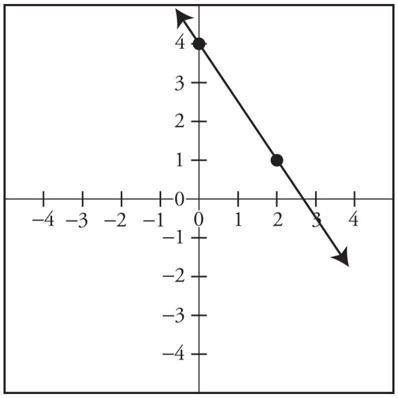


Now you can use the slope to find a second point. This time, the slope is $-\frac{9}{2}$, which is a negative slope.

While positive slopes go up and to the right, negative slopes go down and to the right. You might think of it this way: if the “rise” is negative, that is like a “drop” or “fall.” Now, to find the next point, you need to go *down* 3 units and right 2 units:



That means that (2, 1) is another point on the line. Now that you have two points, you can draw your line:



$$y = \left(-\frac{3}{2}\right)x + 4$$

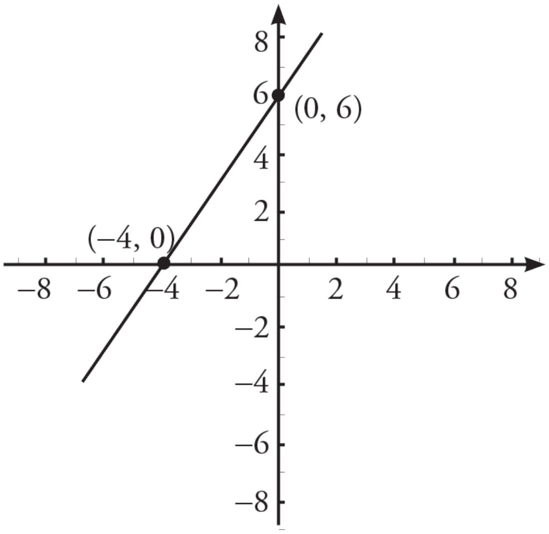
Check Your Skills

13. Draw a coordinate plane and graph the line $y = 2x - 4$. Identify the slope and the y-intercept.

The Intercepts of a Line

A point where a line intersects a coordinate axis is called an **intercept**. There are two types of intercepts: the *x*-intercept, where the line intersects the *x*-axis, and the *y*-intercept, where the line intersects the *y*-axis.

The *x*-intercept is expressed using the ordered pair (x, 0), where x is the point where the line intersects the *x*-axis. **The *x*-intercept is the point on the line at which $y = 0$.** In this diagram, the *x*-intercept is −4, as expressed by the ordered pair (−4, 0).



The *y*-intercept is expressed using the ordered pair (0, y), where y is the point where the line intersects the *y*-axis. **The *y*-intercept is the point on the line at which $x = 0$.** In this diagram, the *y*-intercept is 6, as expressed by the ordered pair (0, 6).

To find *x*-intercepts, plug in 0 for *y*. To find *y*-intercepts, plug in 0 for *x*.

Check Your Skills

14. What are the *x*- and *y*-intercepts of the equation $x - 2y = 8$?

The Intersection of Two Lines

Recall that a line in the coordinate plane is defined by a linear equation relating x and y. That is, if a point (x, y) lies on the line, then those values of x and y satisfy the equation. For instance, the point (3, 2) lies on the line defined by the equation $y = 4x - 10$, because the equation is true when you plug in $x = 3$ and $y = 2$:

$$y = 4x - 10$$
$$2 = 4(3) - 10 = 12 - 10$$
$$2 = 2 \quad \text{True}$$

On the other hand, the point (7, 5) does not lie on that line, because the equation is false when you plug in $x = 7$ and $y = 5$:

$$y = 4x - 10$$
$$5 = 4(7) - 10 = 28 - 10 = 18? \quad \text{False}$$

So, what does it mean when two lines intersect in the coordinate plane? It means that at the point of intersection, BOTH equations representing the lines are true. That is, the pair of numbers (x, y) that represents the point of intersection solves BOTH equations. Finding this point of intersection is equivalent to solving a system of two linear equations. You can find the intersection by using algebra more easily than by graphing the two lines. Try this example:

> At what point does the line represented by $y = 4x - 10$ intersect the line represented by $2x + 3y = 26$?

Because $y = 4x - 10$, replace y in the second equation with $4x - 10$ and solve for x:

$$2x + 3(4x - 10) = 26$$
$$2x + 12x - 30 = 26$$
$$14x = 56$$
$$x = 4$$

Now solve for y. You can use either equation, but the first one is more convenient:

$$y = 4x - 10$$
$$y = 4(4) - 10$$
$$y = 16 - 10 = 6$$

Thus, the point of intersection of the two lines is (4, 6).

If two lines in a plane do not intersect, then the lines are parallel. If this is the case, there is *no* pair of numbers (x, y) that satisfies both equations at the same time.

Two linear equations can represent two lines that intersect at a single point, or they can represent parallel lines that never intersect. There is one other possibility: the two equations might represent the same line. In this case, infinitely many points (x, y) along the line satisfy the two equations (one of which must actually be the other equation in disguise).

The Distance Between Two Points

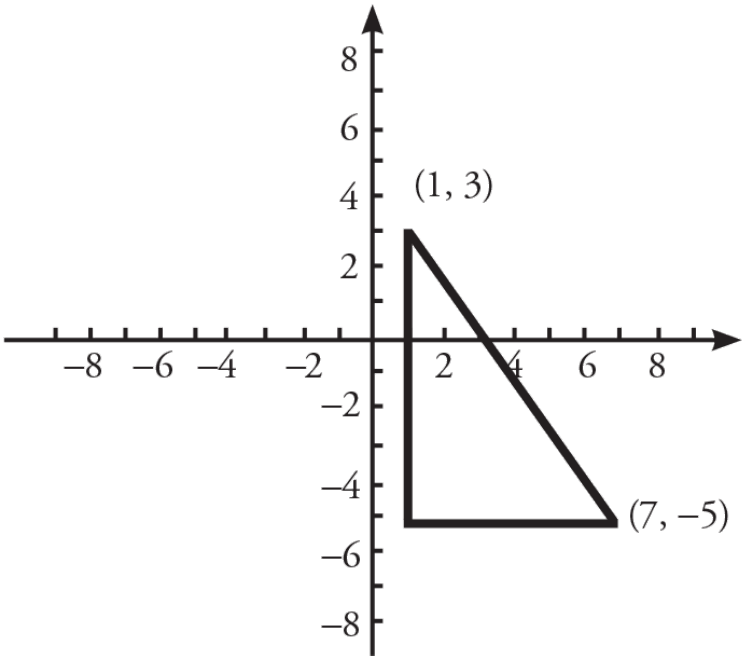
The distance between any two points in the coordinate plane can be calculated by using the Pythagorean theorem. For example:

> What is the distance between the points (1, 3) and (7, −5)?

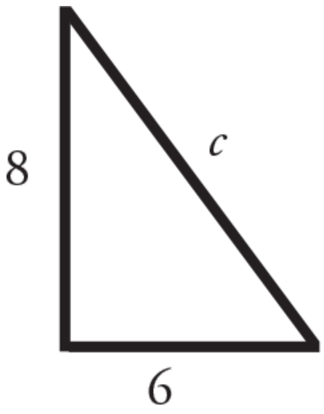
First, draw a right triangle connecting the points.

Second, find the lengths of the two legs of the triangle by calculating the rise and the run:

> The *y*-coordinate changes from 3 to −5, a difference of 8 (the vertical leg).
>
> The *x*-coordinate changes from 1 to 7, a difference of 6 (the horizontal leg).



Third, use the Pythagorean theorem to calculate the length of the diagonal, which is the distance between the points:



$6^2 + 8^2 = c^2$

$36 + 64 = c^2$

$100 = c^2$

$c = 10$

The distance between the two points is 10 units.

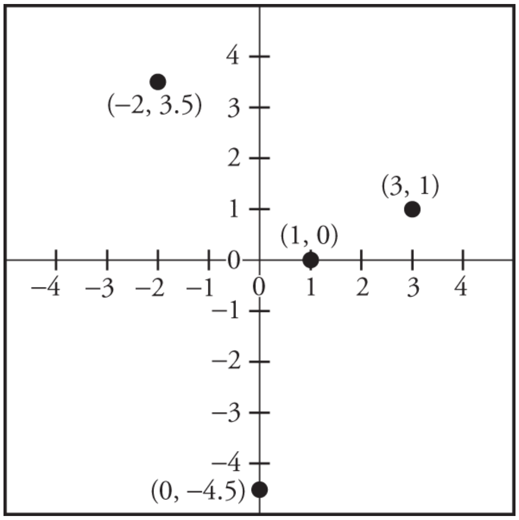
Alternatively, to find the hypotenuse, you may have recognized this triangle as a variation of a 3–4–5 triangle (specifically, a 6–8–10 triangle).

Check Your Skills

15. What is the distance between (−2, −4) and (3, 8)?

Check Your Skills Answer Key

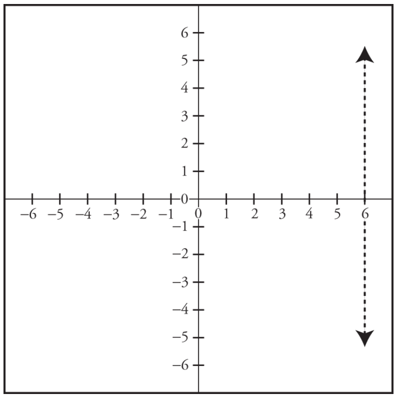
1.



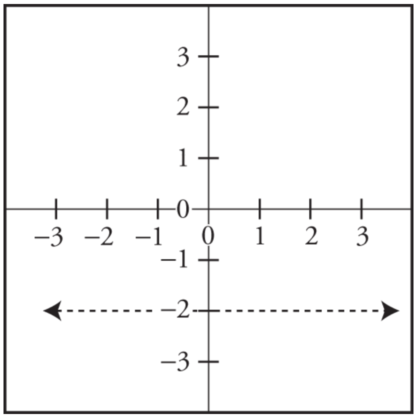
2. 1. (2, −1): **E**
   2. (−1.5, −3): **C**
   3. (−1, 2): **B**
   4. (3, 2): **D**
   5. (2, 3): **A**

3.

$x = 6$

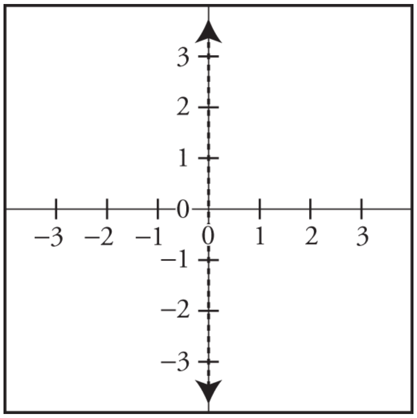


4.



$y = -2$

5.



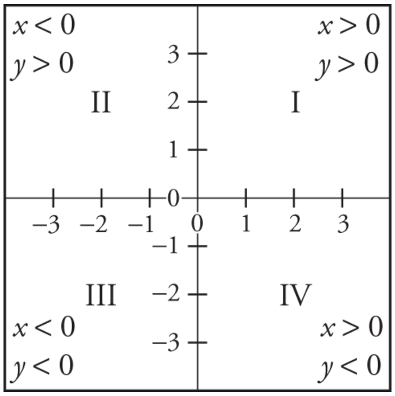
$x = 0$ is the y-axis.

6. (1, −2) is in **Quadrant IV**

(−4.6, 7) is in **Quadrant II**

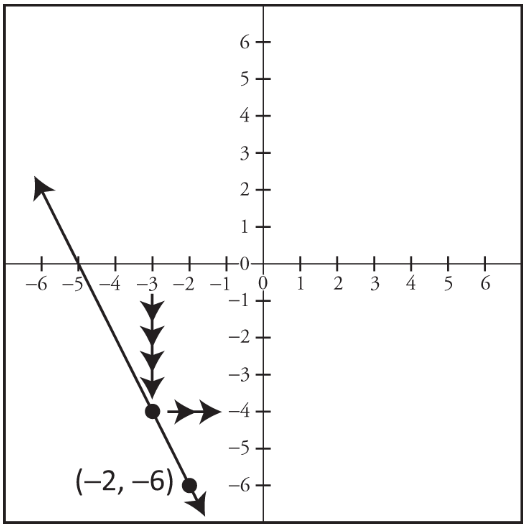
(−1, −2.5) is in **Quadrant III**

(3, 3) is in **Quadrant I**



7. $x < 0$, $y > 0$ indicates **Quadrant II**
$x < 0$, $y < 0$ indicates **Quadrant III**
$y > 0$ indicates **Quadrants I and II**
$x < 0$ indicates **Quadrants II and III**

8. The point on the line with $x = -3$ has a y-coordinate of −4.



9. **False**

The relationship is $y = 2x + 1$, and the point you are testing is (9, 21). So plug in 9 for x and see what you get: $y = 2(9) + 1 = 19$. The point (9, 21) does not lie on the line.

10. **True**

    The relationship is $y = x^2 - 2$, and the point you are testing is (4, 14). So plug in 4 for x and see what you get: $y = (4)^2 - 2 = 14$. The point (4, 14) lies on the curve.

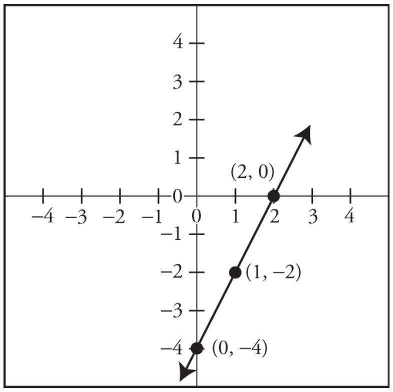
11. **Slope is 3, *y*-intercept is 4**

    The equation $y = 3x + 4$ is already in $y = mx + b$ form, so you can directly find the slope and y-intercept. The slope is 3, and the y-intercept is 4.

12. **Slope is 2.5, *y*-intercept is −6**

    To find the slope and y-intercept of a line, you need the equation to be in $y = mx + b$ form. You need to divide the original equation by 2 to make that happen. So $2y = 5x - 12$ becomes $y = 2.5x - 6$. So the slope is 2.5 (or 5/2) and the y-intercept is −6.

13. **Slope is 2, *y*-intercept is −4**

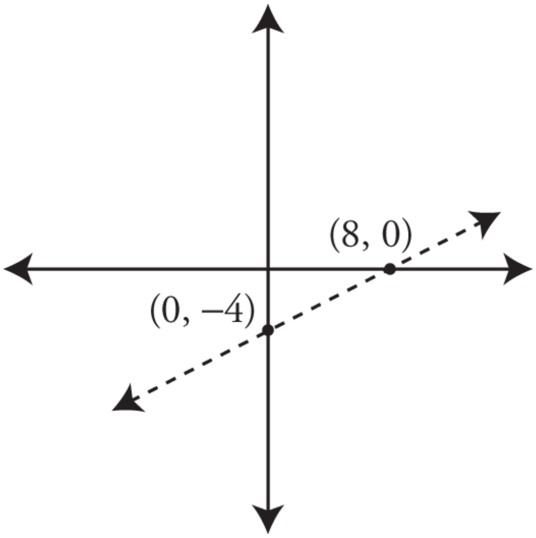


$y = 2x - 4$

slope = 2

y-intercept = −4

14. ***x*-intercept is 8, *y*-intercept is −4**



The line is drawn on the coordinate plane shown here, but you can also answer this question using algebra.

To determine the *x*-intercept, set *y* equal to 0, then solve for *x*:

$x - 2y = 8$

$y = 0$

$x - 0 = 8$

$x = 8$

To determine the *y*-intercept, set *x* equal to 0, then solve for *y*:

$x - 2y = 8$

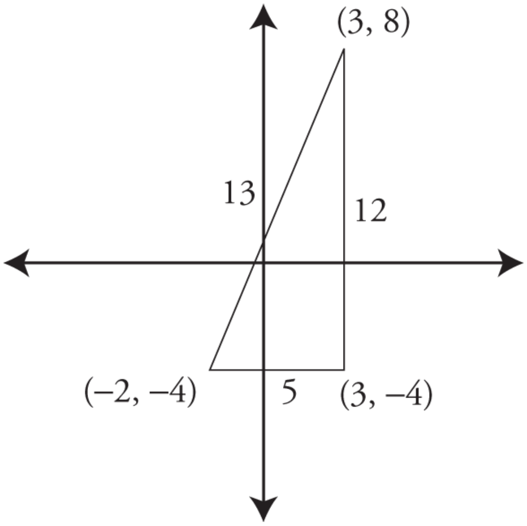
$x = 0$

$0 - 2y = 8$

$-2y = 8$

$y = -4$

15. **13**



The illustration shows the two points. A right triangle has been constructed by finding a point directly below (3, 8) and directly to the right of (−2, −4). This right triangle has legs of 5 (the change from −2 to 3) and 12 (the change from −4 to 8). You can plug those values into the Pythagorean theorem and solve for the hypotenuse:

$$
\begin{aligned}
&A^2 + B^2 = C^2\\
&5^2 + 12^2 = C^2\\
&25 + 144 = C^2\\
&C^2 = 169\\
&C = \sqrt{169} = 13
\end{aligned}
$$

Alternatively, you could recognize the common Pythagorean triple 5–12–13.

Problem Set

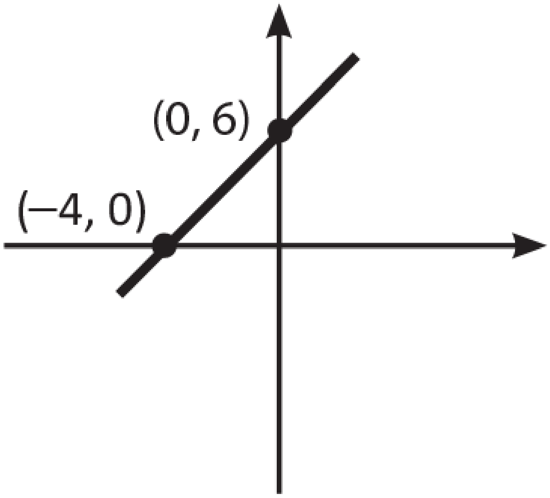
1. A line has the equation $y = 3x + 7$. At which point will this line intersect the y-axis?
2. A line has the equation $x = \frac{y}{80} - 20$. At which point will this line intersect the x-axis?
3. A line has the equation $x = -2y + z$. If (3, 2) is a point on the line, what is z?
4. A line is represented by the equation $y = zx + 18$. If this line intersects the x-axis at $(-3, 0)$, what is z?
5. A line has a slope of 1/6 and intersects the x-axis at $(-24, 0)$. Where does this line intersect the y-axis?
6. Which quadrants, if any, do not contain any points on the line represented by $x - y = 18$?

7. Which quadrants, if any, do not contain any points on the line represented by $x = 10y$?

8. Which quadrants contain points on the line $y = \frac{x}{1,000} + 1,000,000$?

9. Which quadrants contain points on the line represented by $x + 18 = 2y$?

Use this image to answer questions 10 and 11.



10. What is the equation of the line shown?

11. What is the intersection point of the lines defined by the equations $2x + y = 7$ and $3x - 2y = 21$?

12.

Quantity A	**Quantity B**
The y-intercept of the line $\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$	The x-intercept of the line $\frac{2}{9}+\frac{5}{9}=\frac{7}{9}$

13.

Quantity A	**Quantity B**
The slope of the line $2x + 5y = 10$	The slope of the line $5x + 2y = 10$

14.

Quantity A	**Quantity B**
The distance between points $(0, 9)$ and $(-2, 0)$	The distance between points $(3, 9)$ and $(10, 3)$

Solutions

1. **(0, 7)**

   A line intersects the *y*-axis at the *y*-intercept. Because this equation is written in slope-intercept form, the *y*-intercept is easy to identify: 7. Thus, the line intersects the *y*-axis at the point (0, 7).

2. **(−20, 0)**

   A line intersects the *x*-axis at the *x*-intercept, or when the *y*-coordinate is equal to 0. Substitute 0 for *y* and solve for *x*:

   $$x = 0 - 20$$
   $$x = -20$$

3. **7**

   Substitute the coordinates (3, 2) for *x* and *y* and solve for *z*:

   $$3 = -2(2) + z$$
   $$3 = -4 + z$$
   $$z = 7$$

4. **6**

   Substitute the coordinates (−3, 0) for *x* and *y* and solve for *z*:

   $$0 = z(-3) + 18$$
   $$3z = 18$$

$$z = 6$$

5. **(0, 4)**

   Use the information given to find the equation of the line:

$$\begin{aligned} \text{Area} &= \frac{b \times h}{2} \\ 12.5 &= \frac{5x}{2} \\ 25 &= 5x \\ x &= 5 \end{aligned}$$

   The variable *b* represents the *y*-intercept. Therefore, the line intersects the *y*-axis at (0, 4).

6. **II**

   First, rewrite the line in slope-intercept form:

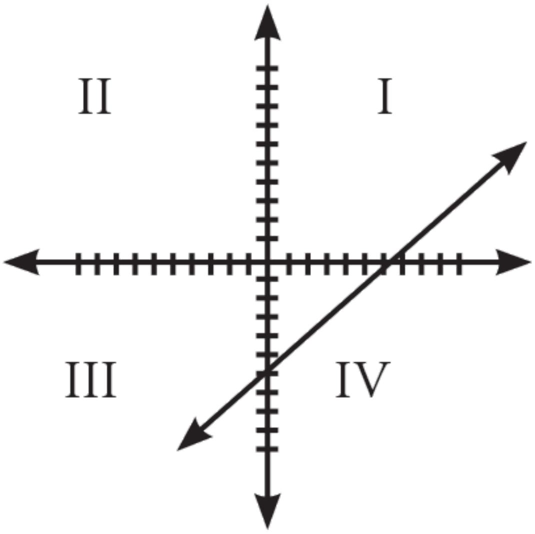
$$y = x - 18$$

   Find the intercepts by setting *x* to 0 and *y* to 0:

$$\begin{aligned} y &= 0 - 18 \\ y &= -18 \end{aligned}$$

$$\begin{aligned} 0 &= x - 18 \\ x &= 18 \end{aligned}$$

   Plot the points: (0, −18), and (18, 0). From the sketch, you can see that the line does not pass through quadrant II.

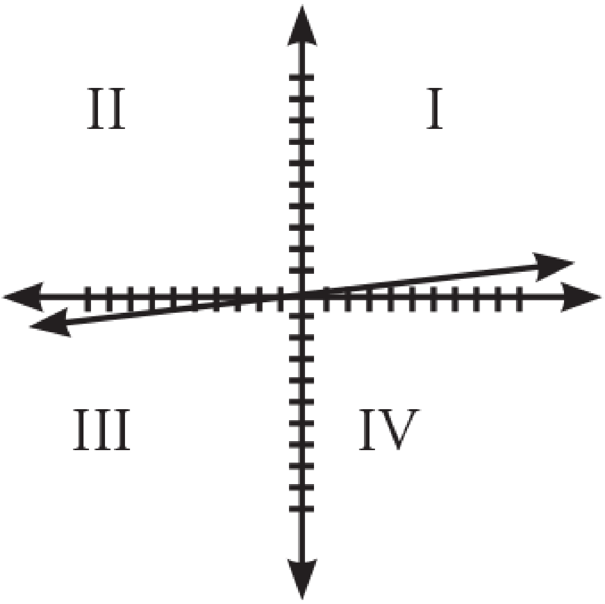


7. **II and IV**

   First, rewrite the line in slope-intercept form:

   $$\frac{a}{b} > 0$$

   Notice from the equation that the *y*-intercept of the line is (0,0). This means that the line crosses the *y*-intercept at the origin, so the *x*- and *y*-intercepts are the same. To find another point on the line, substitute any convenient number for *x*; in this case, 10 would be a convenient, or “smart,” number:



$$\frac{90}{100} = 90\%$$

The point (10, 1) is on the line.

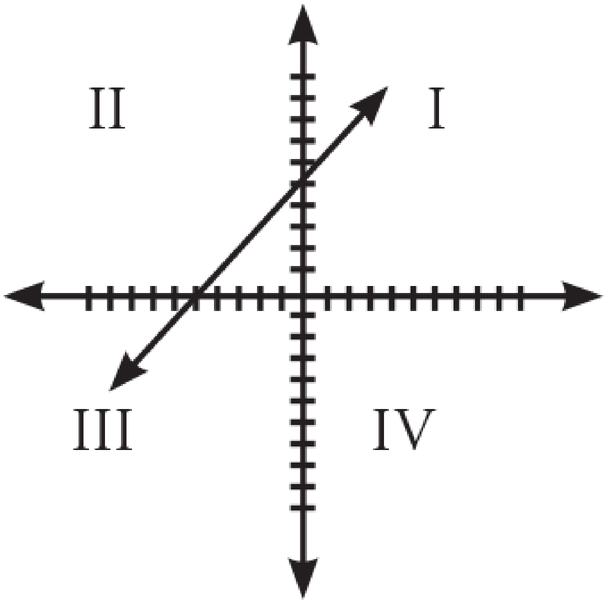
Plot the points: (0, 0) and (10, 1). From the sketch, you can see that the line does not pass through quadrants II and IV.

8. **I, II, and III**

   The line is already written in slope-intercept form:

$$y = \frac{x}{1{,}000} + 1{,}000{,}000$$

   Find the intercepts by setting *x* to 0 and *y* to 0:



$$0 = \frac{x}{1{,}000} + 1{,}000{,}000$$
$$x = -1{,}000{,}000{,}000$$

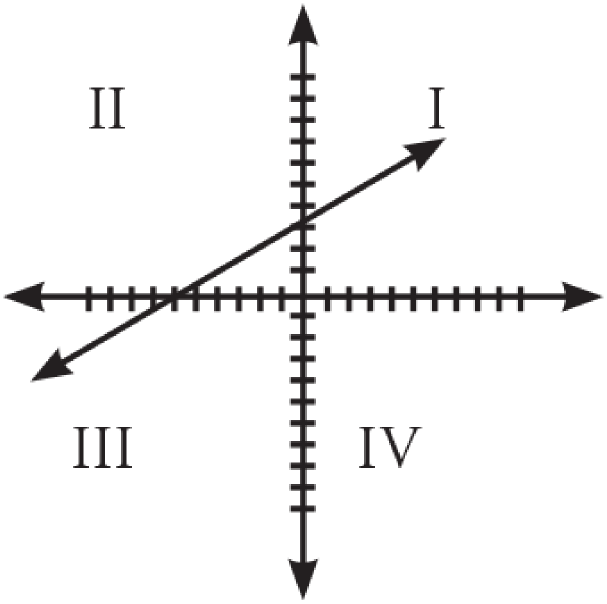
$$y = \frac{0}{1{,}000} + 1{,}000{,}000$$
$$y = 1{,}000{,}000$$

Plot the points: (−1,000,000,000, 0) and (0, 1,000,000). From the sketch, which is obviously not to scale, you can see that the line passes through quadrants I, II, and III.

9. **I, II, and III**
First, rewrite the line in slope-intercept form:

$$y = \frac{x}{2} + 9$$

Find the intercepts by setting *x* to 0 and *y* to 0:



$$0 = \frac{x}{2} + 9$$
$$x = -18$$

$$y = \frac{0}{2} + 9$$
$$y = 9$$

Plot the points: (−18, 0) and (0, 9). From the sketch, you can see that the line passes through quadrants I, II, and III.

10. $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$

First, calculate the slope of the line:

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{6-0}{0-(-4)} = \frac{6}{4} = \frac{3}{2}$$

You can see from the graph that the line crosses the *y*-axis at (0, 6). The equation of the line is:

$$\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$$

11. **(5, −3)**

To find the coordinates of the point of intersection, solve the system of two linear equations. You could turn both equations into slope-intercept form and set them equal to each other, but it is easier to multiply the first equation by 2 and then add the second equation:

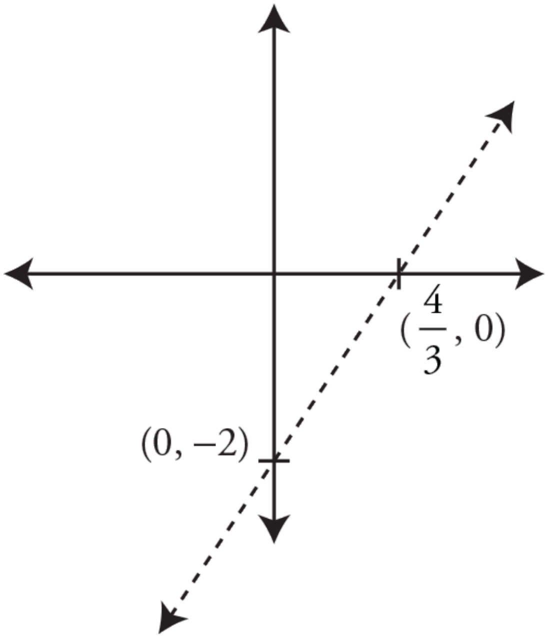
$2x + y = 7$ (first equation)
$4x + 2y = 14$ (first equation multipied by 2)
$3x - 2y = 21$ (second equation)
$7x = 35$ (sum of previous two eqati
$x = 5$

Now plug $x = 5$ into either equation:

$2x + y = 7$ (first equation)
$2(5) + y = 7$
$10 + y = 7$
$y = -3$

Thus, the point (5, −3) is the point of intersection. There is no need to graph the two lines to find the point of intersection.

12. **(B)**



The line is illustrated on the coordinate plane shown. Because the equation is already in slope intercept form ($y = mx + b$), you can read the y-intercept directly from the b position, and use the slope to determine the x-intercept. A slope of 3/2 means that the line corresponding to this equation will rise 3 for every 2 that it runs. You don't need to determine the exact x-intercept to see that it is positive, and so greater than −2.

Alternatively, you could set each variable equal to 0 and determine the intercepts.

To determine the y-intercept, set x equal to 0, then solve for y:

$$y = \frac{3}{2}x - 2$$
$$y = \frac{3}{2}(0) - 2$$
$$y = 0 - 2 = -2$$

To determine the *x*-intercept, set *y* equal to 0, then solve for *x* :

$$y = \frac{3}{2}x - 2$$
$$(0) = \frac{3}{2}x - 2$$
$$2 = \frac{3}{2}x$$
$$\frac{4}{3} = x$$

Quantity A	Quantity B
The *y*-intercept of the line	The *x*-intercept of the line
-2	$\frac{1}{2}$

Therefore, **Quantity B is greater**.

13. **(A)**

The best method would be to put each equation into slope-intercept form (*y* = *mx* + *b*) and see which has the greater value for *m*, which represents the slope. Start with the equation in Quantity A:

$$2x + 5y = 10$$
$$5y = -2x + 10$$
$$y = -\frac{2}{5}x + 2$$

Quantity A	**Quantity B**
The slope of the line $2x + 5y = 10$ is $-\frac{9}{2}$	The slope of the line $5x + 2y = 10$

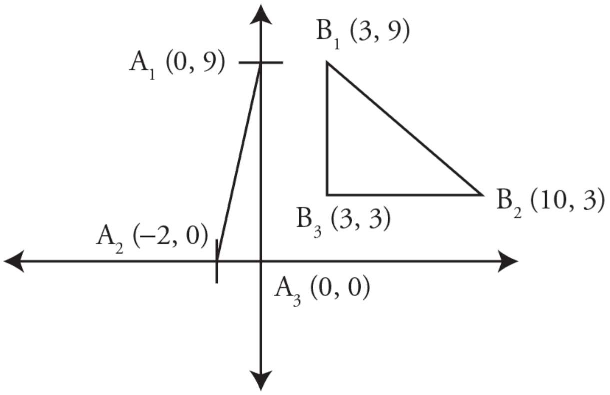
Now find the slope of the equation in Quantity B:

$$5x + 2y = 10$$
$$2y = -5x + 10$$
$$y = -\frac{5}{2}x + 5$$

Quantity A	**Quantity B**
$-\frac{9}{2}$	The slope of the line $5x + 2y = 10$ is $-\frac{9}{2}$

Be careful. Remember that $\frac{127}{255}$ or $\frac{162}{320}$. Therefore, **Quantity A is greater**.

14. **(C)**



The illustration shows the two points from Quantity A, here labeled A_1 and A_2, and the two points from Quantity B, here labeled B_1 and B_2. You can construct a right triangle from the A values by finding a point (0, 0) directly below A_1 and directly to the right of A_2. This right triangle has legs of 2 (the change from −2 to 0) and 9 (the change from 0 to 9). You can plug those values into the Pythagorean theorem and solve for the hypotenuse:

$$a^2 + b^2 = c^2$$
$$(2)^2 + (9)^2 = c^2$$
$$4 + 81 = c^2$$
$$c^2 = 85$$
$$c = \sqrt{85}$$

Quantity A	**Quantity B**
The distance between points (0, 9) and $(-2, 0) = \sqrt{85}$	The distance between points (3, 9) and (10, 3)

You can construct a right triangle from the B values by finding a point (3, 3) directly below B_1 and directly to the left of B_2. This right triangle has legs of 7 (the change from 3 to 10) and 6 (the change from 3 to 9). You can plug those values into the Pythagorean theorem and solve for the hypotenuse:

$$a^2 + b^2 = c^2$$
$$(2)^2 + (9)^2 = c^2$$
$$4 + 81 = c^2$$
$$c^2 = 85$$
$$c = \sqrt{85}$$

Quantity A	**Quantity B**
The distance between points (0, 9) and $(-2, 0) = \sqrt{85}$	The distance between points (3, 9) and $(10, 3) = \sqrt{85}$

Therefore, **the two quantities are equal**.

Unit Four: Number Properties

This Number Properties unit provides a comprehensive analysis of the properties and rules of integers tested on the GRE, addressing topics from prime products to consecutive integers.

In This Unit...

Chapter 15: Divisibility & Primes

Chapter 16: Odds & Evens

Chapter 17: Positives & Negatives

Chapter 18: Exponents

Chapter 19: Roots

Chapter 20: Consecutive Integers

Chapter 21: Number Lines

Chapter 15

DIVISIBILITY & PRIMES

In This Chapter...

Divisibility Rules

Factors

Prime Numbers

Prime Factorization

The Factor Foundation Rule

The Factor/Prime Factorization Connection

Unknown Numbers and Divisibility

Fewer Factors, More Multiples

Divisibility and Addition/Subtraction

Remainders

Chapter 15

Divisibility & Primes

There is a category of problems on the GRE that tests what could broadly be referred to as “Number Properties.” These questions are focused on a very important subset of numbers known as integers. Before we explore divisibility any further, it will be necessary to understand exactly what integers are and how they function.

Integers are whole numbers. That means that they are numbers that do not have any decimals or fractions attached. Some people think of them as counting numbers, that is, 1, 2, 3 … and so on. Integers can be positive, and they can also be negative. For instance, −1, −2, −3 … are all integers as well. And there’s one more important number that qualifies as an integer: 0.

So numbers such as 7, 15,003, −346, and 0 are all integers. Numbers such as 1.3, 3/4, and π are not integers.

Now let’s look at the rules for integers when dealing with the four basic operations: addition, subtraction, multiplication and division.

integer + integer = always an integer	ex.: $4 + 11 = 15$
integer − integer = always an integer	ex. :$-5 - 32 = -37$

integer × integer = always an integer	ex.: 14 × 3 = 42

None of these properties of integers turn out to be very interesting. But what happens when we *divide* an integer by another integer? Well, 18 ÷ 3 = 6, which is an integer, but 12 ÷ 8 = 1.5, which is not an integer.

If an integer divides another integer and the result, or quotient, is an integer, you would say that the first number is divisible by the second. So 18 is divisible by 3 because 18 ÷ 3 equals an integer. On the other hand, you would say that 12 is NOT divisible by 8, because 12 ÷ 8 is not an integer.

Divisibility Rules

The Divisibility Rules are important shortcuts to determine whether an integer is divisible by 2, 3, 4, 5, 6, 8, 9, and 10. You can always use your calculator to test divisibility, but these shortcuts will save you time.

An integer is divisible by:

2 if the integer is even.
Twelve is divisible by 2, but 13 is not. Integers that are divisible by 2 are called “even” and integers that are not are called “odd.” You can tell whether a number is even by checking to see whether the units (ones) digit is 0, 2, 4, 6, or 8. Thus, 1,234,567 is odd, because 7 is odd, whereas 2,345,678 is even, because 8 is even.

3 if the SUM of the integer’s digits is divisible by 3.
For example, 72 is divisible by 3 because the sum of its digits is 9, which is divisible by 3. By contrast, 83 is not divisible by 3, because the sum of its digits is 11, which is not divisible by 3.

4 if the integer is divisible by 2 twice, or if the two-digit number at the end is divisible by 4.
Let's use 28: 28 is divisible by 4 because you can divide it by 2 twice and get an integer result (28 ÷ 2 = 14, and 14 ÷ 2 = 7). For larger numbers, check only the last two digits. For example, 23,456 is divisible by 4 because 56 is

divisible by 4, but 25,678 is not divisible by 4 because 78 is not divisible by 4.

5 if the integer ends in 0 or 5.
You can divide 75 and 80 by 5, but not 77 or 83.

6 if the integer is divisible by both 2 and 3.
For example, 48 is divisible by 6 because it is divisible by 2 (it ends with an 8, which is even) AND by 3 (4 + 8 = 12, which is divisible by 3).

8 if the integer is divisible by 2 three times in succession, or if the three-digit number at the end is divisible by 8.
Thirty-two is divisible by 8 because you can divide it by 2 three times and get an integer result (32 ÷ 2 = 16, 16 ÷ 2 = 8, and 8 ÷ 2 = 4). For larger numbers, check only the last 3 digits. For example, 23,456 is divisible by 8 because 456 is divisible by 8, whereas 23,556 is not divisible by 8 because 556 is not divisible by 8.

9 if the sum of the integer's digits is divisible by 9.
For instance, 4,185 is divisible by 9 since the sum of its digits is 18, which is divisible by 9. By contrast, 3,459 is not divisible by 9, because the sum of its digits is 21, which is not divisible by 9.

10 if the integer ends in 0.
This can be demonstrated with 670, which is divisible by 10, but 675 is not.

The GRE can also test these divisibility rules in reverse. For example, if you are told that a number has a ones digit equal to 0, you can infer that that number is divisible by 10. Similarly, if you are told that the sum of the digits of *x* is equal to 21, you can infer that *x* is divisible by 3 but NOT by 9.

Note also that there is no rule listed for divisibility by 7. The simplest way to check for divisibility by 7, or by any other number not found in this list, is to use the calculator.

Check Your Skills

1. Is 123,456,789 divisible by 2?

2. Is 732 divisible by 3?

3. Is 989 divisible by 9?

4. Is 4,578 divisible by 4?

5. Is 4,578 divisible by 6?

6. Is 603,864 divisible by 8?

Factors

Continue to explore the question of divisibility by asking: What numbers is 6 divisible by? Questions related to divisibility are only interested in positive integers, so you really only have six possible numbers: 1, 2, 3, 4, 5, and 6. You can test to see which numbers 6 is divisible by:

$6 \div 1 = 6$	Any number divided by 1 equals itself, so an integer divided by 1 will be an integer.
$\left.\begin{array}{l} 6 \div 2 = 3 \\ 6 \div 3 = 2 \end{array}\right\rangle$	Note that these form a pair.
$\left.\begin{array}{l} 6 \div 4 = 1.5 \\ 6 \div 5 = 1.2 \end{array}\right\rangle$	Theses are not integers, so 6 is NOT divisible by 4 or by 5.
$6 \div 6 = 1$	Any number divided by itself equals 1, so an integer is always divisible by itself.

So 6 is divisible by 1, 2, 3, and 6. That means that 1, 2, 3, and 6 are **factors** of 6. There are a variety of ways you might see this relationship expressed on the GRE:

2 is a factor of 6	6 is a multiple of 2
2 is a divisor of 6	6 is divisible by 2
2 divides 6	2 goes into 6

Sometimes it will be necessary to find the factors of a number in order to answer a question. An easy way to find all the factors of a small number is to use factor pairs. Factor pairs for any integer are the pairs of factors that, when multiplied together, yield that integer.

Here’s a step-by-step way to find all the factors of the number 60 using a **factor pairs table:**

Make a table with two columns labeled “Small” and “Large.”

Start with 1 in the small column and 60 in the large column. (The first set of factor pairs will always be 1 and the number itself.)

The next number after 1 is 2. If 2 is a factor of 60, then write “2” underneath the “1” in your table. It is, so divide 60 by 2 to find the factor pair: 60 ÷ 2 = 30. Write “30” in the large column.

The next number after 2 is 3. Repeat this process until the numbers in the small and the large columns run into each other. In this case, 6 and 10 are a factor pair. But 7, 8, and 9 are not factors of 60, and the next number after 9 is 10, which appears in the large column, so you can stop.

Small	Large
1	60

Small	Large
2	30
3	20
4	15
5	12
6	10

The advantage of using this method, as opposed to thinking of factors and listing them out, is that this is an organized, methodical approach that makes it easier to find every factor of a number quickly. Let’s practice. (This is also a good opportunity to practice your long division.)

Check Your Skills

7. Find all the factors of 90.

8. Find all the factors of 72.

9. Find all the factors of 105.

10. Find all the factors of 120.

Prime Numbers

Let’s backtrack a little bit and try finding the factors of another small number: 7. The only possibilities are the positive integers less than or equal to 7, so let’s check every possibility.

$$
\begin{array}{ll}
7 \div 1 = 7 & \text{Every number is divisible by 1—no surprise there!} \\
\left.\begin{array}{l} 7 \div 2 = 3.5 \\ 7 \div 3 = 2.33\ldots \\ 7 \div 4 = 1.75 \\ 7 \div 5 = 1.4 \end{array}\right\} & \text{The number 7 is not divisible by } \textit{any} \text{ integer besides 1 and itself.} \\
7 \div 6 = 1.16\ldots & \\
7 \div 7 = 1 & \text{Every number is divisible by itself—boring!}
\end{array}
$$

So 7 only has two factors—1 and itself. Numbers that only have two factors are known as **prime numbers**. As you will see, prime numbers play a very important role in answering questions about divisibility. Because they’re so important, it’s critical that you learn to identify what numbers are prime and what numbers aren’t.

The prime numbers that appear most frequently on the test are prime numbers less than 20. They are 2, 3, 5, 7, 11, 13, 17, and 19. Two things to note about this list: 1 is not prime, and out of *all* the prime numbers, 2 is the *only* even prime number.

The number 2 is prime because it has only two factors—1 and itself. The reason that it’s the only even prime number is that *every* other even number is also divisible by 2, and thus has another factor besides 1 and itself. For instance, you can immediately tell that 12,408 isn’t prime, because we know that it has at least one factor besides 1 and itself: 2.

So every positive integer can be placed into one of two categories—prime or not prime:

Primes	Non-Primes
2, 3, 5, 7, 11, etc.	4, 6, 8, 9, 10, etc.
exactly two factors: 1 and itself	*more than* two factors
ex. $7 = 1 \times 7$ 	ex. $6 = 1 \times 6$ *and* $6 = 2 \times 3$
only one factor pair	more than two factors *and* more than one factor pair

Check Your Skills

11. List all the prime numbers between 20 and 50.

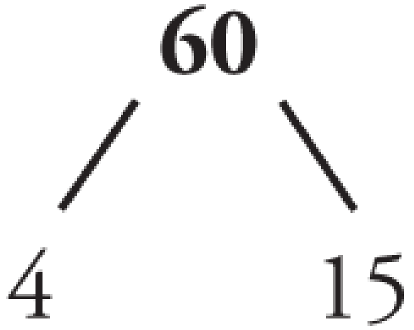
Prime Factorization

Take another look at 60. When you found the factor pairs of 60, you saw that it had 12 factors and 6 factor pairs.

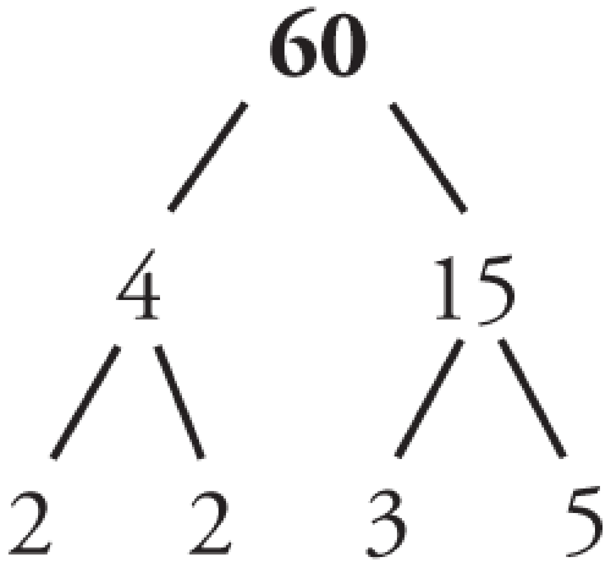
$$\begin{array}{rl} 60 = 1 \times 60 & \text{Always the first factor pair — boring!} \\ \left.\begin{array}{r} \textit{and}\quad 2 \times 30 \\ \textit{and}\quad 3 \times 20 \\ \textit{and}\quad 4 \times 15 \\ \textit{and}\quad 5 \times 12 \\ \textit{and}\quad 6 \times 10 \end{array}\right\} & \begin{array}{l} \text{There are 5 other factor pairs—interesting!} \\ \text{Look at these in a little more detail.} \end{array} \end{array}$$

From here on, pairs will be referred to as boring and interesting factor pairs. These are not technical terms, but the boring factor pair is the factor pair that involves 1 and the number itself. All other pairs are interesting pairs. Keep reading to see why!

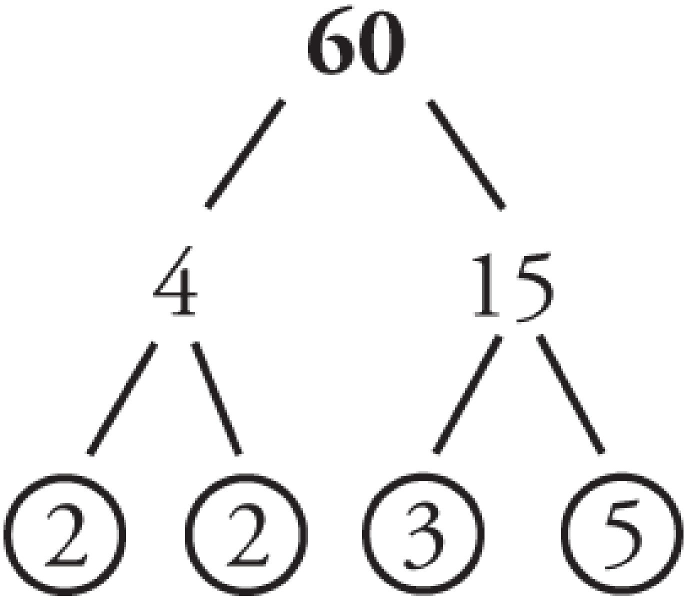
Examine one of these factor pairs—4 × 15. One way to think about this pair is that 60 *breaks down* into 4 and 15. One way to express this relationship visually is to use a **factor tree:**



Now, the question arises—can you go further? Sure! Neither 4 nor 15 is prime, which means they both have factor pairs that you might find *interesting.* For example, 4 breaks down into 2 × 2, and 15 breaks down into 3 × 5:

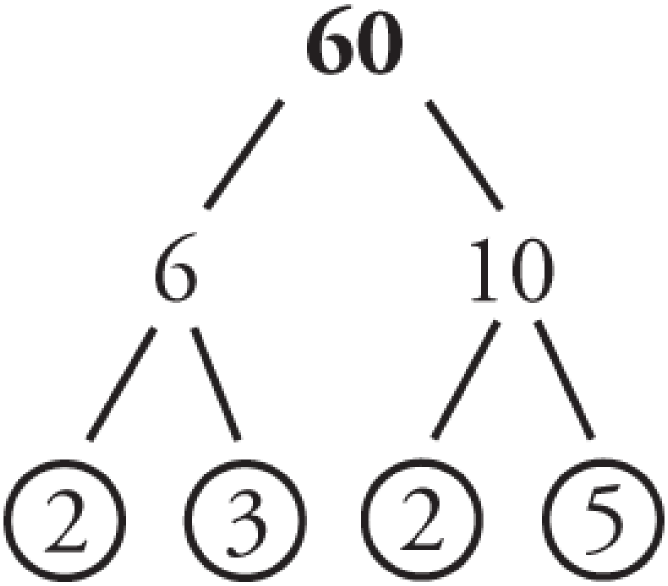


Can you break it down any further? Not with *interesting* factor pairs. You could say that 2 = 2 × 1, for instance, but that doesn't provide you any new information. The reason you can't go any further is that 2, 2, 3, and 5 are all *prime numbers*. Prime numbers only have one boring factor pair. So when you find a prime factor, you will know that that branch of your factor tree has reached its end. You can go one step further and circle every prime number as you go, reminding you that you the branch can’t break down any further. The factor tree for 60 would look like this:



So after breaking down 60 into 4 and 15, and breaking 4 and 15 down, you end up with 60 equals 2 × 2 × 3 × 5.

What if you start with a different factor pair of 60? Create a factor tree for 60 in which the first breakdown you make is 6 × 10:



According to this factor tree 60 equals 2 × 3 × 2 × 5. Notice that, even though they’re in a different order, this is the same group of prime numbers as before. In fact, *any* way you break down 60, you will end up with the same prime factors: two 2’s, one 3, and one 5. Another way to say this is that 2 × 2 × 3 × 5 is the **prime factorization** of 60.

One way to think about prime factors is that they are the DNA of a number. Every number has a unique prime factorization. The only number that can be written as 2 × 2 × 3 × 5 is 60. Breaking down numbers into their prime factors is the key to answering many divisibility problems.

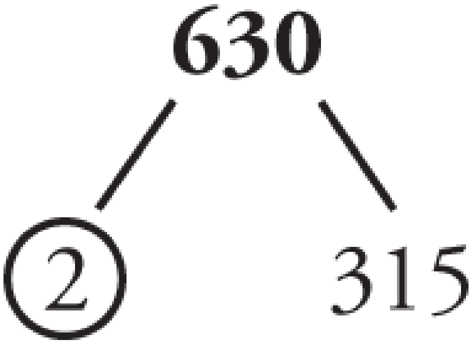
As you proceed through the chapter, pay special attention to what prime factors can tell you about a number and some different types of questions the GRE may ask. But because the prime factorization of a number is so important, first you need a fast, reliable way to find the prime factorization of *any* number.

A factor tree is the best way to find the prime factorization of a number. A number like 60 should be relatively straightforward to break down into primes, but what if you need the prime factorization of 630?

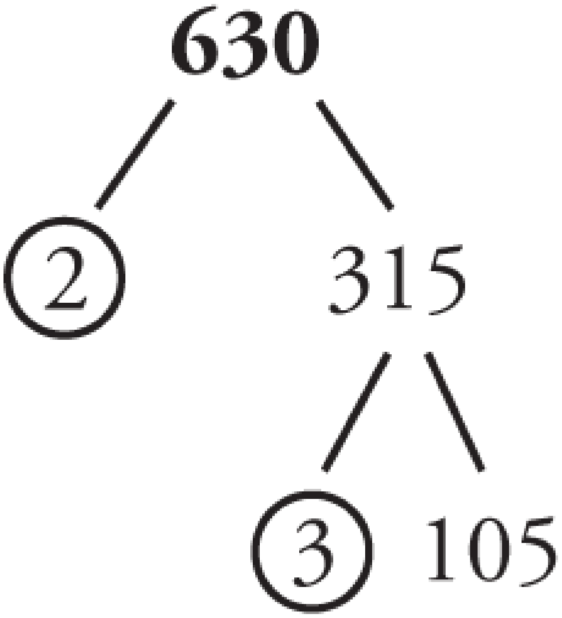
For large numbers, it's often best to start with the smallest prime factors and work your way toward larger primes. This is why it's good to know your divisibility rules.

Take a second to try on your own, then continue through the explanation.

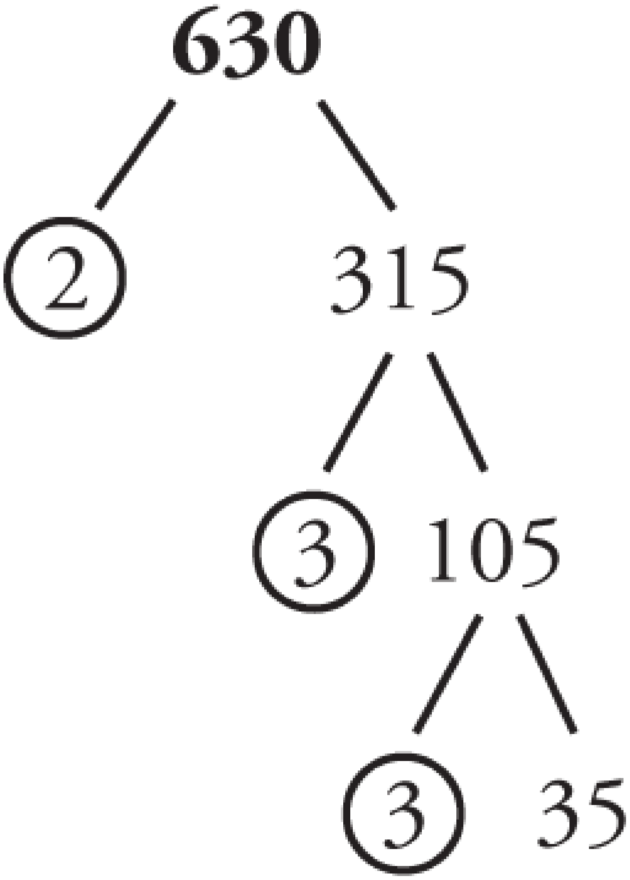
Start by finding the smallest prime number that 630 is divisible by. The smallest prime number is 2. Because 630 is even, it must be divisible by 2: 630 ÷ 2 = 315. So your first breakdown of 630 is into 2 and 315:



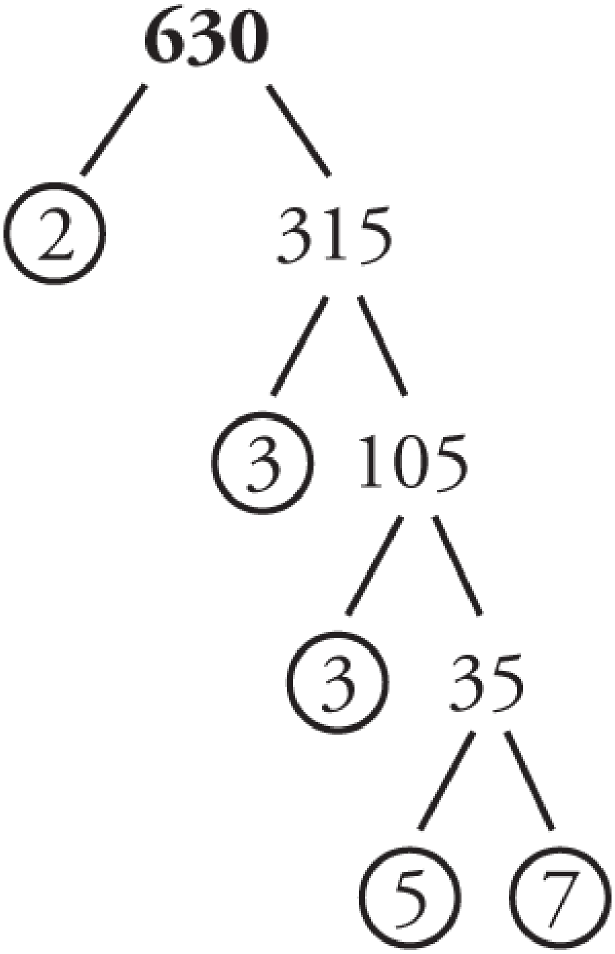
Now you still need to factor 315. It's not even, so it's not divisible by 2. Is it divisible by 3? If the digits of 315 add up to a multiple of 3, it is. Because 3 + 1 + 5 = 9, which is a multiple of 3, then 315 is divisible by 3: 315 ÷ 3 = 105. Your factor tree now looks like this:



If 315 was not divisible by 2, then 105 won’t be either (the reason for this will be discussed later), but 105 might still be divisible by 3. Because 1 + 0 + 5 = 6, then 105 is divisible by 3: 105 ÷ 3 = 35. Your tree now looks like this:



Because 35 is not divisible by 3 (3 + 5 = 8, which is not a multiple of 3), the next number to try is 5. Because 35 ends in a 5, it is divisible by 5: 35 ÷ 5 = 7. Your tree now looks like this:



Every number on the tree has now been broken down as far as it can go. So the prime factorization of 630 is 2 × 3 × 3 × 5 × 7.

Alternatively, you could have split 630 into 63 and 10, because it's easy to see that 630 is divisible by 10. Then you would proceed from there. Either way will get you to the same set of prime factors.

Now it's time to get a little practice doing prime factorizations.

Check Your Skills

12. Find the prime factorization of 90.

13. Find the prime factorization of 72.

14. Find the prime factorization of 105.

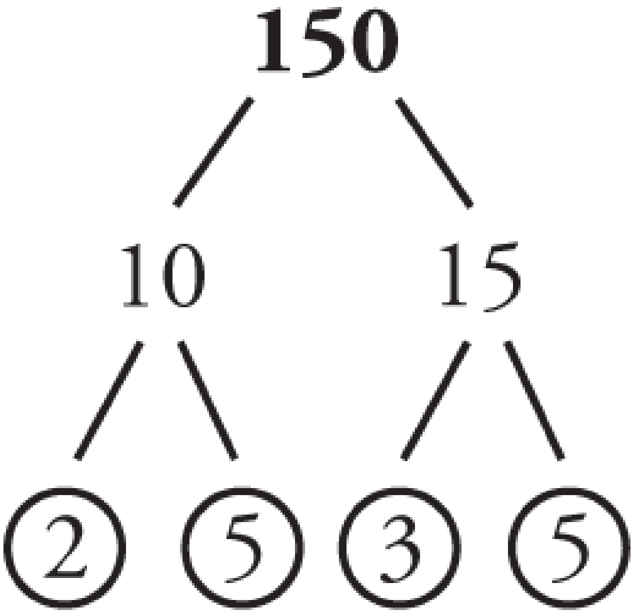
15. Find the prime factorization of 120.

The Factor Foundation Rule

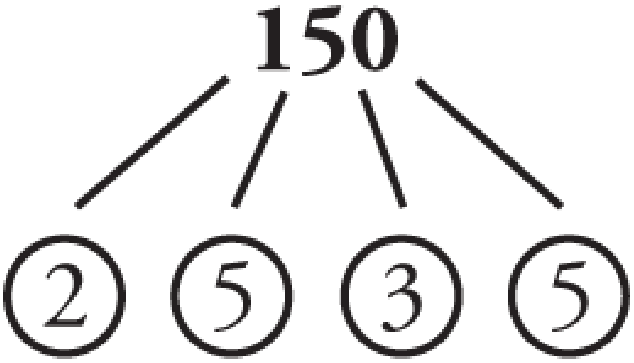
This discussion begins with the **factor foundation rule**. The factor foundation rule states that if *a* is divisible by *b*, and *b* is divisible by *c*, then *a* is divisible by *c* as well. In other words, if you know that 12 is divisible by 6, and 6 is divisible by 3, then 12 is divisible by 3 as well.

This rule also works in reverse to a certain extent. If *d* is divisible by two different primes, *e* and *f*, *d* is also divisible by $e \times f$. In other words, if 20 is divisible by 2 and by 5, then 20 is also divisible by 2 × 5, which is 10.

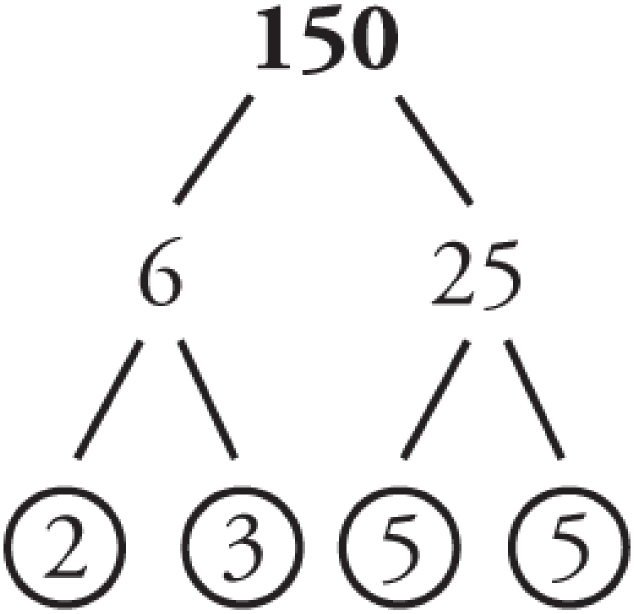
Another way to think of this rule is that divisibility travels up and down the factor tree. Let's walk through the factor tree of 150. Break it down, and then build it back up.



Because 150 is divisible by 10 and by 15, then 150 is also divisible by *everything* that 10 and 15 are divisible by. Because 10 is divisible by 2 and by 5, then 150 is also divisible by 2 and 5. Because 15 is divisible by 3 and by 5, then 150 is also divisible by 3 and 5. Taken all together, the prime factorization of 150 is 2 × 3 × 5 × 5. You could represent that information like this:

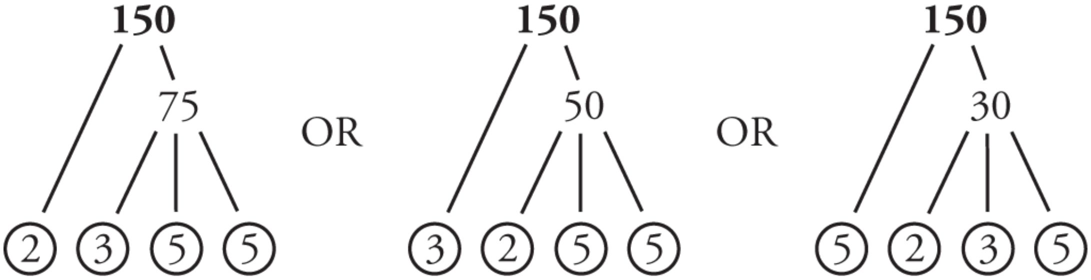


Think of prime factors as building blocks. In the case of 150, you have one 2, one 3, and two 5’s at your disposal to build other factors of 150. In the first example, you went down the tree—from 150 down to 10 and 15, and then down again to 2, 5, 3, and 5. But you can also build upward, starting with the four building blocks. For instance, 2 × 3 = 6, and 5 × 5 = 25, so your tree could also look like this:



(Even though 5 and 5 are not different primes, 5 appears twice on 150's tree. So you are allowed to multiply those two 5's together to produce another factor of 150, namely 25.)

The tree above isn't even the only other possibility. These are all trees that you could build using different combinations of the prime factors.



You began with four prime factors of 150: 2, 3, 5 and 5. But you were able to build different factors by multiplying 2, 3, or even all 4 of those primes together in different combinations. As it turns out, *all* of the factors of a number (except for the number 1) can be built with different combinations of its prime factors.

The Factor/Prime Factorization Connection

Take one more look at the number 60 and its factors. Specifically, look at the prime factorizations of all the factors of 60:

	Small	**Large**	
1	1	60	2 × 2 × 3 × 5
2	2	30	2 × 3 × 5
3	3	20	2 × 2 × 5
2 × 2	4	15	3 × 5
5	5	12	2 × 2 × 3
2 × 3	6	10	2 × 5

All the factors of 60 (with the special exception of 1) are either the prime factors themselves or various products of those prime factors. To say this another way, every non-prime factor of a number can be expressed as the product of some or all of its prime factors. Take a look back at your work for Check Your Skills questions #7–10 and #12–15. Break down all the factor pairs from the first section into their prime factors. This relationship between factors and prime factors is true of every number.

Now that you know why prime factors are so important, it's time for the next step. An important skill on the GRE is to take the given information in a question and go further with it. For example, if a question tells you that a number *n* is even, what else do you know about it? Every even number is a multiple of 2, so *n* is a multiple of 2. These kinds of inferences often provide crucial information necessary to correctly solving problems.

So far, you've been finding factors and prime factors of numbers—but the GRE will sometimes ask divisibility questions about *variables*. In the next section, the discussion of divisibility will bring variables into the picture. But first, recap what you've learned so far and what tools you'll need going forward:

- If *a* is divisible by *b*, and *b* is divisible by *c*, then *a* is divisible by *c* as well (e.g., 100 is divisible by 20, and 20 is divisible by 4, so 100 is divisible by 4 as well).
- If *d* has *e* and *f* as prime factors, *d* is also divisible by *e* × *f* (e.g., 90 is divisible by 5 and by 3, so 90 is also divisible by 5 × 3 = 15). You can let *e* and *f* be the same prime, as long as there are at least two copies of that prime in *d*'s factor tree. (e.g., 98 has two 7's in its factors, and so is divisible by 49).
- Every factor of a number (except the number 1) is either prime or the product of a different combination of that number's prime factors. For example, 30 = 2 × 3 × 5. Its factors are 1, 2, 3, 5, 6 (2 × 3), 10 (2 × 5), 15 (3 × 5), and 30 (2 × 3 × 5).
- To find *all* the factors of a number in an easy, methodical way, set up a factor pairs table.
- To find *all* the prime factors of a number, use a factor tree. With larger numbers, start with the smallest primes and work your way up to larger

primes.

Check Your Skills

16. The prime factorization of a number is 3 × 5. What is the number and what are all of its factors?

17. The prime factorization of a number is 2 × 5 × 7. What is the number and what are all of its factors?

18. The prime factorization of a number is 2 × 3 × 13. What is the number and what are all of its factors?

Unknown Numbers and Divisibility

Say that you are told some unknown positive number *x* is divisible by 6. How can you represent this on paper? There are many ways, depending on the problem. You could say that you know that *x* is a multiple of 6, or you could say that $x = 6 \times$ an integer. You could also represent the information with a factor tree. Careful though—although you've had a lot of practice drawing factor trees, there is one important difference now that you're dealing with an unknown number. You know that *x* is divisible by 6, but *x* may be divisible by other numbers as well. You have to treat what they have told you as incomplete information, and remind yourselves there are other things about *x* you don't know. To represent that on the page, your factor tree could look like this:

Now the question becomes—what else do you know about *x*? If a question on the GRE told you that *x* is divisible by 6, what could you definitely say about *x*? Take a look at these three statements, and for each statement, decide whether it *must* be true, whether it *could* be true, or whether it *cannot* be true.

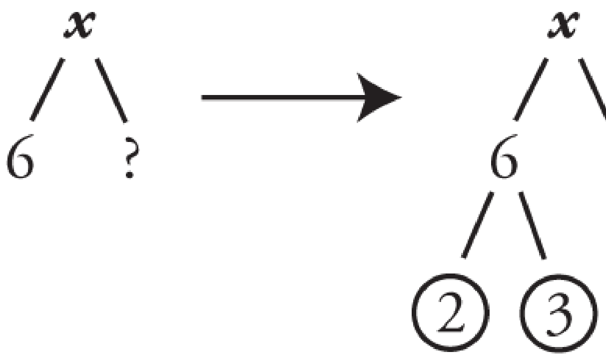
x is divisible by 3
x is even
x is divisible by 12

Deal with each statement one at a time, beginning with Statement I—*x* is divisible by 3. One approach to take here is to think about the multiples of 6. If *x* is divisible by 6, then you know that *x* is a multiple of 6. List out the first several multiples of 6, and see if they're divisible by 3.

$$x \text{ is a number on this list. } \left\{\begin{array}{cr} 6 & 6 \div 3 = 2 \\ 12 & 12 \div 3 = 4 \\ 18 & 18 \div 3 = 6 \\ 24 & 24 \div 3 = 8 \\ \cdots & \cdots \end{array}\right\} \text{ All of these numbers are also divisible by 3.}$$

At this point, you can be fairly certain that *x* is divisible by 3. In fact, listing out possible values of a variable is often a great way to begin answering a question in which you don't know the value of the number you are asked about.

But can you do better than say you're fairly certain *x* is divisible by 3? Is there a way to definitively say *x* *must* be divisible by 3? As it turns out, there is. Look at the factor tree for *x* again:

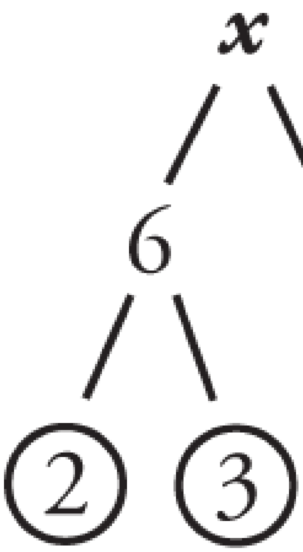


Remember, the ultimate purpose of the factor tree is to break numbers down into their fundamental building blocks: prime numbers. Now that the factor tree is broken down as far as it will go, you can apply the factor foundation rule. Thus, x is divisible by 6, and 6 is divisible by 3, so you can say definitively that x *must* be divisible by 3.

In fact, questions like this one are the reason so much time was spent discussing the factor foundation rule and the connection between prime factors and divisibility. Prime factors provide the foundation for a way to make definite statements about divisibility. With that in mind, look at Statement II.

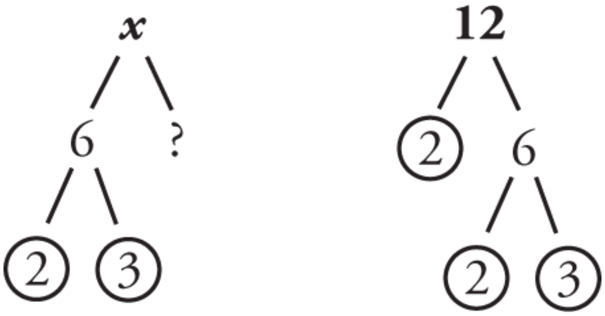
Statement II says x is even. This question is about divisibility, so the question becomes, what is the connection between divisibility and a number being even? Remember, an important part of this test is the ability to make inferences based on the given information.

What's the connection? Well, being even means being divisible by 2. So if you know that x is divisible by 2, then you can guarantee that x is even. Look at the factor tree:



You can once again make use of the factor foundation rule—6 is divisible by 2, so *x* *must* be divisible by 2 as well. And if *x* is divisible by 2, then *x* *must* be even as well.

That just leaves the final statement. Statement III says *x* is divisible by 12. Look at this question from the perspective of factor trees, and compare the factor tree of *x* with the factor tree of 12:

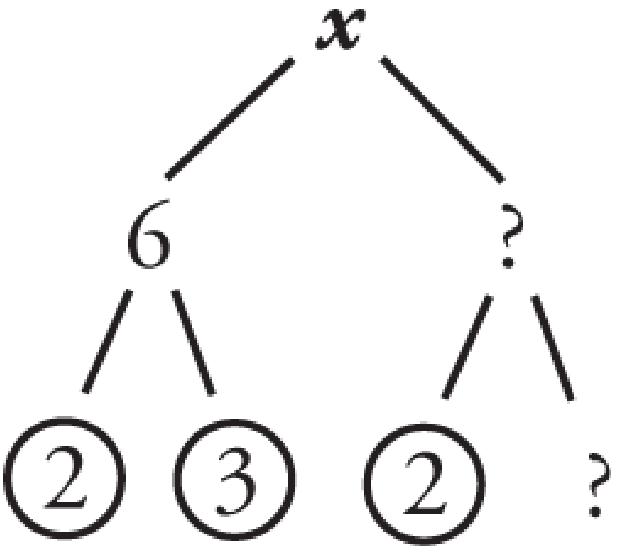


What would you have to know about *x* to guarantee that it is divisible by 12? Well, when 12 is broken down all the way, 12 is 2 × 2 × 3. Thus, 12’s building blocks are two 2’s and a 3. For *x* to be divisible by 12, it would have to also have two 2’s and one 3 among its prime factors. In other words, for

x to be divisible by 12, it has to be divisible by *everything* that 12 is divisible by.

You need x to be divisible by two 2's and one 3 in order to say it *must* be divisible by 12. But looking at your factor tree, there is only one 2 and only one 3. Because there is only one 2, you can't say that x *must* be divisible by 12. But then the question becomes, *could* x be divisible by 12? Think about the question for a second, and then keep reading.

The key to this question is the question mark that you put on x's factor tree. That question mark should remind you that you don't know everything about x. Thus, x could have other prime factors. What if one of those unknown factors was another 2? Then the tree would look like this:



So *if* one of those unknown factors were a 2, then x would be divisible by 12. The key here is that you have no way of knowing for sure whether there is a 2. Thus, x may be divisible by 12, it may not. In other words, x *could* be divisible by 12.

To confirm this, go back to the multiples of 6. You still know that *x* must be a multiple of 6, so start by listing out the first several multiples and see whether they are divisible by 12.

$$x \text{ is a number on this list. } \left\{ \begin{array}{ll} 6 & 6 \div 12 = 0.5 \\ 12 & 12 \div 12 = 1 \\ 18 & 18 \div 12 = 1.5 \\ 24 & 24 \div 12 = 2 \\ \ldots & \ldots \end{array} \right\} \text{ Some, but not all, of these numbers are also divisible by 12.}$$

Once again, some of the possible values of *x* are divisible by 12, and some aren't. The best you can say is that *x* *could* be divisible by 12.

Check Your Skills

For these statements, the following is true: *x* is divisible by 24. For each statement, say whether it *must* be true, *could* be true, or *cannot* be true.

19. *x* is divisible by 6

20. *x* is divisible by 9

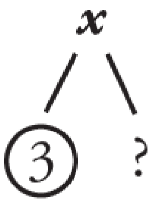
21. *x* is divisible by 8

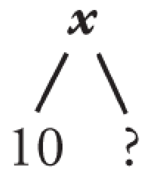
Consider the following question, which has an additional twist this time. Once again, there will be three statements. Decide whether each statement *must* be true, *could* be true, or *cannot* be true. Answer this question on your own, then explore each statement one at a time on the next page.

x is divisible by 3 and by 10.

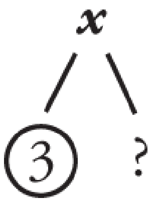
x is divisible by 2
is divisible by 15
x is divisible by 45

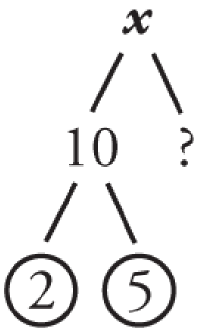
Before diving into the statements, spend a moment to organize the information the question has given you. You know that *x* is divisible by 3 and by 10, so you can create two factor trees to represent this information:





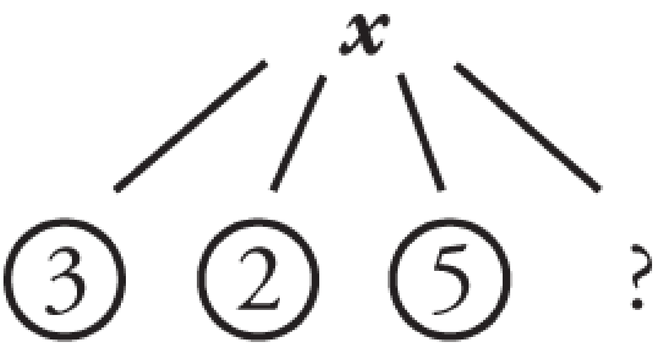
Now that you have your trees, get started with statement I. Statement I says that *x* is divisible by 2. The way to determine whether this statement is true should be fairly familiar by now—use the factor foundation rule. First of all, your factor trees aren't quite finished. Factor trees should always be broken down all the way until every branch ends in a prime number. Really, your factor trees should look like this:





Now you are ready to decide whether statement I is true. Because *x* is divisible by 10, and 10 is divisible by 2, therefore *x* is divisible by 2. Statement I *must* be true.

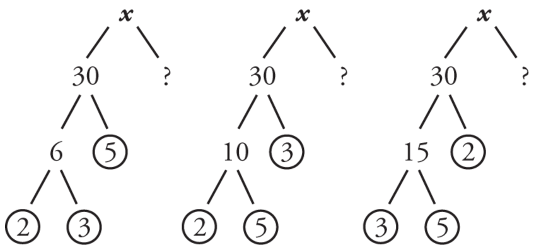
That brings you to statement II. This statement is a little more difficult. It also requires you to take another look at your factor trees. You have two separate trees, but they're giving you information about the same variable —*x*. Neither tree gives you complete information about *x*, but you do know a couple of things with absolute certainty. From the first tree, you know that *x* is divisible by 3, and from the second tree you know that *x* is divisible by 10—which really means you know that *x* is divisible by 2 and by 5. You can actually combine those two pieces of information and represent them on one factor tree, which would look like this:



Now you know three prime factors of *x*: 2, 3, and 5. Return to the statement. Statement II says that *x* is divisible by 15. What do you need to know to say that *x must* be divisible by 15? If you can guarantee that *x* has all the prime factors that 15 has, then you can guarantee that *x* is divisible by 15.

The number 15 breaks down into the prime factors 3 and 5. So to guarantee that *x* is divisible by 15, you need to know it's divisible by 3 and by 5. Looking back up at your factor tree, notice that *x* has both a 3 and a 5, which means that *x* is divisible by 15. Therefore, statement II *must* be true.

You can also look at this question more visually. Remember, prime factors are like building blocks—*x* is divisible by any combination of these prime factors. You can combine the prime factors in a number of different ways, as shown here:

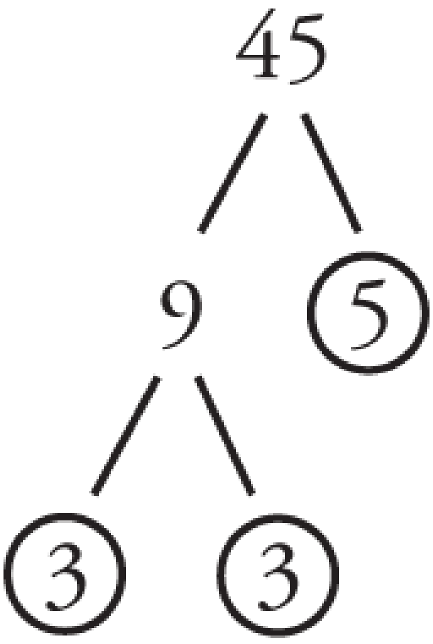


Each of these factor trees can tell you different factors of *x*. But what's really important is what they have in common. No matter what way you combine the prime factors, each tree ultimately leads to 2 × 3 × 5, which equals 30. So you know that *x* is divisible by 30. And if *x* is divisible by 30, it is also divisible by everything 30 is divisible by. You know how to identify every number 30 is divisible by—use a factor pair table. The factor pair table of 30 looks like this:

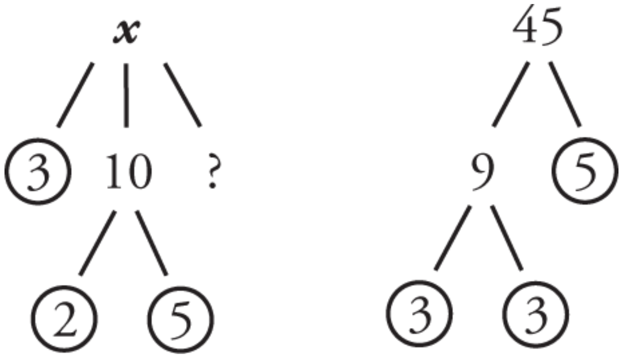
Small	Large
1	30
2	15
3	10
5	6

Again, Statement II says that *x* is divisible by 15. Because *x* is divisible by 30, and 30 is divisible by 15, then *x must* be divisible by 15.

That brings you to Statement III. Statement III says that *x* is divisible by 45. What do you need to know to say that *x must* be divisible by 45? Build a factor tree of 45, which looks like this:



The number 45 is divisible by 3, 3, and 5. For *x* to be divisible by 45, you need to know that it has all the same prime factors. Does it?



The factorization of 45 has one 5 and two 3's. Although *x* has a 5, you only know that *x* has one 3. That means that you can't say for sure that *x* is divisible by 45. However, *x could* be divisible by 45, because you don't know what the question mark contains. If it contains a 3, then *x* is divisible by 45. If it doesn't contain a 3, then *x* is not divisible by 45. Without more information, you can't say for sure either way. So statement III *could* be true.

Now it's time to recap what's been covered in this chapter. When dealing with questions about divisibility, you need a quick, accurate way to identify *all* the factors of a number. A factor pair table provides a reliable way to make sure you find every factor of a number.

Prime factors provide essential information about a number or variable. They are the fundamental building blocks of every number. In order for a number or variable to be divisible by another number, it must "contain" all the same prime factors that the other number contains. In the last example, you could definitely say that *x* was divisible by 15, because *x* contained one 3 and one 5. But you could not say for sure that it was divisible by 45, because 45 has one 5 and two 3's, but *x* is only known to contain one 5 and one 3.

Check Your Skills

For these statements, the following is true: *x* is divisible by 28 and by 15. For each statement, say whether it *must* be true, *could* be true, or *cannot* be true.

22. *x* is divisible by 14.

23. *x* is divisible by 20.

24. *x* is divisible by 24.

Fewer Factors, More Multiples

Sometimes it is easy to confuse factors and multiples. The mnemonic “Fewer Factors, More Multiples” should help you remember the difference. Factors divide into an integer and are therefore less than or equal to that integer. Positive multiples, on the other hand, are 1 times that integer, 2 times that integer, etc. and are therefore greater than or equal to that integer.

Any integer only has a limited number of factors. For example, there are only four factors of 8: 1, 2, 4, and 8. By contrast, there is an infinite number of multiples of an integer. For example, the first five positive multiples of 8 are 8, 16, 24, 32, and 40, but you could go on listing multiples of 8 forever.

Factors, multiples, and divisibility are very closely related concepts. For example, 3 is a factor of 12. This is the same as saying that 12 is a multiple of 3, or that 12 is divisible by 3.

On the GRE, this terminology is often used interchangeably in order to make the problem seem harder than it actually is. Be aware of the different ways that the GRE can phrase information about divisibility. Moreover, try to convert all such statements to the same terminology. For example, **all** of the following statements **say exactly the same thing**:

- 12 is divisible by 3

- 12 is a multiple of 3
- $\frac{23}{7}$ is an integer
- 12 = 3*n*, where *n* is an integer
- 12 items can be shared among 3 people so that each person has the same number of items
- 3 is a divisor of 12, or 3 is a factor of 12
- 3 divides 12
- $\frac{23}{7}$ yields a remainder of 0
- 3 “goes into” 12 evenly

Another term that the GRE sometimes uses is “unique prime factor.” The distinction between a prime factor and a unique prime factor is best illustrated by an example. If you prime factor 12, you end up with two 2’s and one 3, but 12 only has two unique prime factors, 2 and 3, because the two 2’s are the same number. So 100 has two unique prime factors (2 and 5) just as 10 does.

Divisibility and Addition/Subtraction

If you add two multiples of 7, you get another multiple of 7. Try it: 35 + 21 = 56. This should make sense: (5 × 7) + (3 × 7) = (5 + 3) × 7 = 8 × 7.

Likewise, if you subtract two multiples of 7, you get another multiple of 7. Try it: 35 − 21 = 14. Again, it's clear why: (5 × 7) − (3 × 7) = (5 − 3) × 7 = 2 × 7.

This pattern holds true for the multiples of any integer N. **If you add or subtract multiples of *N*, the result is a multiple of *N***. You can restate this principle using any of the disguises above: for instance, if N is a divisor of x and of y, then N is a divisor of $x + y$.

Remainders

The number 17 is not divisible by 5. When you divide 17 by 5, using long division, you get a **remainder**: a number left over. In this case, the remainder is 2, as shown here:

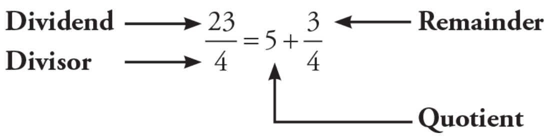
$$\begin{array}{r} 3 \\ 5\,\overline{)\;17} \\ -15 \\ \hline 2 \end{array}$$

You can also write that 17 is 2 more than 15, or 2 more than a multiple of 5. In other words, you can write 17 = 15 + 2 = 3 × 5 + 2. Every number that leaves a remainder of 2 after it is divided by 5 can be written this way: as a multiple of 5, plus 2.

On simpler remainder problems, it is often easiest to pick numbers. Simply add the desired remainder to a multiple of the divisor. For instance, if you need a number that leaves a remainder of 4 after division by 7, first pick a multiple of 7, such as 14. Then add 4 to get 18, which satisfies the requirement (18 = 7 × 2 + 4).

A remainder is defined as the integer portion of the **dividend** (or numerator) that is not evenly divisible by the **divisor** (or denominator).

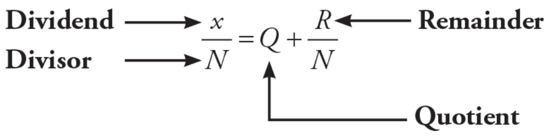
Here is an example written in fractional notation:



$$\frac{23}{4} = 5 + \frac{3}{4}$$

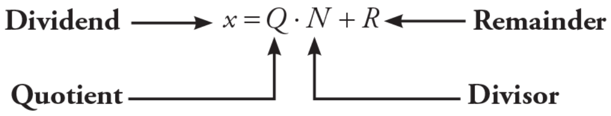
The **quotient** is the resulting integer portion that *can* be divided out (in this case, the quotient is 5). Note that the dividend, divisor, quotient, and remainder will *always* be integers. Sometimes, the quotient may be zero. For instance, when 3 is divided by 5, the remainder is 3 but the quotient is 0 (because 0 is the biggest multiple of 5 that can be divided out of 3).

Algebraically, this relationship can be written as the **Remainder Formula**:



$$\frac{x}{N} = Q + \frac{R}{N}$$

This framework is often easiest to use on GRE problems when you multiply through by the divisor N:



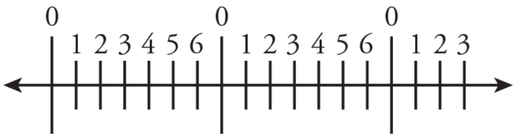
$$x = Q \cdot N + R$$

(Example: 23 = 5 × 4 + 3)

Again, remember that x, Q, N, and R must *all* be integers. It should also be noted that R must be equal to or greater than 0, but less than N (the divisor).

Range of Possible Remainders

When you divide an integer by 7, the remainder could be 0, 1, 2, 3, 4, 5, or 6. Notice that you cannot have a negative remainder or a remainder larger than 7, and that you have exactly 7 possible remainders. You can see these remainders repeating themselves on the Remainder Ruler:



This pattern can be generalized. When you divide an integer by a positive integer N, the possible remainders range from 0 to $(N - 1)$. There are thus N possible remainders. Negative remainders are not possible, nor are remainders equal to or larger than N.

> If $a \div b$ yields a remainder of 3, $c \div d$ yields a remainder of 4, and $a, b, c,$ and d are all integers, what is the smallest possible value for $b + d$?

The remainder must be smaller than the divisor, therefore, 3 must be smaller than b. Because b must be an integer, then b is at least 4. Similarly, 4 must be smaller than d, and d must be an integer, so d must be at least 5. Therefore, the smallest possible value for $b + d$ is $4 + 5 = 9$.

Remainder of 0

If *x* divided by *y* yields a remainder of 0 (commonly referred to as "no remainder"), then *x* is divisible by *y*. Conversely, if x is divisible by y, then *x* divided by *y* yields a remainder of 0 (or “no remainder”).

Similarly, if *x* divided by *y* yields a remainder greater than 0, then *x* is *not* divisible by *y*, and vice versa.

Arithmetic with Remainders

Two useful tips for arithmetic with remainders, if you have the same divisor throughout:

> **You can add and subtract remainders directly, as long as you correct excess or negative remainders**. “Excess remainders” are remainders larger than or equal to the divisor. To correct excess or negative remainders, just add or subtract the divisor. For instance, if *x* leaves a remainder of 4 after division by 7, and *y* leaves a remainder of 2 after division by 7, then *x* + *y* leaves a remainder of 4 + 2 = 6 after division by 7. You do not need to pick numbers or write algebraic expressions for *x* and *y*. Simply write R4 + R2 = R6.
>
> If *x* leaves a remainder of 4 after division by 7 and *z* leaves a remainder of 5 after division by 7, then adding the remainders together yields 9. This number is too high, however. The remainder must be non-negative and less than 7. You can take an additional 7 out of the remainder, because 7 is the **excess** portion. The correct remainder is thus R4 + R5 = R9 = R2 (subtracting a 7 out).
>
> With the same *x* and *z*, subtraction of the remainders gives −1, which is also an unacceptable remainder (it must be non-negative). In this case, add an extra 7 to see that *x* − *z* leaves a remainder of 6 after division by 7. Using R’s, you can write R4 − R5 = R(−1)= R6 (adding a 7 in).

You can multiply remainders, as long as you correct excess remainders at the end.

Again, if *x* has a remainder of 4 upon division by 7 and *z* has a remainder of 5 upon division by 7, then 4 × 5 gives 20. Two additional 7’s can be taken out of this remainder, so *x* × *z* will have remainder 6 upon division by 7. In other words, (R4)(R5) = R20 = R6 (taking out two 7’s). You can prove this by again picking *x* = 25 and *z* = 12 (try the algebraic method on your own!):

$$25 \times 12 = 300 = \underset{\substack{\uparrow \\ \textbf{Quotient}}}{42} \times \underset{\substack{\uparrow \\ \textbf{Divisor}}}{7} + 6 \leftarrow \textbf{Remainder}$$

Check Your Skills

25. What is the remainder when 13 is divided by 6?

26. What’s the first double-digit number that results in a remainder of 4 when divided by 5?

27. If *x* has a remainder of 4 when divided by 9 and *y* has a remainder of 3 when divided by 9, what’s the remainder when $x + y$ is divided by 9?

28. Using the example from #27, what’s the remainder when xy is divided by 9?

Check Your Skills Answer Key

1. **No**
   Is 123,456,789 divisible by 2?

   123,456,789 is an odd number, because it ends in 9, so 123,456,789 is not divisible by 2.

2. **Yes**
   Is 732 divisible by 3?

   The digits of 732 add up to a multiple of 3 (7 + 3 + 2 = 12), so 732 is divisible by 3.

3. **No**
   Is 989 divisible by 9?

   The digits of 989 do not add up to a multiple of 9 (9 + 8 + 9 = 26), so 989 is not divisible by 9.

4. **No**
   Every whole hundred is divisible by 4, so you only need to check the amount “left over.” Because 78 is not divisible by 4, then 4,578 is not divisible by 4.

5. **Yes**

   Any number divisible by both 2 and 3 is divisible by 6. So 4,578 must be divisible by 2, because it ends in an even number. It also must be divisible by 3, because the sum of its digits is a multiple of 3 (4 + 5 + 7 + 8 = 24). Therefore, 4,578 is divisible by 6.

6. **Yes**

   Easiest to use your calculator for this one: 603,864 ÷ 8 = 75,483 with no remainder. 🖩

   Alternatively, evaluate the three-digit number at the end; every whole thousand is divisible by 8, so you only need to check the amount "left over." 864 is divisible by 8 because 864 = 800 + 64 and both 800 and 64 are multiples of 8.

7. Find all the factors of 90.

Small	**Large**
1	90
2	45
3	30
5	18
6	15
9	10

8. Find all the factors of 72.

Small	Large
1	72
2	36
3	24
4	18
6	12
8	9

9. Find all the factors of 105.

Small	Large
1	105
3	35
5	21
7	15

10. Find all the factors of 120.

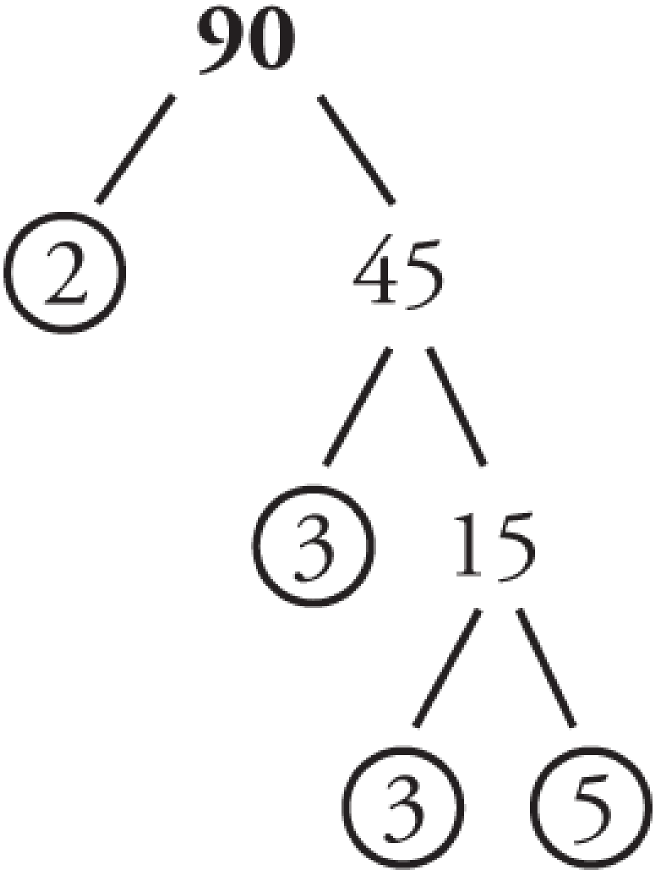
Small	Large
1	120
2	60
3	40

Small	Large
4	30
5	24
6	20
8	15
10	12

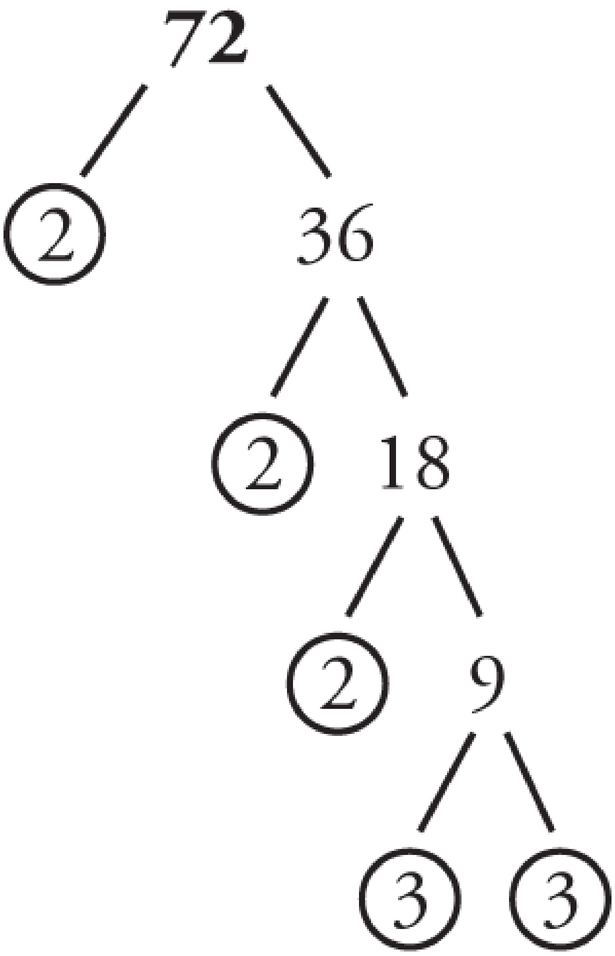
11. List all the prime numbers between 20 and 50.

    23, 29, 31, 37, 41, 43, and 47

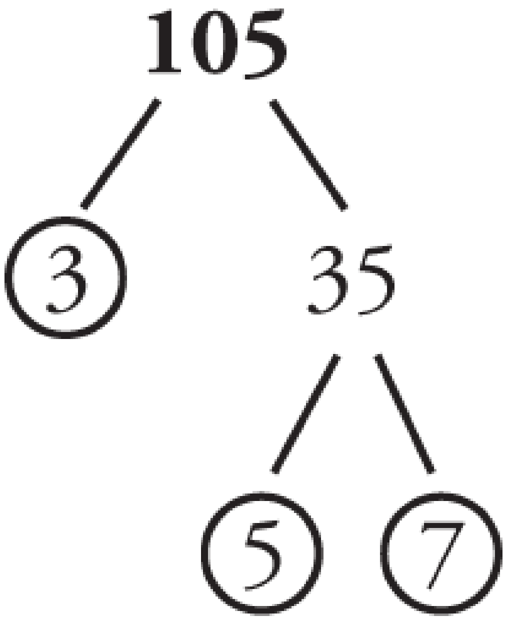
12. Find the prime factorization of 90.



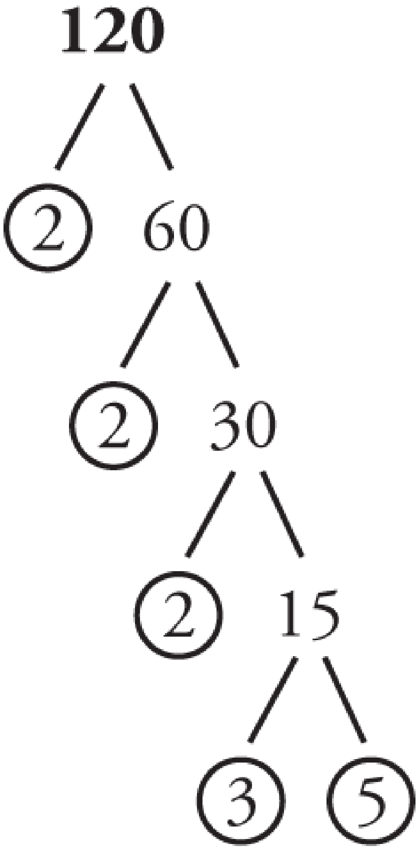
13. Find the prime factorization of 72.



14. Find the prime factorization of 105.



15. Find the prime factorization of 120.



16. The prime factorization of a number is 3×5. What is the number and what are all its factors?

$3 \times 5 = 15$

Small	Large
1	15
3	5

17. The prime factorization of a number is $2 \times 5 \times 7$. What is the number and what are all its factors?

$2 \times 5 \times 7 = 70$

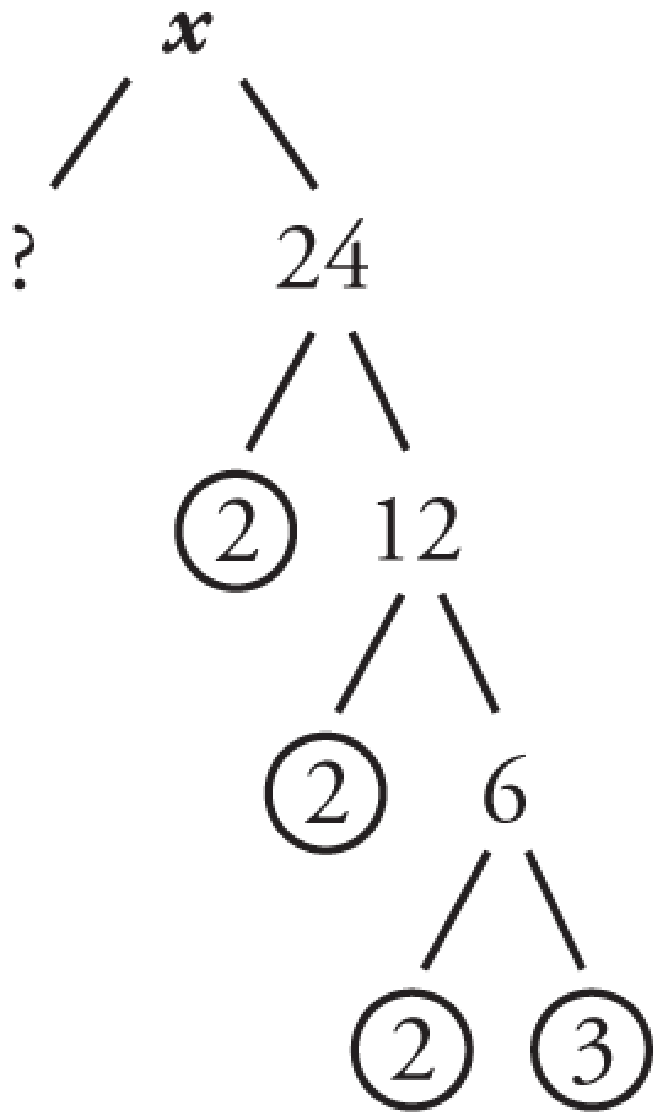
	Small	Large	
1	1	70	$2 \times 5 \times 7$
2	2	35	5×7
5	5	14	2×7
7	7	10	2×5

18. The prime factorization of a number is $2 \times 3 \times 13$. What is the number and what are all its factors?

$2 \times 3 \times 13 = 78$

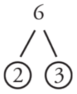
	Small	Large	
1	1	78	2 × 3 × 13
2	2	39	3 × 13
3	3	26	2 × 13
2 × 3	6	13	13

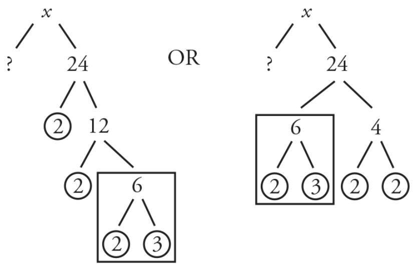
For questions 19–21, x is divisible by 24.



19. **Must Be True**

    x is divisible by 6

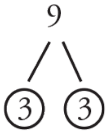


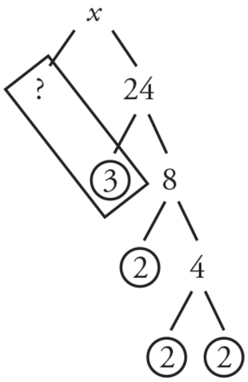


For *x* to be divisible by 6, you need to know that it contains the same prime factors as 6, which contains a 2 and a 3. *x* also contains a 2 and a 3, therefore, *x* *must* therefore be divisible by 6.

20. **Could Be True**

    x is divisible by 9

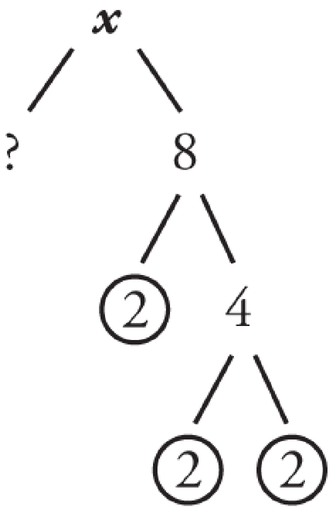




For *x* to be divisible by 9, you need to know that it contains the same prime factors as 9, which contains two 3’s. However, *x* only contains one 3 that you know of. But the question mark means *x* may have other prime factors, and may contain another 3. For this reason, *x* *could* be divisible by 9.

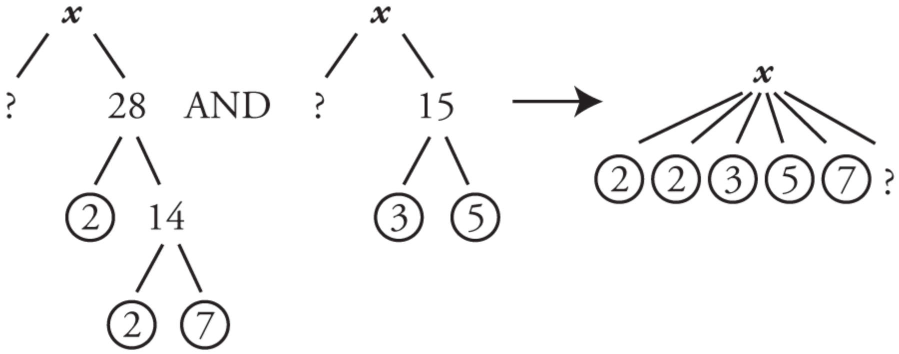
21. **Must Be True**

    x is divisible by 8



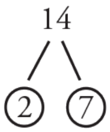
For *x* to be divisible by 8, you need to know that it contains the same prime factors as 8, which contains three 2’s. Because *x* also contains three 2’s, *x must* therefore be divisible by 8.

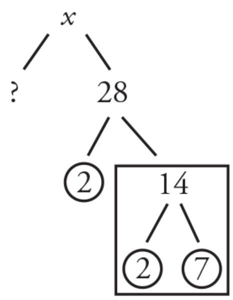
For questions 22–24, *x* is divisible by 28 and by 15.



22. **Must Be True**

x is divisible by 14.

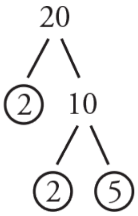


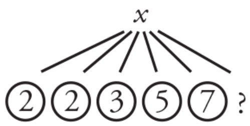


For *x* to be divisible by 14, you need to know that it contains the same prime factors as 14, which contains a 2 and a 7. Because *x* also contains a 2 and a 7, *x must* therefore be divisible by 14.

23. **Must Be True**

x is divisible by 20.

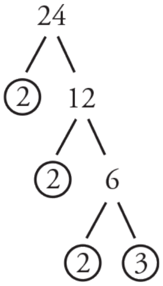


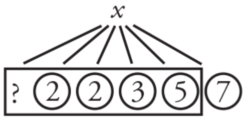


For *x* to be divisible by 20, you need to know that it contains the same prime factors as 20, which contains two 2's and one 5. Because *x* also contains two 2's and a 5, *x must* therefore be divisible by 20.

24. **Could Be True**

x is divisible by 24.





For *x* to be divisible by 24, you need to know that it contains the same prime factors as 24, which contains three 2’s and one 3. However *x* contains one 3, but only two 2’s that you know of. But the question mark means *x* may have other prime factors, and may contain another 2. For this reason, *x could* be divisible by 24.

25. **1**

The number 6 goes into 13 two full times, which means the quotient is 2. Therefore, 2 × 6 = 12, and 12 + 1 = 13. The remainder is 1.

26. **14**

For a number to result in a remainder of 4 when divided by 5, it has to be equal to a multiple of 5, plus 4. The first of these is 4 (5 × 0 + 4 = 4), the second is 9 (5 × 1 + 4 = 9), and the third is 14 (5 × 2 + 4). Thus, 14 is the first double-digit number that produces the required remainder.

27. **7**

Using the Remainder Formula:

$$\frac{x}{9} = Q + \frac{4}{9} \to x = 9Q + 4$$
$$\frac{y}{9} = Q' + \frac{3}{9} \to y = 9Q' + 3$$

Therefore, $x + y = 9(Q + Q') + 7$ and the remainder is 7.

28. **3**

Again using the Remainder Formula:

$$\frac{x}{9} = Q + \frac{4}{9} \rightarrow x = 9Q + 4$$
$$\frac{y}{9} = Q' + \frac{3}{9} \rightarrow y = 9Q' + 3$$

Therefore, $xy = (9Q + 4)(9Q' + 3) = 81QQ' + 27Q + 36Q' + 12$.

Because each of the terms except 12 is divisible by 9, and a 9 can be removed from 12, the correct answer is 12 – 9 = 3.

Problem Set

For problems #1–10, use prime factorization, if appropriate, to answer each question: **Yes**, **No**, or **Cannot Be Determined**. If your answer is **Cannot Be Determined**, use two numerical examples to show how the problem could go either way. All variables in problems 1–12 are assumed to be positive integers unless otherwise indicated.

1. If a is divided by 7 or by 18, an integer results. Is $\frac{a}{42}$ an integer?

2. If 80 is a factor of r, is 15 a factor of r?

3. If 7 is a factor of n and 7 is a factor of p, is $n + p$ divisible by 7?

4. If 8 is not a factor of g, is 8 a factor of $2g$?

5. If j is divisible by 12 and 10, is j divisible by 24?

6. If 12 is a factor of xyz, is 12 a factor of xy?

7. If 6 is a divisor of r and r is a factor of s, is 6 a factor of s?

8. If 24 is a factor of h and 28 is a factor of k, must 21 be a factor of hk?

9. If 6 is not a factor of d, is $12d$ divisible by 6?

10. If 60 is a factor of u, is 18 a factor of u?

11.

Quantity A	**Quantity B**
The number of distinct prime factors of 40	The number of distinct prime factors of 50

12.

Quantity A	**Quantity B**
The product of 12 and an even prime number	The sum of the greatest four factors of 12

13. $x = 20$, $y = 32$, and $z = 12$

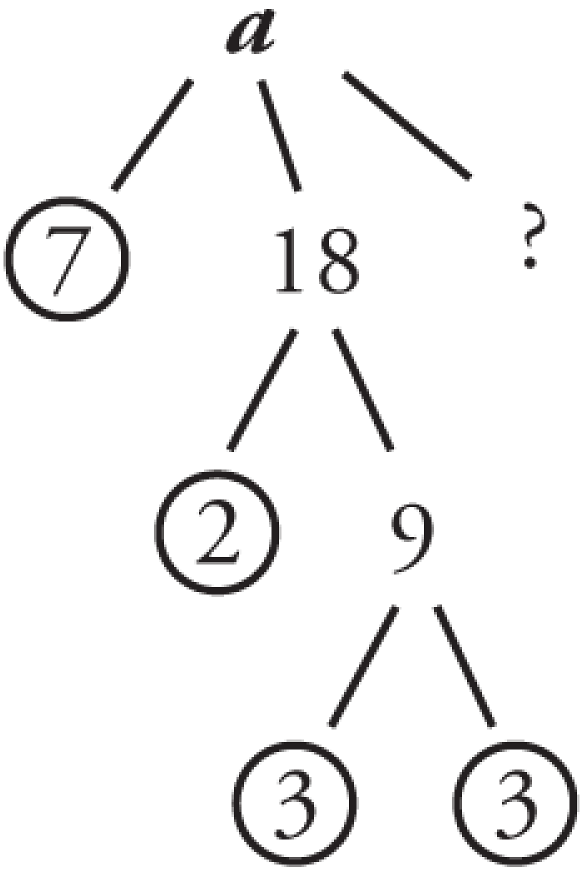
Quantity A	**Quantity B**
The remainder when x is divided by z	The remainder when y is divided by z

14. If a and b are positive integers such that the remainder is 4 when a is divided by b, what is the smallest possible value of $a + b$?

15. If $\frac{x}{y}$ has a remainder of 0 and $\frac{z}{y}$ has a remainder of 3, what is the remainder of $\frac{xz}{y}$?

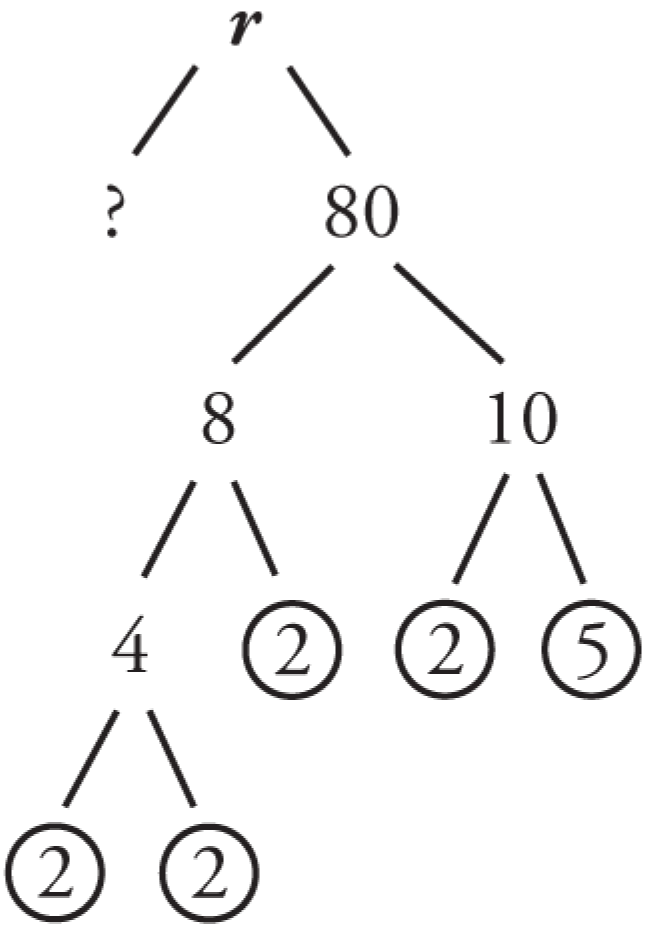
Solutions

1. **Yes**
   If *a* is divisible by 7 and by 18, its prime factors include 2, 3, 3, and 7, as indicated by the factor tree to the right. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of *a*. Thus, 42 = 2 × 3 × 7. Therefore, 42 is also a factor of *a*.



2. **Cannot Be Determined**

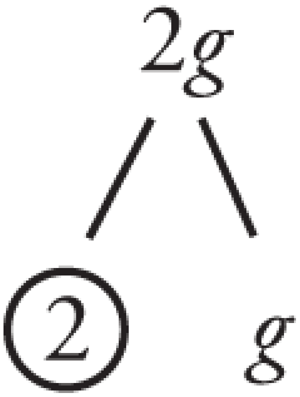
If *r* is divisible by 80, its prime factors include 2, 2, 2, 2, and 5, as indicated by the factor tree below. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of *r*. Thus, 15 = 3 × 5. Because the prime factor 3 is not in the factor tree, you cannot determine whether 15 is a factor of *r*. As numerical examples, you could take *r* = 80, in which case 15 is *not* a factor of *r*, or *r* = 240, in which case 15 *is* a factor of *r*.



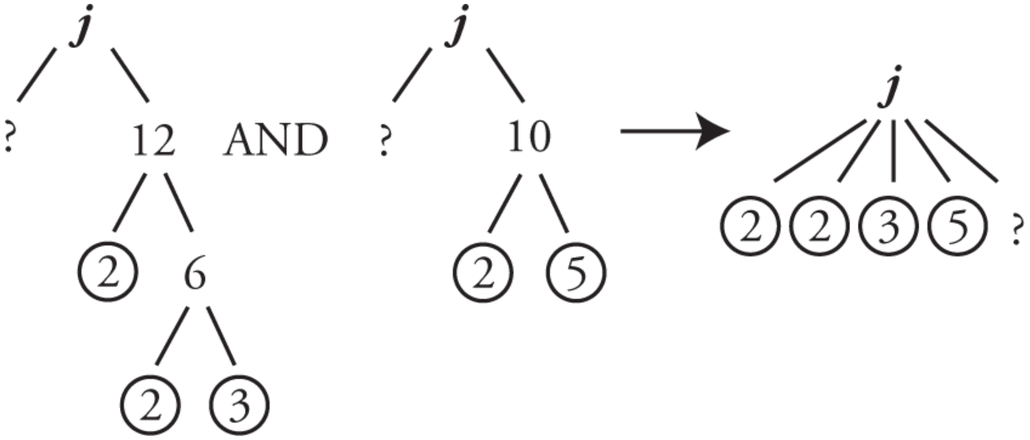
3. **Yes**

If two numbers are both multiples of the same number, then their *sum* is also a multiple of that same number. Because *n* and *p* share the common factor 7, the sum of *n* and *p* must also be divisible by 7.

4. **Cannot Be Determined**
   For 8 to be a factor of 2*g*, you would need two more 2’s in the factor tree. By the Factor Foundation Rule, *g* would need to be divisible by 4. You know that *g* is not divisible by 8, but there are certainly integers that are divisible by 4 and not by 8, such as 4, 12, 20, 28, and so on. However, while you cannot conclude that *g* is *not* divisible by 4, you cannot be certain that *g* *is* divisible by 4, either. As numerical examples, you could take *g* = 5, in which case 8 is *not* a factor of 2*g*, or *g* = 4, in which case 8 *is* a factor of 2*g*.



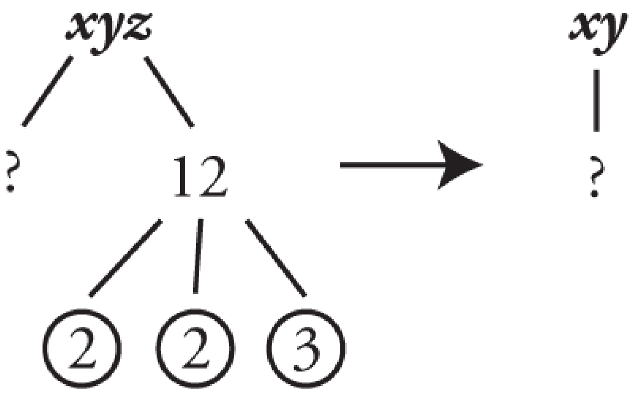
5. **Cannot Be Determined**
   If *j* is divisible by 12 and by 10, its prime factors include 2, 2, 3, and 5, as indicated by the factor tree to the right. There are only two 2’s that are definitely in the prime factorization of *j*, because the 2 in the prime factorization of 10 may be redundant—that is, it may be the same 2 as one of the 2’s in the prime factorization of 12.



Thus, 24 = 2 × 2 × 2 × 3. There are only two 2’s in the prime box of *j*; 24 requires three 2’s. Therefore, 24 is not necessarily a factor of *j*.

As another way to prove that you cannot determine whether 24 is a factor of *j*, consider 60. The number 60 is divisible by both 12 and 10. However, it is not divisible by 24. Therefore, *j* could equal 60, in which case it is not divisible by 24. Alternatively, *j* could equal 120, in which case it is divisible by 24.

6. **Cannot Be Determined**
   If *xyz* is divisible by 12, its prime factors include 2, 2, and 3, as indicated by the factor tree to the right. Those prime factors could all be factors of *x* and *y*, in which case 12 is a factor of *xy*. For example, this is the case when *x* = 20, *y* = 3, and *z* = 7. However, *x* and *y* could be prime or otherwise not divisible by 2, 2, and 3, in which case *xy* is not divisible by 12. For example, this is the case when *x* = 5, *y* = 11, and *z* = 24.

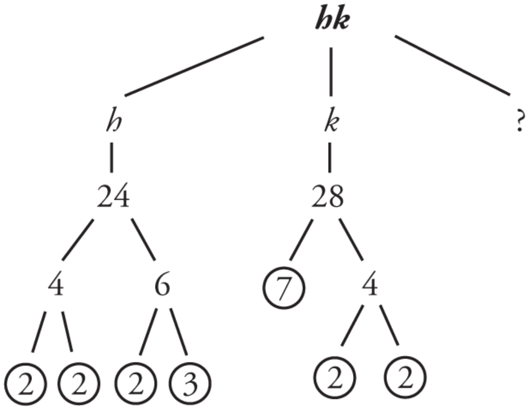


7. **Yes**

   By the Factor Foundation Rule, if 6 is a factor of *r* and *r* is a factor of *s*, then 6 is a factor of *s*.

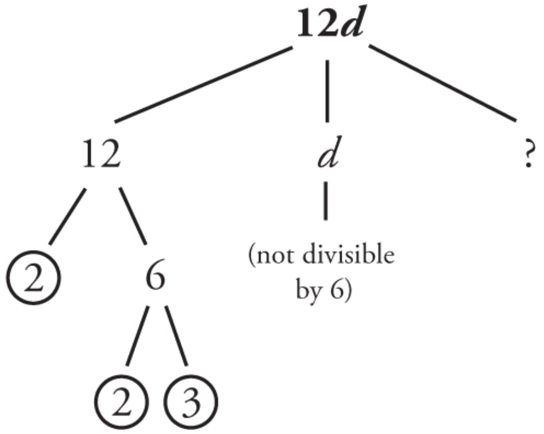
8. **Yes**

   By the Factor Foundation Rule, all the factors of both *h* and *k* must be factors of the product, *hk*. Therefore, the factors of *hk* include 2, 2, 2, 2, 2, 3, and 7, as shown in the combined factor tree to the right. Thus, 21 = 3 × 7. Both 3 and 7 are in the tree. Therefore, 21 is a factor of *hk*.

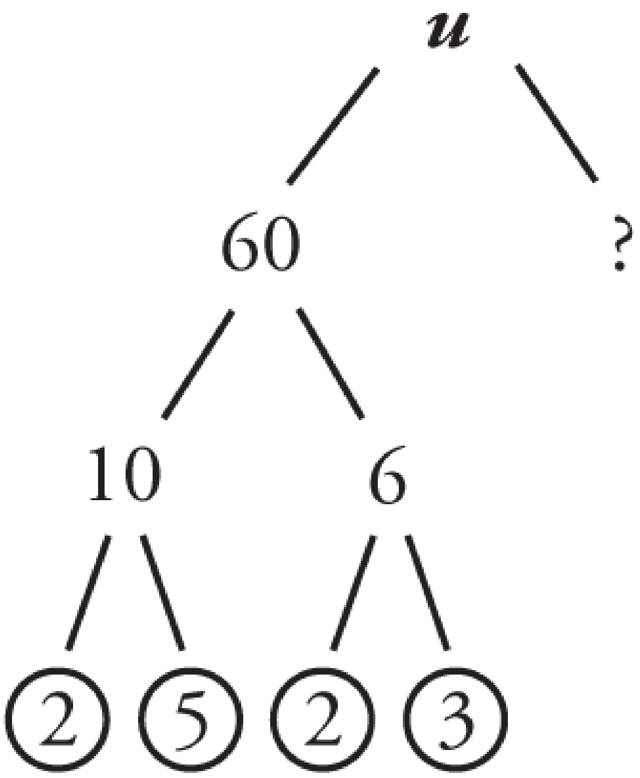


9. **Yes**

The fact that *d* is not divisible by 6 is irrelevant in this case. Because 12 is divisible by 6, 12*d* is also divisible by 6.



10. **Cannot Be Determined**
    If *u* is divisible by 60, its prime factors include 2, 2, 3, and 5, as indicated by the factor tree to the right. Therefore, any integer that can be constructed as a product of any of these prime factors is also a factor of *u*. Next, 18 = 2 × 3 × 3. There is only one 3 in the factor tree, therefore, you cannot determine whether or not 18 is a factor of *u*. As numerical examples, you could take *u* = 60, in which case 18 is not a factor of *u*, or *u* = 180, in which case 18 is a factor of *u*.



11. **(C)**
    The prime factorization of 40 is 2 × 2 × 2 × 5. So 40 has two distinct prime factors: 2 and 5. The prime factorization of 50 is 5 × 5 × 2, so 50 also has two distinct prime factors: 2 and 5. Therefore, **the two quantities are equal**.

12. **(B)**
    Simplify Quantity A first. There is only one even prime number: 2. Therefore, Quantity A is 12 × 2 = 24.

Quantity A	**Quantity B**
The product of 12 and an even prime number =	The sum of the greatest four factors of 12 =
12 × 2 = 24	12 + 6 + 4 + 3 = 25

The four greatest factors of 12 are 12, 6, 4 and 3. Thus, 12 + 6 + 4 + 3 = 25. Therefore, **Quantity B is greater**.

13. **(C)**

When 20 is divided by 12, the result is a quotient of 1 and a remainder of 8 ($12 \times 1 + 8 = 20$).

When 32 is divided by 12, the result is a quotient of 2 and a remainder of 8 ($12 \times 2 + 8 = 32$).

$$x = 20,\ y = 32, \text{ and } z = 12$$

Quantity A	**Quantity B**
8	8

Therefore, **the two quantities are equal.**

14. **9**

Because $\frac{a}{b}$ has a remainder of 4, b must be at least 5 (remember, the remainder must *always* be smaller than the divisor). The smallest possible value for a is 4 (it could also be 9, 14, 19, etc.). Thus, the smallest possible value for $a + b$ is 9.

15. **0**

Because $\frac{x}{y}$ has a remainder of 0, x is divisible by y. Therefore, xz will be divisible by y, and so will have a remainder of 0 when divided by y.

Chapter 16

ODDS & EVENS

In This Chapter...

Arithmetic Rules of Odds & Evens
The Sum of Two Primes
Testing Odd & Even Cases

Chapter 16

Odds & Evens

Even numbers are integers that are divisible by 2. Odd numbers are integers that are not divisible by 2. All integers are either even or odd.

Evens: 0, 2, 4, 6, 8, 10, 12 …

Odds: 1, 3, 5, 7, 9, 11 …

Notice that zero is even, because 0/2 = integer.

Consecutive integers alternate between even and odd: 9, 10, 11, 12, 13 …

O, E, O, E, O …

Negative integers are also either even or odd:

Evens: −2, −4, −6, −8, −10, −12 …

Odds: −1, −3, −5, −7, −9, −11 …

Arithmetic Rules of Odds & Evens

The GRE tests your knowledge of how odd and even numbers combine through addition, subtraction, multiplication, and division. Rules for adding, subtracting, multiplying, and dividing odd and even numbers can be derived by simply picking numbers and testing them out. While this is certainly a valid strategy, it also pays to memorize the following rules for operating with odds and evens, as they are extremely useful for certain GRE math questions.

Addition and Subtraction

Add or subtract 2 odds or 2 evens, and the result is EVEN: $7 + 11 = 18$ and $14 - 6 = 8$.

Add or subtract an odd with an even, and the result is ODD: $7 + 8 = 15$ and $12 - 5 = 7$.

Multiplication

When you multiply integers, if *any* of the integers are even, the result is even: $3 \times \mathbf{8} \times 9 \times 13 = 2{,}808$.

Likewise, if *none* of the integers are even, then the result is odd: $3 \times 5 \times 7 = 105$.

If you multiply together several even integers, the result will be divisible by higher and higher powers of 2. This is because each even number will contribute at least one 2 to the factors of the product.

For example, if there are *two* even integers in a set of integers being multiplied together, the result will be divisible by 4: **2** × 5 × **6** = 60 (divisible by 4).

If there are THREE even integers in a set of integers being multiplied together, the result will be divisible by 8: **2** × 5 × **6** × **10** = 600 (divisible by 8).

To summarize so far:

Odd ± Even = **Odd**

Odd ± Odd = **Even**

Even ± Even = **Even**

Odd × Odd = **Odd**

Even × Even = **Even** (and divisible by 4)

Odd × Even = **Even**

Division

There are no guaranteed outcomes in division, because the division of two integers may not yield an integer result. There are several potential outcomes, depending upon the value of the dividend and divisor.

Divisibility of Odds & Evens

	Even?	Odd?	Non-Integer?

	Even?	**Odd?**	**Non-Integer?**
Even ÷ Even	✓ Example: 12 ÷ 2 = 6	✓ Example: 12 ÷ 4 = 3	✓ Example: 12 ÷ 8 = 1.5
Even ÷ Odd	✓ Example: 12 ÷ 3 = 4	✗	✓ Example: 12 ÷ 5 = 2.4
Odd ÷ Even	✗	✗	✓ Example: 9 ÷ 6 = 1.5
Odd ÷ Odd	✗	✓ Example: 15 ÷ 5 = 3	✓ Example: 15 ÷ 25 = 0.6

An odd number divided by any other integer *cannot* produce an even integer. Also, an odd number divided by an even number *cannot* produce an integer, because the odd number will never be divisible by the factor of 2 concealed within the even number.

Check Your Skills

For questions #1–3, say whether the expression will be odd or even.

1. $1{,}007{,}425 \times 305{,}313 + 2$

2. $5 \times 778 \times 3 \times 4 + 1$

3. The sum of four consecutive integers.

4. Will the product of two odd integers divided by a multiple of two be an integer?

The Sum of Two Primes

Notice that all prime numbers are odd, except the number 2. (All larger even numbers are divisible by 2, so they cannot be prime.) Thus, the sum of any two primes will be even (“Add two odds …”), unless one of those primes is the number 2. So, if you see a sum of two primes that is odd, one of those primes must be the number 2. Conversely, if you know that 2 *cannot* be one of the primes in the sum, then the sum of the two primes must be even.

> If *a* and *b* are both prime numbers greater than 10, which of the following may be true? Indicate all that apply.
>
> $\boxed{\text{A}}$ *ab* is an even number.
> $\boxed{\text{B}}$ The difference between *a* and *b* equals 117.
> $\boxed{\text{C}}$ The sum of *a* and *b* is even.

Because *a* and *b* are both prime numbers greater than 10, they must both be odd. Therefore, *ab* must be an odd number, so choice (A) cannot be true. Similarly, if *a* and *b* are both odd, then $a - b$ cannot equal 117 (an odd number). This difference must be even. Therefore, choice (B) cannot be true. Finally, because *a* and *b* are both odd, $a + b$ must be even, so choice (C) will always be true.

Check Your Skills

5. The difference between the factors of prime number x is 1. The difference between the factors of prime number y is 2. Is xy even?

Testing Odd & Even Cases

Sometimes multiple variables can be odd or even, and you need to determine the implications of each possible scenario. In that case, set up a table listing all the possible odd/even combinations of the variables, and determine what effect that would have on the question.

> If a, b, and c are integers and $ab + c$ is odd, which of the following must be true? Indicate all that apply.
>
> $\boxed{\text{A}}$ $a + c$ is odd
>
> $\boxed{\text{B}}$ $b + c$ is odd
>
> $\boxed{\text{C}}$ abc is even

Here, a, b, and c could all possibly be odd or even. Some combinations of odds & evens for a, b, and c will lead to an odd result. Other combinations will lead to an even result. You need to test each possible combination to see what the result will be for each. Set up a table, as shown, and fill in the possibilities.

Scenario	***a***	***b***	***c***	***ab* + *c***
1	Odd	Odd	Even	O × O + E = O
2	Odd	Even	Odd	O × E + O = O
3	Even	Odd	Odd	E × O + O = O
4	Even	Even	Odd	E × E + O = O

If *c* is even, *ab* must be odd in order for *ab* + *c* to be odd. For that, *a* and *b* must each be odd. If *c* is odd, *ab* must be even in order for *ab* + *c* to be odd. There are three ways that could happen — either *a* or *b* could be even, or both could be even. You can conclude that choice (A) need not be true (Scenario 2 yields *a* + *c* = even), choice (B) need not be true (Scenario 3 yields *b* + *c* = even), and choice (C) must be true (all four working scenarios yield *abc* = even). Therefore, the only correct answer is choice (C).

Check Your Skills

6. If x and y are integers, and $\frac{x}{y}$ is even, which of the following could be true?

   Indicate all that apply.

   - ☐ A xy is odd
   - ☐ B xy is even
   - ☐ C $x + y$ is odd

7. x, y, and z are integers. If xyz is even, $x + z$ is odd, and $y + z$ is odd, z is:

   Choose just one answer.

(A) Even

(B) Odd

(C) Indeterminable (could be even or odd)

Check Your Skills Answer Key

1. **Odd**
   You have an odd multiplied by an odd, which always results in an odd. Then add an even to the odd, which also results in an odd.

2. **Odd**
   At least one of the numbers multiplied together is even, meaning the product will be even. When you add an odd to that even, you get an odd.

3. **Even**
   Because integers go back and forth between evens and odds, the sum of any four consecutive integers can be expressed as Even + Even + Odd + Odd. Taking these one by one, start with Even + Even = Even. Then add an Odd to that Even, resulting in an Odd. Finally, add another Odd to that Odd, resulting in an Even.

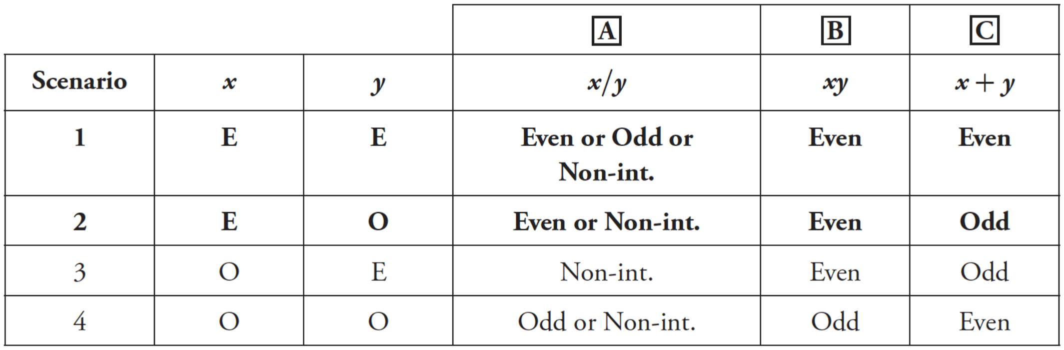
4. **No**
   The product of two odd integers is always odd. Any multiple of two is even, and as the chart showed, an odd divided by an even cannot be an integer.

5. **Yes**

Prime numbers only have two factors: 1 and themselves. So if the difference between the factors of a prime number is 1, its factors must be 1 and 2. This means $x = 2$. By the same logic, y must be equal to 3 ($3 - 1 = 2$). The product of 2 and 3 is 6, so xy is even.

6. **(B)** and **(C)**

If x/y is even, then either x and y are both even, or x is even and y is odd. Make a chart:



			$\boxed{A}$	$\boxed{B}$	$\boxed{C}$
Scenario	x	y	x/y	xy	$x + y$
1	**E**	**E**	**Even or Odd or Non-int.**	**Even**	**Even**
2	**E**	**O**	**Even or Non-int.**	**Even**	**Odd**
3	O	E	Non-int.	Even	Odd
4	O	O	Odd or Non-int.	Odd	Even

The question stem stipulates that x/y is even. This is only possible in the first two scenarios. In both of those situations, xy is even. This means that choice (A) is untrue, but choice (B) is true. While $x + y$ can be either even or odd, that means that it *could* be odd, so choice (C) also works.

7. **(C)**

Indeterminable. Once again, make a chart.

The more restrictive constraints are the odd sums, which require either odd + even or even + odd. Thus, x and y have opposite status, as do y and z. There are only two such cases, as shown here:

Scenario	***x***	***y***	***z***	***xyz***	***x* + *z***	***y* + *z***
1	**E**	**E**	**O**	**E**	**O**	**O**
2	**O**	**O**	**E**	**E**	**O**	**O**

Therefore, *xyz* is even in either case, since there is always at least one even term in the product. As you can see, *z* is even in one case and odd in the other.

Problem Set

For problems #1–15, answer each question **Odd**, **Even**, or **Cannot Be Determined**. Try to explain each answer using the rules you learned in this section. All variables in problems #1–15 are assumed to be integers unless otherwise indicated.

1. If n is odd, p is even, and q is odd, what is $n + p + q$?
2. If r is a prime number greater than 2, and s is odd, what is rs?
3. If t is odd, what is t^4?
4. If u is even and w is odd, what is $u + uw$?
5. If $x \div y$ yields an odd integer, what is x?
6. If $a + b$ is even, what is ab?

7. If c, d, and e are consecutive integers, what is cde?

8. If f and g are prime numbers, what is $f + g$?

9. If h is even, j is odd, and k is odd, what is $k(h + j)$?

10. If m is odd, what is $m^2 + m$?

11. If n, p, q, and r are consecutive integers, what is their sum?

12. If $t = s - 3$, what is $s + t$?

13. If u is odd and w is even, what is $(uw)^2 + u$?

14. If xy is even and z is even, what is $x + z$?

15. If a, b, and c are consecutive integers, what is $a + b + c$?

16. 202 divided by some prime number *x* yields an odd number. 411 multiplied by some prime number *y* yields an even number.

Quantity A	**Quantity B**
x	y

17.

Quantity A	**Quantity B**
The tenths digit of the product of two even integers divided by 4	The tenths digit of the product of an even and an odd integer divided by 4

18. x is a non-negative even integer.

Quantity A	**Quantity B**
x	1

Solutions

1. **Even**
   O + E = O. O + O = E. If in doubt, try plugging in actual numbers: 7 + 2 + 3 = 12 (even).

2. **Odd**
   Multiplying odd by odd equals odd: O × O = O. If in doubt, try plugging in actual numbers: 3 × 5 = 15 (odd).

3. **Odd**
   Again, multiplying the odds: O × O × O × O = O. If in doubt, try plugging in actual numbers: 3 × 3 × 3 × 3 = 81 (odd).

4. **Even**
   uw is even. Therefore, E + E = E.

5. **Cannot Be Determined**
   There are no guaranteed outcomes in division. For example, 6 ÷ 2 = 3 = odd, but 3 ÷ 1 = 3 = odd. Thus, *x* could be even or odd.

6. **Cannot Be Determined**
   If *a* + *b* is even, *a* and *b* are either both odd or both even. If they are both odd, *ab* is odd. If they are both even, *ab* is even. Therefore, you cannot determine whether *ab* is odd or even.

7. **Even**
   At least one of the consecutive integers, c, d, or e, must be even. Therefore, the product cde must be even.

8. **Cannot Be Determined**
   If either f or g is 2, then $f + g$ will be odd. If f and g are odd primes, or if f and g are both 2, then $f + g$ will be even. Therefore, you cannot determine whether $f + g$ is odd or even.

9. **Odd**
   $h + j$ must be odd (E + O = O). Therefore, $k(h + j)$ must be odd (O × O = O).

10. **Even**
    m^2 must be odd (O × O = O). Therefore, $m^2 + m$ must be even (O + O = E).

11. **Even**
    If n, p, q, and r are consecutive integers, two of them must be odd and two of them must be even. You can pair them up to add them: O + O = E and E + E = E. Adding the pairs, you will see that the sum must be even: E + E = E.

12. **Odd**
    If s is even, then t must be odd. If s is odd, then t must be even. Either way, the sum must be odd: E + O = O, or O + E = O.

13. **Odd**

$(uw)^2$ must be even. Therefore, E + O = O.

14. **Cannot Be Determined**

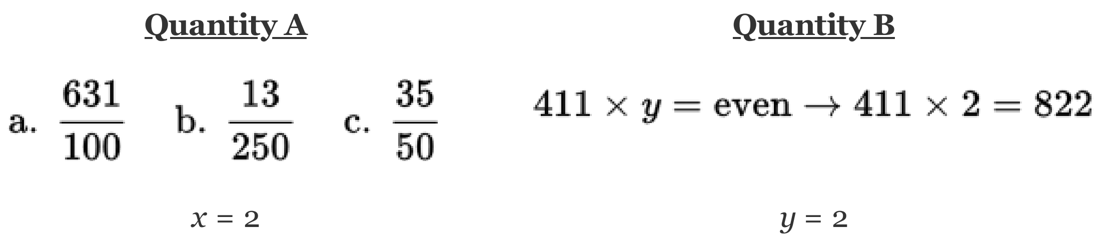
    If xy is even, then either x or y (or both x and y) must be even. Given that z is even, $x + z$ could be O + E or E + E. Therefore, you cannot determine whether $x + z$ is odd or even.

15. **Cannot Be Determined**

    If a, b, and c are consecutive, then there could be either one or two even integers in the set. $a + b + c$ could be O + E + O or E + O + E. In the first case, the sum is even; in the second, the sum is odd. Therefore, you cannot determine whether $a + b + c$ is odd or even.

16. **(C)**

    An even divided by an odd can never yield an odd quotient. This means the prime number x must be even (because otherwise you'd have 202/odd, which wouldn't yield an odd quotient). The only even prime number is 2, so $x = 2$. Similarly, an odd times an odd will always be odd, so y must be even. The only prime even number is 2, so $y = 2$.



Quantity A	Quantity B
a. $\frac{631}{100}$ b. $\frac{13}{250}$ c. $\frac{35}{50}$	$411 \times y = \text{even} \rightarrow 411 \times 2 = 822$
$x = 2$	$y = 2$

17. **(D)**

This question could be solved either by trying out numbers or making a chart. For Quantity A, the product of two even integers will always divide evenly by 4 because each even number has a 2 in its prime tree. For instance, 2 × 2 = 4, 2 × 4 = 8, and 2 × 6 = 12. All of these numbers are divisible by 4, and the integer that results after dividing by 4 will always have a zero in the tenths digit.

$$4 \div 4 = 1.0,\ 8 \div 4 = 2.0,\ 12 \div 4 = 3.0$$

The product of an even and an odd integer *could* be divisible by 4, which would mean that its tenths digit was 0. For example, 4 × 5 = 20, and $\frac{20}{4} = 5$, the tenths digit of which is 0. However, it could also *not* be divisible by 4. For example, 2 × 5 = 10, and $\frac{1}{2} + \frac{3}{4} =$, the tenths digit of which is 5. Because the two quantities could be equal or different, the answer must be choice (D). Therefore, **the relationship cannot be determined from the information given**.

18. **(D)**

    Always be careful when dealing with evens and odds. While 0 is neither positive nor negative, it *is* even. Thus, the first possible value of *x* here is 0, not 2. Thus, *x* could be either less than or greater than 1. Therefore, **the relationship cannot be determined from the information given**.

Chapter 17

POSITIVES & NEGATIVES

In This Chapter...

Absolute Value: Absolutely Positive

A Double Negative = A Positive

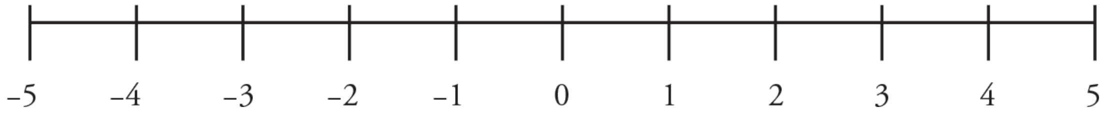
Multiplying & Dividing Signed Numbers

Testing Positive & Negative Cases

Chapter 17

Positives & Negatives

Numbers can be either positive or negative (except the number 0, which is neither). A number line illustrates this idea:



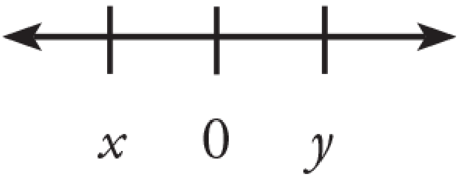
Negative numbers are all to the left of 0. Positive numbers are all to the right of 0.

Note that a variable (such as x) can have either a positive or a negative value, unless there is evidence otherwise. The variable x is not necessarily positive, nor is $-x$ necessarily negative.

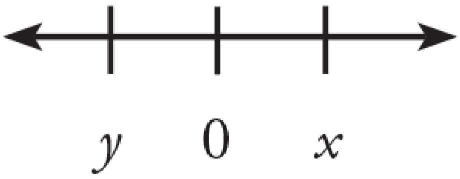
Absolute Value: Absolutely Positive

The absolute value of a number answers this question: **How far away is the number from 0 on the number line?** For example, the number 5 is exactly 5 units away from 0, so the absolute value of 5 equals 5. Mathematically, this is written using the symbol for absolute value: $|5| = 5$. To find the absolute value of −5, look at the number line above: −5 is also exactly 5 units away from 0. Thus, the absolute value of −5 equals 5, or, in mathematical symbols, $|-5| = 5$. Notice that absolute value is always positive, because it disregards the direction (positive or negative) from which the number approaches 0 on the number line. When you interpret a number in an absolute value sign, just think: Absolutely positive! (Except, of course, for 0, because $|0| = 0$, which is the smallest possible absolute value.)

On the number line shown, note that 5 and −5 are the same distance from 0, which is located halfway between them. In general, if two numbers are opposites of each other, then they have the same absolute value, and 0 is halfway between. If $x = -y$, then you have either



OR



(You cannot tell which variable is positive without more information.)

A Double Negative = A Positive

A double negative occurs when a minus sign is in front of a negative number (which already has its own negative sign). For example:

What is $7 - (-3)$?

Subtracting a negative number is equivalent to adding the corresponding positive number:

$7 - (-3) = 7 + 3 = 10$.

This is a very easy step to miss, especially when the double negative is somewhat hidden. For instance:

What is $7 - (12 - 9)$?

Many people will make the mistake of computing this as $7 - 12 - 9 = -14$. However, notice that the second term in the expression in parentheses has a double negative. Therefore, this expression should be calculated as $7 - 12 + 9 = 4$.

Check Your Skills

1. Does $|-5| + |5| + |-5| = |15|$?

2. If $4y - (x - 4) = 4x + (-y + 4)$, and neither x nor $y = 0$, what is $\frac{x}{y}$?

Multiplying & Dividing Signed Numbers

When you multiply or divide two numbers, positive or negative, follow one simple rule:

If **S**igns are the **S**ame, the answer is po**S**itive
but if **N**ot, the answer is **N**egative.

$7 \times 8 = 56$ and $(-7) \times (-8) = 56$
$(-7) \times 8 = -56$ and $7 \times (-8) = -56$
$56 \div 7 = 8$ and $-56 \div (-8) = 7$
$56 \div (-7) = -8$ and $-56 \div 8 = -7$

That is, positive × positive or negative × negative will result in a positive. In contrast, positive × negative will result in a negative.

This principle can be extended to multiplication and division by more than two numbers. For example, if three numbers are multiplied together, the result will be positive if there are NO negative numbers, or *two* negative numbers. The result will be negative if there is *one* or *three* negative numbers.

This pattern can be summarized as follows. When you multiply or divide a group of nonzero numbers, the result will be positive if you have an EVEN number of negative numbers. The result will be negative if you have an ODD number of negative numbers.

Check Your Skills

3. Is the product $-12 \times -15 \times 3 \times 4 \times 5 \times -2$ positive or negative?

4. If $xy \neq 0$, is $-x \times -y$ definitely positive?

Testing Positive & Negative Cases

Some Positives & Negatives problems deal with multiple variables, each of which can be positive or negative. In these situations, you should set up a table listing all the possible positive/negative combinations of the variables, and determine what effect that would have on the question. For example:

> If $ab > 0$, which of the following must be negative?
>
>) $a + b$
>
>) $|a| + b$
>
>) $b - a$
>
>) $\frac{a}{b}$
>
>) $-\frac{a}{b}$

One way to solve problems such as this one is to test numbers systematically. In this example, you can list both of the two possible positive/negative combinations of *a* and *b* that meet the criteria established in the question. Then, test each of the combinations in each of the answer choices. You can use a chart such as the one below to keep track of your work, choosing simple values (e.g., 3 and 6) to make calculations quickly:

	Criterion: $ab > 0$	**A** $a + b$	**B** $\lvert a \rvert + b$	**C** $b - a$	**D** $\frac{a}{b}$	**E** $-\frac{a}{b}$
+, + $a = 3$ $b = 6$	YES	POS	POS	POS	POS	[✓ NEG]
−, − $a = -3$ $b = -6$	YES	NEG	NEG	NEG	POS	[✓ NEG]

Notice that if more than one answer choice gives you the desired result for all cases, you can try another pair of numbers and test those answer choices again.

Another approach to this problem is to determine what you know from the fact that $ab > 0$. If $ab > 0$, then the signs of a and b must both be the same (both positive or both negative). This should lead you to answer choice (E), because $-\frac{a}{b}$ must be negative if a and b have the same sign.

Check Your Skills

5. $|x| > |y|$. Which of the following must be true? Indicate all that apply.

$\boxed{\text{A}}$ xy is positive

$\boxed{\text{B}}$ $x + y > 0$

$\boxed{\text{C}}$ $x^2 > y^2$

6. If $ab < 0$, $a > b$, and $a > -b$, which of the following must be true?

(A) $a/b > 0$

(B) $a + b < 0$

(C) $b - (-a) > 0$

(D) $a/b = 1$

(E) $a - b < 0$

Check Your Skills Answer Key

1. **Yes**

   The absolute values of 5 and −5 are both 5, and $5 + 5 + 5 = 15$.

2. **1**

   This question is easy as long as you pay close attention to the signs. The left side of the equation will have a double negative (in front of the 4), so it simplifies to $4y - x + 4$. The right side has no double negative (in fact, the parentheses are unnecessary), so it simplifies to $4x - y + 4$.

   The equation now reads:

   $$4y - x + 4 = 4x - y + 4.$$

   Continue to simplify: $5y = 5x$, or $x = y$. If x and y are equal, then $\frac{x}{y} = 1$.

3. **Negative**

   Here there are three negative numbers and three positive numbers. The product of two of the negative numbers will be positive, and the third negative number will make the final product negative.

4. **No**

Because the two negative signs multiply to a positive, you can say that $(-x)(-y) = xy$. However, always be careful when dealing with variables, because x or y could themselves be negative. If x and y are both positive or both negative, their product will be positive. But if x is positive and y is negative, the product will be negative.

5. **(C)**
What's more fun than making a chart? Nothing. Try looking at the four possible situations with x and y that maintain the requirement that the absolute value of x is greater than the absolute value of y:

Scenario	x	y	xy	$x + y$	x^2	y^2
1	5	3	15	8	25	9
2	−5	3	−15	−2	25	9
3	5	−3	−15	2	25	9
4	−5	−3	15	−8	25	9

The second scenario here gets rid of both choice (A) and choice (B), because xy and $x + y$ are both negative. However, in every case, $x^2 > y^2$, so choice (C) alone is necessarily true. You could also solve this by looking at each statement while keeping in mind what you know if the absolute value of x is greater than the absolute value of y. You don't know if xy is positive because x or y could be negative.

6. **(C)**
Though a bit tricky, the given information here tells you everything you need to know about a and b. Instead of testing numbers to get through

the answer choices (as above), test numbers to make sense of the given information. First, if $ab < 0$, then the two variables must have opposite signs. Second, if $a > b$, then a must be the positive number, and b the negative number. Finally, if $a > -b$, a must have a larger absolute value than b (if $a = 4$ and $b = -5$, then $-b > a$, which is the opposite of what you want).

Now, it should be enough just to walk through the answer choices.

$$\frac{a}{b} < 1$$

Untrue: If a and b have opposite signs, then the quotient will be negative.

$a + b < 0$

Untrue: You know that a is positive and has a larger absolute value than b. No matter what a and b are, their sum has to be positive.

$b - (-a) > 0$

True: This actually simplifies to look like the equation in answer choice (B), though with the sign switched. You already know this has to be true.

$\frac{a}{b} < 1$.

Untrue: The quotient of a positive and a negative number must be negative, so can never be 1.

$a - b < 0$

Untrue: If you subtract a negative number from a positive number, you'll be left with an even bigger positive number.

Problem Set

Solve problems #1–5. *Don't use the calculator.*

1. Evaluate $2|x - y| + |z + w|$ if $x = 2$, $y = 5$, $z = -3$, and $w = 8$.

2. Simplify $66 \div (-33) \times |-9|$

3. Simplify $\frac{-30}{5} - \frac{18 - 9}{-3}$

4. Simplify $\frac{20 \times (-7)}{-35 \times (-2)}$

5. When is $|x - 4|$ equal to $4 - x$?

In problems 6–15, decide whether the expression described is **Positive**, **Negative**, or **Cannot Be Determined**. If you answer

Cannot Be Determined, give numerical examples to show how the problem could be either positive or negative.

6. The product of three negative numbers

7. The quotient of one negative and one positive number

8. xy, given that $x < 0$ and $y \neq 0$

9. $|x| \times y^2$, given that $xy \neq 0$

10. $\frac{x}{y} \div z$, given that x, y, and z are negative

11. $\frac{|ab|}{b}$, given that $b < a < 0$

12. $-4|d|$, given that $d \neq 0$

13. $\frac{rst}{w}$, given that $r < s < 0 < w < t$

14. $h^4k^3m^2$, given that $k < 0$ and $hm \neq 0$

15. $\frac{3}{10} \times \frac{6}{7} =$, given that $xyz > 0$

16.

$xy > 0$

Quantity A	**Quantity B**
$\frac{x}{\lvert x \rvert}$	$3\frac{1}{2}$

17.

Quantity A	**Quantity B**
$-a \times -a \times a \times a$	-1

18.

$|x| = |y|, x \neq 0$

Quantity A	**Quantity B**
$x + y$	$2x$

Solutions

1. **11**

   $2|x - y| + |z + w| = 2|2 - 5| + |-3 + 8| = 2|-3| + |5| = 2(3) + 5 = 11$. Note that when you deal with more complicated absolute value expressions, such as $|x - y|$ in this example, you should NEVER change individual signs to "+" signs! For instance, in this problem $|x - y| = |2 - 5|$, not $|2 + 5|$.

2. **−18**

   In division, use the Same Sign rule. In this case, the signs are not the same. Therefore, 66 ÷ (−33) yields a negative number (−2). Then, multiply by the absolute value of −9, which is 9. To multiply −2 × 9, use the Same Sign rule: the signs are not the same, so the answer is negative. Remember to apply division and multiplication from left to right: first the division, then the multiplication.

3. **−3**

   This is a two-step subtraction problem. Use the Same Sign rule for both steps. In the first step, $\frac{-30}{5}$, the signs are different; therefore, the answer is −6. In the second step, $\frac{18-9}{-3} = \frac{9}{-3}$, the signs are again different. That result is −3. The final answer is $-6 - (-3) = -3$.

4. **−2**

The sign of the first product, 20 × (−7), is negative (by the Same Sign rule). The sign of the second product, −35 × (−2), is positive (by the Same Sign rule). Applying the Same Sign rule to the final division problem, the final answer must be negative.

5. ***x* ≤ 4**
   Absolute value brackets can only do one of two things to the expression inside of them: (1) leave the expression unchanged, whenever the expression is 0 or positive, or (2) change the sign of the whole expression, whenever the expression is 0 or negative. (Notice that both outcomes occur when the expression is 0, because "negative 0" and "positive 0" are equal.) In this case, the sign of the whole expression $x - 4$ is being changed, resulting in $-(x - 4) = 4 - x$. This will happen only if the expression $x - 4$ is 0 or negative. Therefore, $x - 4 \leq 0$, or $x \leq 4$.

6. **Negative**
   The product of the first two negative numbers is positive. This positive product times the third negative is negative.

7. **Negative**
   By the Same Sign rule, the quotient of a negative and a positive number must be negative.

8. **Cannot Be Determined**
   x is negative. However, y could be either positive or negative. Thus, there is no way to determine whether the product xy is positive or

negative. Numerical examples are $x = -2$ and $y = 3$ or -3, leading to $xy = -6$ or 6.

9. **Positive**
   $|x|$ is positive because absolute value can never be negative, and $x \neq 0$ (since $xy \neq 0$). Also, y^2 is positive because y^2 will be either positive × positive or negative × negative (and $y \neq 0$). The product of two positive numbers is positive, by the Same Sign rule.

10. **Negative**
    Do this problem in two steps: First, a negative number divided by a negative number yields a positive number (by the Same Sign rule). Second, a positive number divided by a negative number yields a negative number (again, by the Same Sign rule).

11. **Negative**
    a and b are both negative. Therefore, this problem is a positive number (by the definition of absolute value) divided by a negative number. By the Same Sign rule, the answer will be negative.

12. **Negative**
    You do not need to know the sign of d to solve this problem. Because d is within the absolute value symbols, you can treat the expression $|d|$ as a positive number (since you know that $d \neq 0$). By the Same Sign rule, a negative number times a positive number yields a negative number.

13. **Positive**

r and s are negative; w and t are positive. Therefore, rst is a positive number. A positive number divided by another positive number yields a positive number.

14. **Negative**

    Nonzero numbers raised to even exponents always yield positive numbers. Therefore, h^4 and m^2 are both positive. Because k is negative, k^3 is negative. Therefore, the final product, $h^4k^3m^2$, is the product of two positives and a negative, which is negative.

15. **Negative**

    Simplifying the original fraction yields: $\frac{-x}{yz}$.

    If the product xyz is positive, then there are two possible scenarios: (1) all the integers are positive, or (2) two of the integers are negative and the third is positive. Test out both scenarios, using real numbers. In the first case, the end result is negative. In the second case, the two negative integers will essentially cancel each other out. Again, the end result is negative.

16. **(C)**

    If $xy > 0$, x and y have the same sign. You already know that the denominator of both fractions described in the quantities will be positive. The numerator will either be positive for both, or negative for both. If both x and y are positive, the quantities simplify like this:

$$xy > 0$$

Quantity A	Quantity B
$\frac{x}{\lvert x\rvert} \rightarrow \frac{\text{positive } x}{\text{positive } x} \rightarrow 1$	$\frac{x}{\lvert x\rvert} \rightarrow \frac{\text{positive } x}{\text{positive } x} \rightarrow 1$

In this case, both quantities equal 1. If *x* and *y* are both negative, the quantities simplify like this:

$$xy > 0$$

Quantity A	Quantity B
$\frac{x}{\lvert x\rvert} \rightarrow \frac{\text{negative } x}{\text{positive } x} \rightarrow -1$	$\frac{y}{\lvert y\rvert} \rightarrow \frac{\text{negative } y}{\text{positive } y} \rightarrow -1$

In this case, both quantities equal −1. Either way, the values in the **two quantities are equal**.

17. **(A)**
If *a* is positive, then −*a* is negative, and Quantity A can be rewritten as (negative) × (negative) × (positive) × (positive), which will result in a positive product.

If *a* is negative, then −*a* is positive, and Quantity A can be rewritten as (positive) × (positive) × (negative) × (negative), which will result in a positive product.

In either of these situations, the quantities look like this:

Quantity A	Quantity B

positive	−1

Quantity A will be greater.

The other possibility is that a is 0. If a is 0, then Quantity A looks like this:

Quantity A	**Quantity B**
$0 \times 0 \times 0 \times 0 = 0$	-1

Therefore, **Quantity A is greater** in either scenario.

18. **(D)**
If $|x| = |y|$, then the two numbers could either be equal (positive or negative) or opposite (one positive and one negative). The following chart shows all the possible arrangements if $|x| = |y| = 3$:

x	y	Quantity A = $x + y$		Quantity B = $2x$
3	3	6	=	6
3	−3	0	<	6
−3	3	0	>	−6
−3	−3	−6	=	−6

Alternatively, you could reason that if x and y are the same sign, then $x = y$. Substitute x for y in Quantity A:

$$|x| = |y|, x \neq 0$$

Quantity A	Quantity B
$x + (x) = 2x$	$2x$

If x and y are the same sign, the quantities are equal.

If x and y have opposite signs, then $-x = y$. Substitute $-x$ for y in Quantity A:

$$|x| = |y|, x \neq 0$$

Quantity A	Quantity B
$x + (-x) = 0$	$2x$

If x does not equal 0, then the values in the two quantities will be different. The correct answer is (D). Thus, **the relationship cannot be determined from the information given**.

Chapter 18

EXPONENTS

In This Chapter...

Wow, That Increased Exponentially!
All About the Base
All About the Exponent
Combining Exponential Terms
Rewriting Bases
Simplifying Exponential Expressions
Rules of Exponents
Common Exponent Errors

Chapter 18

Exponents

The mathematical expression 4^3 consists of a base (4) and an exponent (3).

The expression is read as “four to the third power.” The base (4) is multiplied by itself as many times as the power requires (3).

Thus, 4^3 equals $4 \times 4 \times 4 = 64$.

Two exponents have special names: the exponent 2 is called the square, and the exponent 3 is called the cube. For example, 5^2 can be read as five to the second power, or as five squared ($5^2 = 5 \times 5 = 25$); 5^3 can be read as five to the third power, or as five cubed ($5^3 = 5 \times 5 \times 5 = 125$).

Wow, That Increased Exponentially!

Have you ever heard the expression: “Wow, that increased exponentially!”? This phrase captures the essence of exponents. When a positive number greater than 1 increases exponentially, it does not merely increase; it increases a whole lot in a short amount of time.

An important property of exponents is that the greater the exponent, the faster the rate of increase. Consider the following progression:

$5^1 = 5$	
$5^2 = 25$	Increased by 20
$5^3 = 125$	Increased by 100
$5^4 = 625$	Increased by 500

The important thing to remember is that for positive bases bigger than 1, the greater the exponent, the faster the rate of increase.

All About the Base

The Sign of the Base

The base of an exponential expression may be either positive or negative. With a negative base, simply multiply the negative number as many times as the exponent requires.

For example:

$(-4)^2 = (-4) \times (-4) = 16$ $(-4)^3 = (-4) \times (-4) \times (-4) = -64$

Consider this problem:

If $x^2 = 16$, is x equal to 4?

Your initial inclination is probably to say yes. However, x may not be 4; it may be −4. Thus, you cannot answer the question without additional information. You must be told that x is positive in order to affirm that x is 4. Beware whenever you see an even exponent on the test. Another important thing to remember is that according to the Order of Operations rules, (PEMDAS), exponents have higher precedence than subtraction, so -4^2 means $-(4^2) = -16$, not $(-4)^2 = 16$.

The Even Exponent Is Dangerous: It Hides the Sign of the Base!

One of the GRE’s most common tricks involves the even exponent. In many cases, when an integer is raised to a power, the answer keeps the original sign of the base. For example:

$3^2 = 9$	$(-3)^3 = -27$	$3^3 = 27$
(positive base, positive result)	(negative base, negative result)	(positive base, positive result)

However, any base raised to an even power will always result in a positive answer. This is because even if the underlying base is negative, there will be an *even* number of negative signs in the product, and an even number of negative signs in a product makes the product positive. For example:

$3^2 = 9$	$(-3)^2 = 9$	$(-3)^4 = 81$
(positive base, positive result)	(negative base, positive result	(negative base, positive result)

Therefore, when a base is raised to an even exponent, the resulting answer may either keep or change the original sign of the base. Whether $x = 3$ or -3, $x^2 = 9$. This makes even exponents extremely dangerous, and the GRE loves to try to trick you with them.

Note that odd exponents are harmless, since they always keep the original sign of the base. For example, if you have the equation $x^3 = 64$, you can be sure that $x = 4$. You know that x is not −4 because $(-4)^3$ would yield −64.

Check Your Skills

1. If $x \times x \times x = -27$, what is x?

2. If $x^2 \times x^3 \times x = 64$, what is x?

A Base of 0, 1, or −1

- The number 0 raised to any positive power always yields 0.
- The number 1 raised to any power at all always yields 1.
- The number −1 raised to any power yields 1 when the exponent is even, and yields −1 when the exponent is odd.

For example, $0^3 = 0 \times 0 \times 0 = 0$ and $0^4 = 0 \times 0 \times 0 \times 0 = 0$.

Similarly, $1^3 = 1 \times 1 \times 1 = 1$ and $1^4 = 1 \times 1 \times 1 \times 1 = 1$.

Finally, $(-1)^3 = (-1) \times (-1) \times (-1) = -1$, but $(-1)^4 = (-1) \times (-1) \times (-1) \times (-1) = 1$.

Thus, if you are told that $x^6 = x^7 = x^{15}$, you know that x must be either 0 or 1. Do not try to do algebra on the equation. Simply plug 0 and 1 to check that the equation makes sense. Note that −1 does not fit the equation, since $(-1)^6 = 1$, but $(-1)^7 = -1$.

Of course, if you are told that $x^6 = x^8 = x^{10}$, x could be 0, 1, *or* −1. Any one of these three values fits the equation as given. (See why even exponents are so dangerous?)

Check Your Skills

3. If $x \neq 0$ and $(x^4)(x^{-4}) = y$, what is y?

4. If $x^3 - x = 0$ and $x^2 + x^2 = 2$, what is x?

A Fractional Base

When the base of an exponential expression is a positive proper fraction (in other words, a fraction between 0 and 1), an interesting thing occurs: as the exponent increases, the value of the expression decreases. For example:

$$\left(\frac{3}{4}\right)^1 = \frac{3}{4} \qquad \left(\frac{3}{4}\right)^2 = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16} \qquad \left(\frac{3}{4}\right)^3 = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} = \frac{27}{64}$$

Notice that $\frac{10}{22} \approx \frac{10}{20} = \frac{1}{2}$. Increasing powers cause positive fractions to decrease.

You could also distribute the exponent before multiplying. For example:

$$\left(\frac{3}{4}\right)^1 = \frac{3^1}{4^1} = \frac{3}{4} \qquad \left(\frac{3}{4}\right)^2 = \frac{3^2}{4^2} = \frac{9}{16} \qquad \left(\frac{3}{4}\right)^3 = \frac{3^3}{4^3} = \frac{27}{64}$$

Note that, just like proper fractions, decimals between 0 and 1 decrease as their exponent increases:

$$(0.6)^2 = 0.36 \qquad (0.5)^4 = 0.0625 \qquad (0.1)^5 = 0.00001$$

Check Your Skills

5. Which is bigger, $\left(\frac{3}{4}\right)^2$ or $(0.8)^2$?

6. Which is bigger, $\frac{23}{7}$ or $\left(\frac{10}{7}\right)^2$?

A Compound Base

When the base of an exponential expression is a product, you can multiply the base together and then raise it to the exponent, *or* you can distribute the exponent to each number in the base. For example:

$$(2 \times 5)^3 = (10)^3 = 1{,}000 \quad \text{or} \quad (2 \times 5)^3 = 2^3 \times 5^3 = 8 \times 125 = 1{,}000$$

You cannot do this with a sum or a difference, however. You must add or subtract the numbers inside the parentheses first:

$(2 + 5)^3 = (7)^3 = 343$ $(2 + 5)^3 \neq 2^3 + 5^3$, which is $8 + 125 = 133$

$(5 - 2)^4 = 3^4 = 81$ $(5 - 2)^4 \neq 5^4 - 2^4$, which is $625 - 16 = 609$

All About the Exponent

The Sign of the Exponent

An exponent is not always positive. What happens if the exponent is negative? Take a look below:

$$5^{-1} = \frac{1}{5^1} = \frac{1}{5} \qquad \frac{1}{4^{-2}} = \frac{1}{\frac{1}{4^2}} = 4^2 = 16 \qquad (-2)^{-3} = \frac{1}{(-2)^3} = -\frac{1}{8}$$

Very simply, negative exponents mean “put the term containing the exponent in the denominator of a fraction, and make the exponent positive.” In other words, divide by the base a certain number of times, rather than multiply. An expression with a negative exponent is the reciprocal of what that expression would be with a positive exponent. **When you see a negative exponent, think reciprocal.** For example:

$$\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^{3} = \frac{64}{27}$$

An Exponent of 1

Any base raised to the exponent of 1 keeps the original base. This is fairly intuitive, as shown here:

$$3^1 = 3 \qquad 4^1 = 4 \qquad (-6)^1 = -6 \qquad \left(-\frac{1}{2}\right)^1 = -\frac{1}{2}$$

However, a fact that is not always obvious is that **any number that does not have an exponent implicitly has an exponent of 1**. For example:

$$3 \times 3^4 = ?$$

In this case, just pretend that the "3" term has an exponent of 1 and proceed:

$3^1 \times 3^4 = 3^{(1+4)} = 3^5$ Likewise, $3 \times 3^x = 3^1 \times 3^x = 3^{(1+x)} = 3^{x+1}$

Rule: When you see a base without an exponent, write in an exponent of 1.

An Exponent of 0

By definition, any nonzero base raised to the 0 power yields 1. This may not seem intuitive. For example:

$$3^0 = 1 \qquad 4^0 = 1 \qquad (-6)^0 = 1 \qquad \left(-\frac{1}{2}\right)^0 = 1$$

To understand this fact, think of division of a number by itself, which is one way a 0 exponent could occur:

$$\frac{3^7}{3^7} = 3^{(7-7)} = 3^0 = 1$$

When you divide 3^7 by itself, the result equals 1. Also, by applying the subtraction rule of exponents, you can see that 3^7 divided by itself yields 3^0. Therefore, 3^0 MUST equal 1.

Note also that 0^0 is indeterminate and *never* appears on the GRE. Zero is the ONLY number that, when raised to the 0 power, does not necessarily equal 1.

Rule: Any nonzero base raised to the power of zero (e.g., 3^0) is equal to 1.

Check Your Skills

7. If $2 \times 2^x = 16$, what is x?

8. If $\frac{5^{y+2}}{5^3} = 1$, what is y?

9. if $\begin{aligned} 3,200 &= 100x \\ x &= 32 \end{aligned}$, what is y?

Combining Exponential Terms

Imagine that you have a string of five *a*’s (all multiplied together, not added), and want to multiply this by a string of three *a*’s (again, all multiplied together). How many *a*’s would you end up with?

Write it out:

$$(a \times a \times a \times a \times a) \times (a \times a \times a) = a \times a \times a \times a \times a \times a \times a \times a$$

If you wrote each element of this equation exponentially, it would read:

$$\boxed{a^5 \quad \times \quad a^3 \quad = \quad a^8}$$

“*a* to the fifth times *a* cubed equals *a* to the eighth”

This leads to the first exponent rule:

> **When multiplying exponential terms that share a common base, add the exponents.**
>
> Other examples:

Exponentially	Written Out

Exponentially	Written Out
$7^3 \times 7^2 = 7^5$	$(7 \times 7 \times 7) \times (7 \times 7) = 7 \times 7 \times 7 \times 7 \times 7$
$5 \times 5^2 \times 5^3 = 5^6$	$5 \times (5 \times 5) \times (5 \times 5 \times 5) = 5 \times 5 \times 5 \times 5 \times 5 \times 5$
$f^3 \times f^1 = f^4$	$(f \times f \times f) \times f = f \times f \times f \times f$

Now imagine that you are dividing a string of five *a*’s by a string of three *a*’s. (Again, these are strings of multiplied *a*’s.) What would be the result? Write it out again:

$$\frac{a \times a \times a \times a \times a}{a \times a \times a}$$ *You can cancel out from top and bottom →*

$$\frac{a \times \cancel{a} \times \cancel{a} \times \cancel{a} \times a}{\cancel{a} \times \cancel{a} \times \cancel{a}} \rightarrow a \times a$$

If you wrote this out exponentially, it would read:

$\boxed{a^5 \div a^3 = a^2}$ “*a* to the fifth divided by *a* cubed equals *a* squared”

This leads to the second exponent rule:

When dividing exponential terms with a common base, subtract the exponents.

Other examples:

Exponentially	Written Out
$7^5 \div 7^2 = 7^3$	$(7 \times 7 \times 7 \times 7 \times 7) / (7 \times 7) = 7 \times 7 \times 7$
$5^5 \div 5^4 = 5$	$(5 \times 5 \times 5 \times 5 \times 5) / (5 \times 5 \times 5 \times 5) = 5$
$f^4 \div f^1 = f^3$	$f \times f \times f \times f) / (f) = f \times f \times f$

These are the first two exponent rules:

Rule Book: Multiplying and Dividing Like Base with Different Exponents	
When multiplying exponential terms that share a common base, add the exponents. $\boldsymbol{a^3 \times a^2 = a^5}$	**When dividing exponential terms with a common base, subtract the exponents.** $\boldsymbol{a^5 \div a^2 = a^3}$

Check Your Skills

Simplify the following expressions by combining like terms.

10. $b^5 \times b^7$

11. $(x^3)(x^4)$

12. $\dfrac{y^5}{y^2}$

13. $\dfrac{d^8}{d^7}$

These are the most commonly used rules, but there are some other important things to know about exponents.

ADDITIONAL EXPONENT RULES

> **When something with an exponent is raised to another power, multiply the two exponents together:**
>
> $$\left(a^2\right)^4 = a^8$$
>
> If you have four pairs of *a*’s, you will have a total of eight *a*’s:
>
> $$(a \times a) \times (a \times a) \times (a \times a) \times (a \times a) = a \times a \times a \times a \times a \times a \times a \times a = a^8$$
>
> It is important to remember that the exponent rules just discussed apply to negative exponents as well as to positive exponents. For instance, there are two ways to combine the expression $2^5 \times 2^{-3}$:

The first way is to rewrite the negative exponent as a positive exponent, and then combine:

$$2^5 \times 2^{-3} = 2^5 \times \frac{1}{2^3} = \frac{2^5}{2^3} = 2^{5-3} = 2^2 = 4$$

Add the exponents directly:

$$2^5 \times 2^{-3} = 2^{5+(-3)} = 2^2 = 4$$

Check Your Skills

Simplify the following expressions.

14. $\left(x^3\right)^4$

15. $\left(5^2\right)^3$

Rewriting Bases

So now you know how to combine exponential expressions when they share a common base. But what can you do when presented with an expression such as $5^3 \times 25^2$? At first, it may seem that no further simplification is possible.

The trick here is to realize that 25 is actually 5^2. Because they are equivalent values, you can replace 25 with 5^2 and see what you get.

You can write $5^3 \times (5^2)^2$ as $5^3 \times 5^4$. This expression can now be combined and you end up with 5^7.

When dealing with exponential expressions, you need to be on the lookout for perfect squares and perfect cubes that can be rewritten. In the last example, 25 is a perfect square and can be rewritten as 5^2. In general, it is good to know all the perfect squares up to 15^2, the perfect cubes up to 6^3, and the powers of 2 and 3. Here's a brief list of some of the numbers likely to appear on the GRE:

The powers of 2: 2, 4, 8, 16, 32, 64, 128
The powers of 3: 3, 9, 27, 81

$4^2 = 16$
$5^2 = 25$

$10^2 = 100$
$11^2 = 121$

$2^3 = 8$
$3^3 = 27$

$6^2 = 36$
$7^2 = 49$
$8^2 = 64$
$9^2 = 81$
$12^2 = 144$
$13^2 = 169$
$14^2 = 196$
$15^2 = 225$
$4^3 = 64$
$5^3 = 125$

Let’s try another example. How would you combine the expression $2^3 \times 8^4$? Try it out for yourself.

Again, the key is to recognize that 8 is 2^3. The expression can be rewritten as $2^3 \times (2^3)^4$, which becomes $2^3 \times 2^{12}$, which equals 2^{15}.

Alternatively, you could replace 2^3 with 8^1. The expression can be rewritten as $8^1 \times 8^4$, which equals 8^5.

Check Your Skills

Combine the following expressions.

16. $2^4 \times 16^3$

17. $7^5 \times 49^8$

18. $9^3 \times 81^3$

Simplifying Exponential Expressions

Now that you have the basics down for working with bases and exponents, what about working with multiple exponential expressions at the same time? If two (or more) exponential terms in an expression have a base in common or an exponent in common, you can often simplify the expression. (In this section, “simplify,” means “reduce to one term.”)

When Can You Simplify Exponential Expressions?

> You can only *simplify* exponential expressions that are linked by multiplication or division. You cannot *simplify* expressions linked by addition or subtraction (although, in some cases, you can *factor* them and otherwise manipulate them).
>
> You can simplify exponential expressions linked by multiplication or division if they have either a base or an exponent in common.

How Can You Simplify Them?

Use the exponent rules described earlier. If you forget these rules, you can derive them on the test by writing out the example exponential expressions. For example:

These expressions CANNOT be simplified:	**These expressions CAN be simplified:**	**Here’s how:**

$7^4 + 7^6$	$(7^4)(7^6)$	$(7^4)(7^6) = 7^{4+6} = 7^{10}$
$3^4 + 12^4$	$(3^4)(12^4)$	$(3^4)(12^4) = (3 \times 12)^4 = 36^4$
$6^5 - 6^3$	$\frac{y^5}{y^2}$	$\frac{6^5}{6^3} = 6^{5-3} = 6^2$
$12^7 - 3^7$	$\frac{12^7}{3^7}$	$\frac{12^7}{3^7} = \left(\frac{12}{3}\right)^7 = 4^7$

You can simplify all the expressions in the middle column to a single term, because the terms are multiplied or divided. The expressions in the left column *cannot be simplified*, because the terms are added or subtracted. However, they *can be factored* whenever the base is the same. For example, $7^4 + 7^6$ can be factored because the two terms in the expression have a factor in common. What factor exactly do they have in common? Both terms contain 7^4. If you factor 7^4 out of each term, you are left with $7^4(7^2 + 1) = 7^4(50)$.

The terms can *also* be factored whenever the exponent is the same and the terms contain something in common in the base. For example, $3^4 + 12^4$ can be factored because $12^4 = (2 \times 2 \times 3)^4$. Thus, both bases contain 3^4, and the factored expression is $3^4(1 + 4^4) = 3^4(257)$.

Likewise, $6^5 - 6^3$ can be factored as $6^3(6^2 - 1)$. $6^3(35)$ and $12^7 - 3^7$ can be factored as $3^7(4^7 - 1)$.

On the GRE, it generally pays to factor exponential terms that have something in common in the bases. For example:

> If $x = 4^{20} + 4^{21} + 4^{22}$, what is the largest prime factor of x?

All three terms contain 4^{20}, so you can factor the expression: $x = 4^{20}(4^0 + 4^1 + 4^2)$. Therefore, $x = 4^{20}(1 + 4 + 16) = 4^{20}(21) = 4^{20}(3 \times 7)$. The largest prime factor of x is 7.

Rules of Exponents

Exponent Rule	Examples
$x^a \times x^b = x^{a+b}$	$c^3 \times c^5 = c^8$ $\quad 3^5 \times 3^8 = 3^{13}$ $5(5^n) = 5^1(5^n) = 5^{n+1}$
$a^x \times b^x = (ab)^x$	$2^4 \times 3^4 = 6^4$ $\quad 12^5 = 2^{10} \times 3^5$
$\frac{x^a}{x^b} = x^{(a-b)}$	$\frac{2^5}{2^{11}} = \frac{1}{2^6} = 2^{-6}$ $\quad \frac{x^{10}}{x^3} = x^7$
$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$	$\left(\frac{10}{2}\right)^6 = \frac{10^6}{2^6} = 5^6$ $\quad \frac{3^5}{9^5} = \left(\frac{3}{9}\right)^5 = \left(\frac{1}{3}\right)^5$
$(a^x)^y = a^{xy} = (a^y)^x$	$(3^2)^4 = 3^{2\cdot 4} = 3^8 = 3^{4\cdot 2} = (3^4)^2$
$x^{-a} = \frac{1}{x^a}$	$\left(\frac{3}{2}\right)^{-2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$ $\quad 2x^{-4} = \frac{2}{x^4}$
$a^x + a^x + a^x = 3a^x$	$3^4 + 3^4 + 3^4 = 3 \times 3^4 = 3^5$ $3^x + 3^x + 3^x = 3 \times 3^x = 3^{x+1}$

Check Your Skills

19. Can these expressions be simplified (i.e., reduced to a single term)?

(A) $x^2 + x^2$

(B) $x^2 \times y^2$

(C) $2(2^n + 3^n)$

Common Exponent Errors

Study this list of common errors carefully and identify any mistakes that you occasionally make. Note the numerical examples given.

INCORRECT	CORRECT
$(x+y)^2 = x^2 + y^2$? $(3+2)^2 = 3^2 + 2^2 = 13$?	$(x+y)^2 = x^2 + 2xy + y^2$ $(3+2)^2 = 5^2 = 25$
$a^x \times b^y = (ab)^{x+y}$? $2^4 \times 3^5 = (2 \times 3)^{4+5}$?	Cannot be simplified further (different bases **and** different exponents)
$a^x \times a^y = a^{xy}$? $5^4 \times 5^3 = 5^{12}$?	$a^x \cdot a^y = a^{x+y}$ $5^4 \times 5^3 = 5^7$
$(a^x)^y = a^{(x+y)}$? $(7^4)^3 = 7^7$?	$(a^x)^y = a^{xy}$ $(7^4)^3 = 7^{12}$
$a^x + a^y = a^{x+y}$? $x^3 + x^2 = x^5$?	Cannot be simplified further (addition **and** different exponents)
$a^x + a^x = a^{2x}$? $2^x + 2^x = 2^{2x}$?	$a^x + a^x = 2a^x$ $2^x + 2^x = 2(2^x) = 2^{x+1}$
$a \times a^x = a^{2x}$? $5 \times 5^z = 25^z$?	$a \times a^x = a^{x+1}$ $5 \times 5^z = 5^{z+1}$
$-x^2 = x^2$? $-4^2 = 16$?	$-x^2$ cannot be simplified further $-4^2 = -16$ **Compare: $(-x)^2 = x^2$ and $(-4)^2 = 16$**

INCORRECT	CORRECT
$a \times b^x = (a \times b)^x$? $2 \times 3^4 = (2 \times 3)^4$?	Cannot be simplified further

Check Your Skills Answer Key

1. **−3**

   If a number is raised to an odd power ($x \cdot x \cdot x = x^3$), the result will have the same sign as the original base. This means that x must be −3.

2. **2 or −2**

   You have an even power here ($x^2 \cdot x^3 \cdot x = x^6$), so the base could be positive or negative. This means x could be either 2 or −2.

3. **1**

   Whenever you multiply two terms with the same base, add the exponents: $4 + (-4) = 0$, and $x^0 = 1$. This means $y = 1$.

4. **1, −1**

   The rules tell you that if $x^3 = x$ (which you get if we add x to both sides of the first equation), x can be 0, −1, *or* 1. If $x = 0$, the second equation, $0 + 0 = 2$, is false. If $x = 1$ or −1, the second equation works, so $x = 1$ or −1.

5. **$(0.8)^2$**

   Every fraction gets smaller the higher you raise its power, so both of these will get smaller. They will also get smaller at a faster rate depending on how small they already are. For example, $\frac{1}{2} + \frac{3}{4} =$, which is smaller than 0.8. This means that $(0.75)^2 < (0.8)^2$. You can also think of it this way: 75% of 75 will be smaller than 80% of 80, because a smaller percent of a smaller starting number will be even less.

6. $\left(\frac{10}{7}\right)^2$

   Even though $\frac{23}{7}$ is being presented to you in fractional form, it is an improper fraction, meaning its value is greater than 1. When something greater than 1 is raised to a power, it gets bigger. This means that $\left(\frac{10}{7}\right)^2 > \frac{10}{7}$.

7. **3**

   Quick thinking about powers of 2 should lead you to $2^4 = 16$. The equation can be rewritten as $2^1 \times 2^x = 2^4$. Now you can ignore the bases, because the powers should add up: $x + 1 = 4$, so $x = 3$.

8. **1**

   Anything raised to the 0 power equals 1, so the expression on the left side of this equation must be equivalent to 5^0. When dividing terms with the same base, subtract exponents: $5^{y+2-3} = 5^0$.

   Now ignore the bases: $y + 2 - 3 = 0$, so $y = 1$.

9. **1**

   The $\frac{1}{2}$ can be rewritten as 2^{-1}, and $\frac{1}{2}$ can be rewritten as 2^{-2}, so the equation becomes $2^{-y} = 2^{(y-2)}$. Now you can ignore the bases, so $-y = y -$

2. $2y = 2$, and $y = 1$.

Alternatively, you could distribute the exponent, then cross-multiply:

$$\frac{1}{2^y} = \frac{2^y}{4}$$
$$4 = (2^y)(2^y)$$
$$2^2 = 2^{2y}$$
$$2 = 2y$$
$$y = 1$$

10. **b^{12}**

$b^5 \times b^7 = b^{(5+7)} = b^{12}$

11. **x^7**

$(x^3)(x^4) = x^{(3+4)} = x^7$

12. **y^3**

$$\frac{y^5}{y^2} = y^{(5-2)} = y^3$$

13. **d**

$$\frac{d^8}{d^7} = d^{(8-7)} = d$$

14. **x^{12}**

$$\left(x^3\right)^4 = x^{3\times4} = x^{12}$$

15. **5^6**

$$\boldsymbol{W\ \Phi\ F} = \left(\sqrt{F}\right)^{W}$$

16. **2^{16}**

$2^4 \times 16^3 = 2^4 \times (2^4)^3 = 2^4 \times 2^{4 \times 3} = 2^4 \times 2^{12} = 2^{4 + 12} = 2^{16}$

17. **7^{21}**

$7^5 \times 49^8 = 7^5 \times (7^2)^8 = 7^5 \times 7^{2 \times 8} = 7^5 \times 7^{16} = 7^{5 + 16} = 7^{21}$

18. **3^{18} or 9^9**

$9^3 \times 81^3 = (3^2)^3 \times (3^4)^3 = 3^{2 \times 3} \times 3^{4 \times 3} = 3^6 \times 3^{12} = 3^{6 + 12} = 3^{18}$

or $9^3 \times (9^2)^3 = 9^3 \times 9^6 = 9^9$

)19. This cannot be simplified, except to say $2x^2$. You can't combine the bases or powers in any more interesting way.

) Even though you have two different variables here, the rules hold, and you can multiply the bases and maintain the power: $(xy)^2$.

) Half of this expression can be simplified, namely the part that involves the common base, 2: $2^{n+1} + 2 \times 3^n$. It may not be much prettier, but at least you've joined up the common terms.

Problem Set

Simplify or otherwise reduce the following expressions using the rules of exponents.

1. 2^{-5}
2. $\dfrac{y^5}{y^2}$
3. $8^4(5^4)$
4. $2^4 \times 2^5 \div 2^7 - 2^4$

Solve the following problems.

5. $\dfrac{9^4}{3^4} + \left(4^2\right)^3$

6. Does $a^2 + a^4 = a^6$ for all values of a?

7. $x^3 < x^2$. Describe the possible values of x.

8. If $x^4 = 16$, what is $|x|$?

9. If $y^5 > 0$, is $y < 0$?

10. If $b > a > 0$ and $c \neq 0$, is $a^2b^3c^4$ positive?

11. Simplify: $\dfrac{y^2 \times y^5}{\left(y^2\right)^4}$

12. If $r^3 + |r| = 0$, what are the possible values of r?

13.

Quantity A	**Quantity B**
2^y	$\left(\frac{1}{2}\right)^{-y}$

14.

Quantity A	**Quantity B**
$3^3 \times 9^6 \times 2^4 \times 4^2$	$9^3 \times 3^6 \times 2^2 \times 4^4$

15.

$y > 1$

Quantity A	**Quantity B**
$(0.99)^y$	$0.99 \times y$

Solutions

1. $\frac{23}{7}$

   Remember that a negative exponent yields the reciprocal of the same expression with a positive exponent. Thus, $2^{-5} = \frac{1}{2^5} = \frac{1}{32}$.

2. **49**

   $$\frac{7^6}{7^4} = 7^{6-4} = 7^2 = 49$$

3. **40^4**

   $8^4(5^4) = 40^4$

4. **−12**

   $$\frac{2^4 \times 2^5}{2^7} - 2^4 = 2^{(4+5-7)} - 2^4 = 2^2 - 2^4 = 2^2\left(1-2^2\right) = 4(1-4) = -12$$

5. **4,177**

   $$\frac{9^4}{3^4} + \left(4^2\right)^3 = 3^4 + 4^6 = 81 + 4,096 = 4,1$$

6. **No**

Remember, you cannot combine exponential expressions linked by addition.

7. **Any non-zero number less than 1**
As positive proper fractions are multiplied, their value decreases. For example, $\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^2$. Also, any negative number will make this inequality true. A negative number cubed is negative. Any negative number squared is positive. For example, $(-3)^3 < (-3)^2$. The number zero itself, however, does not work, because $0^3 = 0^2$.

This could be determined algebraically:

$$\begin{aligned} x^3 &< x^2 \\ x^3 - x^2 &< 0 \\ x^2(x-1) &< 0 \end{aligned}$$

x^2 is positive for all $x \neq 0$, so $x^2(x - 1)$ is negative when $(x - 1)$ is negative: $x < 1$.

8. **2**
The possible values for x are 2 and −2. The absolute value of both 2 and −2 is 2.

9. **No**
An integer raised to an odd exponent retains the original sign of the base. Therefore, if y^5 is positive, y is positive.

10. **Yes**

    b and a are both positive numbers. Whether c is positive or negative, c^4 is positive. (Recall that any number raised to an even power is positive.) Therefore, the product $a^2b^3c^4$ is the product of three positive numbers, which will be positive.

11. $\mathbf{\frac{1}{y}}$

    $$\frac{y^2 \times y^5}{\left(y^2\right)^4} = \frac{y^7}{y^8} = y^{7-8} = y^{-1} = \frac{1}{y}$$

12. **0, −1**

    If $r^3 + |r| = 0$, then r^3 must be the opposite of $|r|$. The only values for which this would be true are 0, which is the opposite of itself, and −1, whose opposite is 1.

13. **(C)**

    When you raise a number to a negative power, that's the same as raising its reciprocal to the positive version of that power. For instance, $3^{-2} = \left(\frac{1}{3}\right)^2$, because $\frac{1}{2}$ is the reciprocal of 3. The reciprocal of $\frac{1}{2}$ is 2, so Quantity B can be rewritten.

Quantity A	**Quantity B**
2^y	$\left(\frac{1}{2}\right)^{-y} = (2)^y$

Therefore, **the two quantities are equal**.

14. **(A)**

The goal with exponent questions is always to get the same bases, the simplest versions of which will always be prime. Each quantity has the same four bases: 2, 3, 4, and 9. Because 2 and 3 are already prime, you need to manipulate 4 and 9: 4 = 2^2 and 9 = 3^2. Rewrite the quantities:

Quantity A	Quantity B
$3^3 \times 9^6 \times 2^4 \times 4^2 =$	$9^3 \times 3^6 \times 2^2 \times 4^4 =$
$3^3 \times (3^2)^6 \times 2^4 \times (2^2)^2$	$(3^2)^3 \times 3^6 \times 2^2 \times (2^2)^4$

Now terms can be combined using the exponent rules.

Quantity A	Quantity B
$3^3 \times (3^2)^6 \times 2^4 \times (2^2)^2 =$	$(3^2)^3 \times 3^6 \times 2^2 \times (2^2)^4 =$
$3^3 \times 3^{12} \times 2^4 \times 2^4 =$	$3^6 \times 3^6 \times 2^2 \times 2^8 =$
$3^{15} \times 2^8$	$3^{12} \times 2^{10}$

Now divide away common terms. Both quantities contain the product $3^{12} \cdot 2^8$.

Quantity A	Quantity B
$\frac{3^{15} \times 2^8}{3^{12} \times 2^8} = 3^3 = 27$	$\frac{3^{12} \times 2^{10}}{3^{12} \times 2^8} = 2^2 = 4$

Therefore, **Quantity A is greater**.

15. **(B)**

Any number less than 1 raised to a power greater than 1 will get smaller, so even though you don’t know the value of y, you do know that the value in Quantity A will be less than 0.99:

$$y > 1$$

Quantity A	**Quantity B**
$(0.99)^y$, which must be less than 0.99	$0.99 \times y$

Conversely, any positive number multiplied by a number greater than 1 will get bigger. You don’t know the value in Quantity B, but you know that it will be larger than 0.99:

$$y > 1$$

Quantity A	**Quantity B**
$(0.99)^y$, which must be less than 0.99	$0.99 \times y$, which must be greater than 0.99

Therefore, **Quantity B is greater**.

Chapter 19

ROOTS

In This Chapter...

Multiplication and Division of Roots

Simplifying Roots

Solving Algebraic Equations Involving Exponential Terms

Chapter 19

Roots

This chapter discusses some of the ways roots are incorporated into expressions and equations and the ways you are allowed to manipulate them. You may be tempted to use the on-screen calculator when you see a root expression, but it's often much easier to go without. You just need to know your roots rules.

Before getting into some of the more complicated rules, it is important to remember that any square root times itself will equal whatever is inside the square root, for instance: $\sqrt{2} \times \sqrt{2} = 2$, $\sqrt{18} \times \sqrt{18} = 18$. You can even apply this rule to variables: $\sqrt{y} \times \sqrt{y} = y$. So the first rule for roots is:

$$\sqrt{x} \times \sqrt{x} = x$$

Multiplication and Division of Roots

Suppose you were to see the equation $3 + \sqrt{4} = x$, and you were asked to solve for *x*. What would you do? Well, $\sqrt{4} = 2$. And by the way, the radical sign always indicates the positive root, so $\sqrt{4}$ never equals −2, by definition. This way, the result of the square-root operation is always just one number. So anyway, you can rewrite the equation as 3 + 2 = *x*, and you would know that *x* = 5. Because 4 is a perfect square, you were able to simply evaluate the root, and continue to solve the problem. But what if the equation were $\sqrt{8} \times \sqrt{2} = x$, and you were asked to find *x*? What would you do then? Neither 8 nor 2 is a perfect square, so you can't easily find a value for either root.

It is important to realize that, on the GRE, sometimes you will be able to evaluate roots (when asked to take the square root of a perfect square or the cube root of a perfect cube), but other times it will be necessary to manipulate the roots. Up next is a discussion of the different ways that you are allowed to manipulate roots, followed by some examples of how these manipulations may help you arrive at a correct answer on GRE questions involving roots.

Go back to the previous question: If $\sqrt{8} \times \sqrt{2} = x$, what is *x*?

When two roots are multiplied by each other, you can do the multiplication within a single root. What that means is that you can rewrite $\sqrt{8} \times \sqrt{2}$ as $\sqrt{8 \times 2},$ which equals $\sqrt[3]{64}$. And $\sqrt[3]{64}$ equals 4, which means that $x = 4$.

This property also works for division.

If $x = \dfrac{\sqrt{27}}{\sqrt{3}}$, what is x?

You can divide the numbers inside the square roots and put them inside one square root. So $\dfrac{\sqrt{27}}{\sqrt{3}}$ becomes $\sqrt{\dfrac{27}{3}}$, which becomes $\sqrt{9}.$ And $\sqrt{9}$ equals 3, so $x = 3$.

Note that these rules apply if there are any number of roots being multiplied or divided. These rules can also be combined with each other. For instance, $\dfrac{\sqrt{15} \times \sqrt{12}}{\sqrt{5}}$ becomes $\sqrt{\dfrac{6}{1}} = \sqrt{6}$ The numbers inside can be combined, and ultimately you end up with $\sqrt{36},$ which equals 6.

Check Your Skills

Solve for x.

1. $x = \sqrt{20} \times \sqrt{5}$

2. $x = \sqrt{20} \times \sqrt{5}$

3. $x = \sqrt{2} \times \sqrt{6} \times \sqrt{12}$

4. $x = \dfrac{\sqrt{384}}{\sqrt{2} \times \sqrt{3}}$

Simplifying Roots

Just as multiple roots can be combined to create one root, you can also take one root and break it apart into multiple roots. You may be asking, why would you ever want to do that? Well, suppose a question asked: If $\sqrt{8} \times \sqrt{2} = x,$ what is *x*? You would combine them, and say that *x* equals $\sqrt[3]{64}$. Unfortunately, $\sqrt[3]{64}$ will never be a correct answer on the GRE. The reason is that $\sqrt[3]{64}$ can be simplified, and correct answers on the GRE are presented in their simplest forms. So now the question becomes, how can you simplify $\sqrt[3]{64}$?

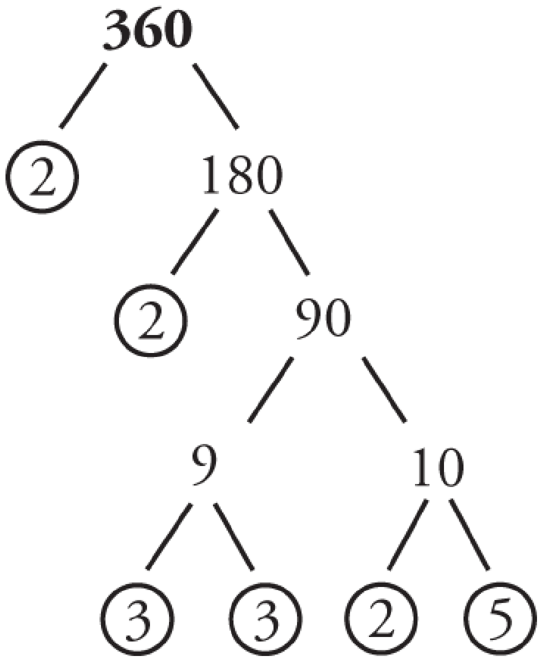
What if you were to rewrite $\sqrt[3]{64}$ as $\sqrt{4 \times 3}$? As mentioned, you could also break this apart into two separate roots that are multiplied together, namely $\sqrt{4} \times \sqrt{3}.$ And you already know that $\sqrt{4}$ equals 2, so you could simplify to $2\sqrt{3}.$ And in fact, that is the simplified form of $\sqrt[3]{64}$, and could potentially appear as the correct answer to a question on the GRE. Just to recap, the progression of simplifying $\sqrt[3]{64}$ was as follows:

$$\sqrt{12} \rightarrow \sqrt{4 \times 3} \rightarrow \sqrt{4} \times \sqrt{3} \rightarrow 2\sqrt{3}$$

Now the question becomes, how can you simplify *any* square root? What if you don’t notice that 12 equals 4 times 3, and 4 is a perfect square? Amazingly enough, the method for simplifying square roots will involve something you’re probably quite comfortable with at this point—prime factorizations.

Take a look at the prime factorization of 12. The prime factorization of 12 is 2 × 2 × 3. So $\sqrt[3]{64}$ can be rewritten as $\sqrt{2 \times 2 \times 3}.$ Recall the first roots rule—any root times itself will equal the number inside. If $\sqrt[3]{64}$ can be rewritten as $\sqrt{2 \times 2 \times 3},$ you can take that one step further and say it is $\sqrt{2} \times \sqrt{2} \times \sqrt{3}.$ And you know that $\sqrt{y} \times \sqrt{y} = y.$

You can generalize from this example and say that when you take the prime factorization of a number inside a square root, any prime factor that you can pair off can effectively be brought out of the square root. Try another example to practice applying this concept. What is the simplified form of $16\sqrt{2}$? Start by taking the prime factorization of 360:



360 = 2 × 2 × 2 × 3 × 3 × 5

Again, you are looking for primes that you can pair off and ultimately remove from the square root. In this case, you have a pair of 2's and a pair of 3's, so you can separate them:

$$\sqrt{360} \rightarrow \sqrt{2 \times 2 \times 2 \times 3 \times 3 \times 5} \rightarrow \sqrt{2 \times 2} \times \sqrt{3 \times 3} \times \sqrt{2 \times 5}$$

Notice that the prime factorization of 360 included three 2's. Two 2's could be paired off, but that still left one 2 without a partner, therefore $\sqrt{2 \times 5}$ represents the prime factors that cannot be paired off. This expression can now be simplified to $2 \times 3 \times \sqrt{2 \times 5}$, which is $16\sqrt{2}$.

You might have seen right away that 360 = 36 × 10, so $\sqrt{360} = \sqrt{36 \times 10} = \sqrt{36} \times \sqrt{10} = 6\sqrt{10}.$ The advantage of the prime factor method is that it will always work, even when you don't spot a shortcut.

Check Your Skills

Simplify the following roots.

5. $\sqrt[3]{64}$

6. $\sqrt[3]{64}$

7. $\sqrt{441}$

Solving Algebraic Equations Involving Exponential Terms

GRE exponent problems sometimes give you an equation, and ask you to solve for either an unknown base or an unknown exponent.

UNKNOWN BASE

The key to solving algebraic expressions with an unknown base is to make use of the fact that exponents and roots are inverses, just as multiplication and division are, and so can be used to effectively cancel each other out. In the equation $x^3 = 8$, x is raised to the third power, so to eliminate the exponent you can take the cube root of both sides of the equation.

$$\sqrt[3]{x^3} = x \;\; \text{SO} \;\;\; \sqrt[3]{8} = 2 = x$$

This process also works in reverse. If you are presented with the equation $\sqrt{4 \times 3}?$ you can eliminate the square root by squaring both sides. Square root and squaring cancel each other out in the same way that cube root and raising something to the third power cancel each other out. So to solve this equation, you can square both sides and get $(\sqrt{x})^2 = 6^2,$ which can be simplified to $x = 36$.

There is one additional danger. Remember that when solving an equation where a variable has been squared, you should be on the lookout for two solutions. To solve for y in the equation $y^2 = 100$, you need to remember that y can equal either 10 or −10.

Unknown Base	Unknown Exponent
$x^3 = 8$	$2^x = 8$

Check Your Skills

Solve the following equations.

8. $x^3 = 64$

9. $\sqrt[3]{x} = 6$

10. $x^2 = 121$

UNKNOWN EXPONENT

Unlike examples in the previous section, you can't make use of the relationship between exponents and roots to help solve for the variable in the equation $2^x = 8$. Instead, the key is to once again recognize that 8 is equivalent to 2^3, and rewrite the equation so that you have the same base on both sides of the equal sign. If you replace 8 with its equivalent value, the equation becomes $2^x = 2^3$.

Now that you have the same base on both sides of the equation, there is only one way for the value of the expression on the left side of the equation to equal the value of the expression on the right side of the equation—the exponents must be equal. You can effectively ignore the bases and set the exponents equal to each other. You now know that $x = 3$.

By the way, when you see the expression 2^x, always call it "two TO THE *x*th power" or "two TO THE *x*." Never call it "two *x*." "Two *x*" is $2x$, or 2 times *x*, which is simply a different expression. Don't get lazy with names; that's how you can confuse one expression for another.

The process of finding the same base on each side of the equation can be applied to more complicated exponents as well. Take a look at the equation $3^{x+2} = 27$. Once again, you must first rewrite one of the bases so that the bases are the same on both sides of the equation. Because 27 is equivalent to 3^3, the equation can be rewritten as $3^{x+2} = 3^3$. You can now ignore the bases (because they are the same) and set the exponents equal to each other: $x + 2 = 3$, which means that $x = 1$.

Check Your Skills

Solve for *x* in the following equations.

11. $2^x = 64$

12. $7^{x-2} = 49$

13. $5^{3x} = 125$

Check Your Skills Answer Key

1. **10**

$$x = \sqrt{20} \times \sqrt{5} = \sqrt{20 \times 5} = \sqrt{100} = 10$$

2. **2**

$$x = \frac{\sqrt{20}}{\sqrt{5}} = \sqrt{\frac{20}{5}} = \sqrt{4} = 2$$

3. **12**

$$x = \sqrt{2} \times \sqrt{6} \times \sqrt{12} = \sqrt{2 \times 6 \times 12} = \sqrt{144} = 12$$

4. **8**

$$x = \frac{\sqrt{384}}{\sqrt{2} \times \sqrt{3}} = \sqrt{\frac{384}{2 \times 3}} = \sqrt{\frac{384}{6}} = \sqrt{64} = 8$$

5. $5\sqrt{3}$

$$\sqrt{75} \rightarrow \sqrt{3 \times 5 \times 5} \rightarrow \sqrt{5 \times 5} \times \sqrt{3} = 5\sqrt{3}$$

6. $\sqrt[3]{64}$

$$\sqrt{96} = \sqrt{2 \times 2 \times 2 \times 2 \times 2 \times 3} = \sqrt{2 \times 2} \times \sqrt{2 \times 2} \times \sqrt{2 \times 3} = 2 \times 2 \times \sqrt{6} = 4\sqrt{6}$$

7. **21**

$$\sqrt{441} \rightarrow \sqrt{3 \times 3 \times 7 \times 7} \rightarrow \sqrt{3 \times 3} \times \sqrt{7 \times 7} = 3 \times 7 = 21$$

8. **4**

$$x^3 = 64$$

$$\sqrt[3]{x^3} = \sqrt[3]{64}$$

$$x = 4$$

9. **216**

$$\sqrt[3]{x} = 6$$

$$\left(\sqrt[3]{x}\right)^3 = (6)^3$$

$$x = 216$$

10. **11 or −11**

$$x^2 = 121$$

$$\sqrt{x^2} = \sqrt{121}$$

$$x = 11 \quad \text{OR} \quad -11$$

11. **6**

$$2^x = 64$$

$$2^x = 2^6$$

$$x = 6$$

12. **4**

$$7^{x-2} = 49$$

$$7^{x-2} = 7^2$$

$$x - 2 = 2$$

$$x = 4$$

13. **1**

$$5^{3x} = 125$$

$$5^{3x} = 5^3$$

$$3x = 3$$

$$x = 1$$

Problem Set

1.

Quantity A	**Quantity B**
$\sqrt{30} \times \sqrt{5}$	12

2.

$36 < x < 49$

Quantity A	**Quantity B**
$2^{\sqrt{x}}$	4^3

3.

Quantity A	**Quantity B**
$\dfrac{\sqrt{6} \times \sqrt{18}}{\sqrt{9}}$	$\dfrac{\sqrt{6} \times \sqrt{18}}{\sqrt{9}}$

Solutions

1. **(A)**

   One of the root rules is that when two individual roots are multiplied together, you can carry out that multiplication under a single root sign:

   $$\sqrt{30} \times \sqrt{5} = \sqrt{30 \times 5} = \sqrt{150}$$

   While this can be simplified $\left(\sqrt{150} = \sqrt{25 \times 6} = 5\sqrt{6},\right)$ you're actually better off leaving it as is:

Quantity A	**Quantity B**
$\sqrt{30} \times \sqrt{5} = \sqrt{150}$	12

   Now square both quantities:

Quantity A	**Quantity B**
$\left(\sqrt{150}\right)^2 = 150$	$(12)^2 = 144$

   Therefore, Quantity A is greater.

2. **(A)**

The common information tells you that *x* is between 36 and 49, which means the square root of *x* must be between 6 and 7. Rewrite Quantity A:

$36 < x < 49$

Quantity A	Quantity B
$2^6 < 2^{\sqrt{x}} < 2^7$	4^3

Now rewrite Quantity B so that it has a base of 2 instead of a base of 4:

$36 < x < 49$

Quantity A	Quantity B
$2^6 < 2^{\sqrt{x}} < 2^7$	$4^3 = (2^2)^3 = 2^6$

The value in Quantity A must be greater than 2^6, and so must be greater than the value in Quantity B. Therefore, **Quantity A is greater**.

3. **(B)**
Simplify both quantities by combining the roots into one root.

Quantity A	Quantity B
$\sqrt{\frac{6 \times 18}{9}}$	$\sqrt{\frac{6 \times 18}{9}}$

Now simplify the fractions underneath each root.

Quantity A	**Quantity B**
$\sqrt{\frac{6 \times \cancel{18}^{2}}{\cancel{9}_{1}}} = \sqrt{12}$	$\sqrt{\frac{6 \times \cancel{18}^{2}}{\cancel{9}_{1}}} = \sqrt{12}$

Because $\sqrt[3]{64}$ is larger than $\sqrt[3]{64}$, **Quantity B is greater.**

Chapter 20

CONSECUTIVE INTEGERS

In This Chapter…

Evenly Spaced Sequences

Properties of Evenly Spaced Sequences

Counting Integers: Add 1 Before You Are Done

Chapter 20

Consecutive Integers

Consecutive integers are integers that follow one after another from a given starting point, without skipping any integers. For example, 4, 5, 6, and 7 are consecutive integers, but 4, 6, 7, and 9 are not. There are many other types of related consecutive patterns. For example:

Consecutive Even Integers: 8, 10, 12, 14 (8, 10, 14, and 16 is incorrect, as it skips 12)

Consecutive Primes: 11, 13, 17, 19 (11, 13, 15, and 17 is wrong, as 15 is not prime)

Evenly Spaced Sequences

To understand consecutive integers, first consider **evenly spaced sequences**. These are sequences of numbers whose values go up or down by the same amount (the **increment**) from one item in the sequence to the next. For instance, the sequence {4, 7, 10, 13, 16} is evenly spaced because each value increases by 3 over the previous value.

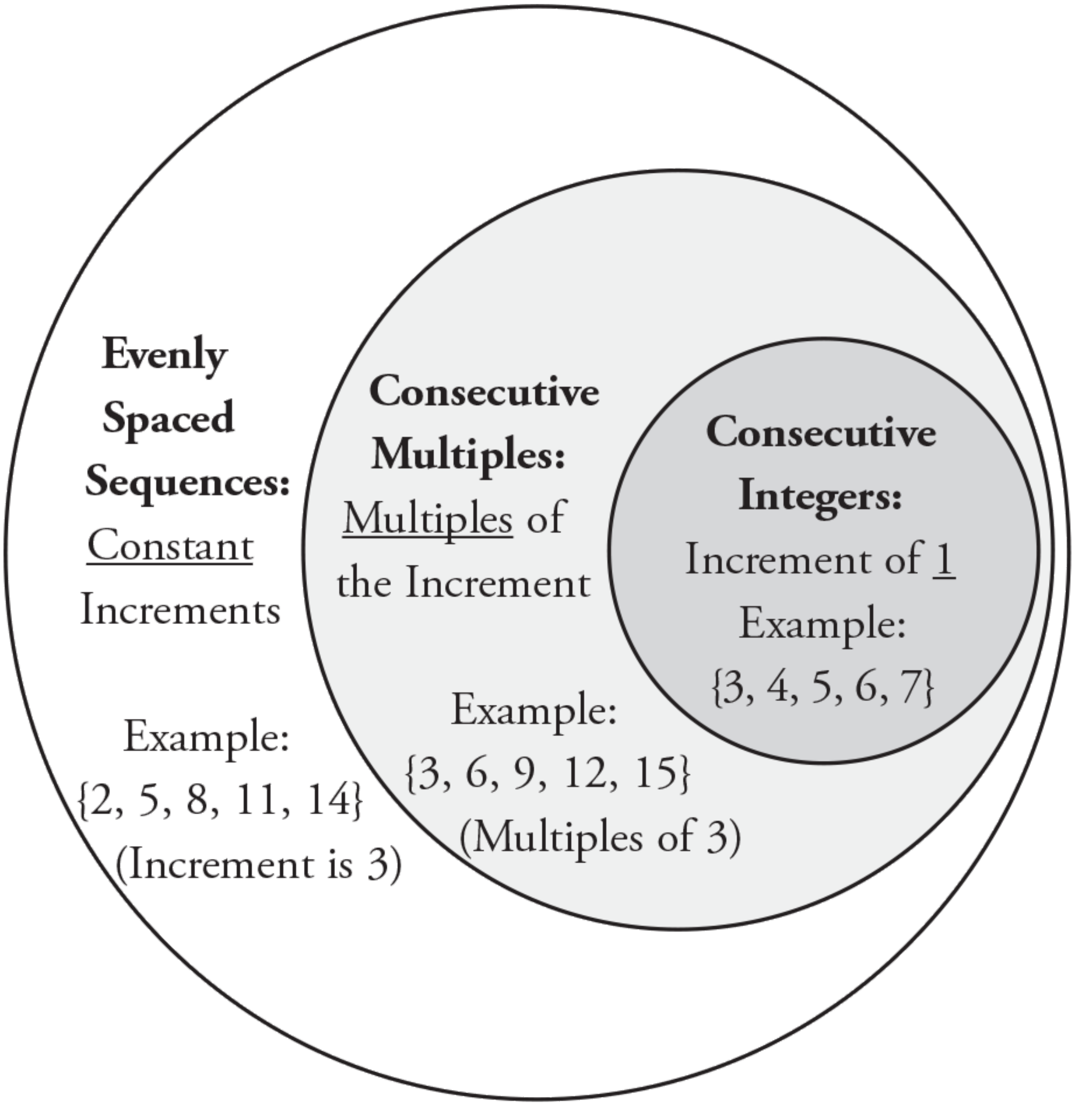
Consecutive multiples are special cases of evenly spaced sequences: all of the values in the sequence are multiples of the increment. For example, {12, 16, 20, 24} is a sequence of consecutive multiples because the values increase from one to the next by 4, and each element is a multiple of 4. Note that sequences of consecutive multiples *must* be composed of integers.

Consecutive integers are special cases of consecutive multiples: all of the values in the sequence increase by 1, and all integers are multiples of 1. For example, {12, 13, 14, 15, 16} is a sequence of consecutive integers because the values increase from one to the next by 1, and each element is an integer.

The relations among evenly spaced sequences, consecutive multiples, and consecutive integers are displayed in the following diagram.

- All sequences of consecutive integers are sequences of consecutive multiples.
- All sequences of consecutive multiples are evenly spaced sequences.
- All evenly spaced sequences are fully defined if the following three parameters are known:

  The smallest (**first**) or largest (**last**) number in the sequence
  The **increment** (always 1 for consecutive integers)
  The **number of items** in the sequence



Check Your Skills

1. Which of the following are evenly spaced sequences?

(A) $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}$

(B) $x, x - 4, x - 8, x - 12, x - 16$

(C) $\sqrt{4}, \sqrt{9}, \sqrt{16}, \sqrt{25}, \sqrt{36}$

(D) $5^1, 5^2, 5^3, 5^4, 5^5$

(E) $y, 2y, 3y, 4y, 5y$

Properties of Evenly Spaced Sequences

The following three properties apply to *all* evenly spaced sequences. However, just because a sequence has these properties does not necessarily mean that the sequence is evenly spaced.

The **arithmetic mean** (average) and **median** are equal to each other. In other words, the average of the elements in the set can be found by figuring out the median, or "middle number."

What is the arithmetic mean of 4, 8, 12, 16, and 20?

In this example there are five consecutive multiples of 4. The median is the third largest, or 12. Since this is an evenly spaced set, the arithmetic mean (average) is also 12.

What is the arithmetic mean of 4, 8, 12, 16, 20, and 24?

In this example there are six consecutive multiples of 4. The median is the arithmetic mean (average) of the third largest and fourth largest, or the average of 12 and 16. Thus, the median is 14. Because this is an evenly spaced sequence, the average is also 14.

The **mean** and **median** of a sequence are equal to the **average** of the first and last terms.

What is the arithmetic mean of 4, 8, 12, 16, and 20?

In this example, 20 is the largest (last) number and 4 is the smallest (first). The arithmetic mean and median are therefore equal to (20 + 4) ÷ 2 which is 12.

What is the arithmetic mean of 4, 8, 12, 16, 20, and 24?

In this example, 24 is the largest (last) number and 4 is the smallest (first). The arithmetic mean and median are therefore equal to (24 + 4) ÷ 2 which is 14.

Thus, for all evenly spaced sequences, just remember: the average equals **(First** + **Last)** ÷ **2**.

The **sum** of the elements in the sequence equals the **arithmetic mean** (average) times the **number of items** in the sequence.

This property applies to all sequences, but it takes on special significance in the case of evenly spaced sequences because the "average" is not only the arithmetic mean, but also the median.

What is the sum of 4, 8, 12, 16, and 20?

You have already calculated the average above; it is equal to 12. There are 5 terms, so the sum equals 12 × 5 which is 60.

What is the sum of 4, 8, 12, 16, 20, and 24?

You have already calculated the average above; it is equal to 14. There are six terms, so the sum equals 14 × 6 which is 84.

Check Your Skills

2. What is the sum of the numbers 13, 14, 15, and 16?

3. If $x = 3$, what is the sum of $2x$, $(2x + 1)$, $(2x + 2)$, $(2x + 3)$, and $(2x + 4)$?

Counting Integers: Add 1 Before You Are Done

How many integers are there from 6 to 10? Four, right? Wrong! There are actually five integers from 6 to 10. Count them and you will see: 6, 7, 8, 9, 10. It is easy to forget that you have to include extremes. In this case, both extremes (the numbers 6 and 10) must be counted.

You don't have to methodically count each term in a long consecutive pattern. Just remember that if both extremes should be counted, you need to take the difference of the last and first numbers and **add 1 before you are done**.

> How many integers are there from 14 to 765, inclusive?

Just remember: for consecutive integers, the formula is **(Last** − **First** + **1)**. Thus: 765 − 14, plus 1, yields 752.

This works easily enough if you are dealing with consecutive integers. Sometimes, however, the question will ask about consecutive multiples. For example, “How many multiples of 4 …” or “How many even numbers …” are examples of sequences of consecutive multiples.

In this case, if you just subtract the largest number from the smallest and add 1, you will be overcounting. For example, “All of the even integers between 12 and 24” yields 12, 14, 16, 18, 20, 22, and 24. That is seven even integers. However, (Last − First + 1) would yield (24 − 12 + 1) = 13, which is too large. How do you amend this? Because the items in the list are going up by increments of 2 (counting only the even numbers), you need to divide (Last − First) by 2. Then, add the 1 before you are done:

> (Last − First) ÷ Increment + 1 = (24 − 12) ÷ 2 + 1 = 6 + 1 = 7

Just remember: for consecutive multiples, the formula is **(Last − First) ÷ Increment + 1**. The bigger the increment, the smaller the result, because there is a larger gap between the numbers you are counting.

Sometimes, however, it is easier to list the terms of a consecutive pattern and count them, especially if the list is short or if one or both of the extremes are omitted.

> How many multiples of 7 are there between 100 and 150?

Note that the first and last items in the sequences are omitted—they must be determined by you. Here it may be easiest to list the multiples: 105, 112, 119, 126, 133, 140, 147. Count the number of terms to get the answer: 7. Alternatively, you could note that 105 is the first number, 147 is the last number, and 7 is the increment, thus:

> Number of terms = (Last − First) ÷ Increment + 1 = (147 − 105) ÷ 7 + 1 = 6 + 1 = 7

Check Your Skills

4. How many integers are there from 1,002 to 10,001?

5. How many multiples of 11 are there between 55 and 144, exclusive?

Check Your Skills Answer Key

1. **Not evenly spaced:** Even though the number inside the square root is going up by the same interval every time, the actual value is not. For example, $\sqrt{9} = 3$ and $48\sqrt{3}\,\text{cm}^2$ That's a difference of 0.4. But $\sqrt{2} \approx 1.4$ and $\sqrt{4} = 2.$ That's a difference of 0.3. Because the interval is changing, this is not an evenly spaced sequence.

   Evenly spaced: No matter what *x* is, this sequence will end up being evenly spaced. For example, if *x* were 3, *x* − 4 would equal −1, which is a difference of 4. Then *x* − 8 would equal −5 and *x* − 12 would equal −9, which is also a difference of 4. The interval is unchanged, so this is an evenly spaced sequence.

   Evenly spaced: This is the opposite of example (a). In this case, the terms inside the square root signs are not creating an evenly spaced sequence, but the actual values are. For the terms $\sqrt{4} = 2$ and $\sqrt{9} = 3,$ there's a difference of 1. For $\sqrt[3]{64} = 4$ and $\sqrt[3]{64} = 4$, there's also a difference of 1. The interval is unchanged, so this is an evenly spaced sequence (and a sequence of consecutive integers at that).

   Not evenly spaced: In this question, 5^1 = 5 and 5^2 = 25. That's a difference of 20. But 5^3 = 125, which is 100 larger than the previous term. Again, the interval is changing, so this is not an evenly spaced sequence.

Evenly spaced: No matter what y is, this will be an evenly spaced sequence. For example, if $y = 5$, then $2y = 10$, which is a difference of 5. Then $3y = 15$ and $4y = 20$, which is also a difference of 5. Even if y is set equal to 0 or 1, the result would still be considered evenly spaced (the difference between every term would be the same, namely 0 or 1).

2. **58**
While you could easily add these up, try using the properties of evenly spaced sequences. The average/median is going to be $\frac{\text{first term} + \text{last term}}{2}$, or $\frac{5}{16} < \frac{5}{15}$ This equals $\frac{23}{7}$, which is 14.5. You now have to multiply this by the number of terms: 4 × 14.5 = 58.

3. **40**
While you could plug 3 in for x everywhere, why waste the time? As soon as you notice that this is an evenly spaced sequence, you know the middle term is the average, and all you need to do is multiply by the number of terms in the sequence: $2x + 2 = 8$ and $8 \times 5 = 40$.

4. **9,000**
10,001 − 1,002 + 1 = 9,000.

5. **8**
Remember that the words *inclusive* and *exclusive* tell you whether or not to include the extremes. In this case, you won’t be including either 55 or 144 (though 144 isn’t actually a multiple of 11, so doesn’t end up mattering). You can solve either through counting or the equation:

Counting: 66, 77, 88, 99, 110, 121, 132, 143 = 8 terms

Equation: $143 - 66 = 77, \frac{77}{11} = 7, 7 + 1 = 8$ terms

Problem Set

Solve these problems using the rules for consecutive integers.

1. How many primes are there from 10 to 41, inclusive?

2. Will the average of six consecutive integers be an integer?

3. If the sum of a sequence of 10 consecutive integers is 195, what is the average of the sequence?

4. How many terms are there in the sequence of consecutive integers from −18 to 33, inclusive?

5. Set A is comprised of all the even numbers between 0 and 20, inclusive.

Quantity A	**Quantity B**
The sum of all the numbers in Set A	150

6.

Quantity A	**Quantity B**
The number of multiples of 7 between 50 and 100, inclusive	The number of multiples of 9 between 30 and 90, inclusive

7. Set A is comprised of the following terms: ($3x$), ($3x - 4$), ($3x - 8$), ($3x - 12$), ($3x - 16$), and ($3x - 20$).

Quantity A	**Quantity B**
The sum of all the terms in Set A	$18x - 70$

Solutions

1. **9**
   The primes from 10 to 41, inclusive, are: 11, 13, 17, 19, 23, 29, 31, 37, and 41. Note that the primes are NOT evenly spaced, so you have to list them and count them manually.

2. **No**
   For any sequence of consecutive integers with an **even** number of items, the average is NEVER an integer. For example, if you pick 4, 5, 6, 7, 8, and 9:

   $$\frac{4+5+6+7+8+9}{6} = \frac{39}{6} = 6.5$$

   Since any set of consecutive integers is evenly spaced, the average is equal to the median. And if the set also has an even number of terms, the median is equal to the average of the two middle terms (i.e., exactly in between two consecutive integers). In the example above, the two middle terms are 6 and 7, giving you a median of 6.5, which is also the average of this evenly spaced set.

3. **19.5**
   Average $= \frac{\text{Sum}}{\text{\# of terms}}$. In this problem, you have $\frac{5}{14} \times \frac{7}{20} =$ as the average.

4. **52**

   Calculate 33 − (−18) = 51. Then add 1 before you are done: 51 + 1 = 52.

5. **(B)**

   There are two ways to do this question. First, try to estimate the sum. Notice that there will be 11 terms in Set A, and one of them is 0. Even if every term in the set were 20, the sum would only be 220 (11 × 20 = 220). But half of the terms are less than 10. It is unlikely that Quantity A will be bigger than 150.

   To do it mathematically, use the equation for the sum of an evenly spaced sequence. You can find the median by adding up the first and last terms and dividing by 2:

$$\$500 = \frac{1}{24}x$$

   You can then find the number of terms by subtracting the first term from the last term, dividing by the interval (in this case, 2) and adding 1: $\frac{x}{5} = \frac{13}{20} - \frac{1}{4}$ and 10 + 1 = 11 terms.

   Finally, the sum of the terms will be the average value of the terms (10) times the number of terms (11): 10 × 11 = 110.

   Set A is comprised of all the even numbers between 0 and 20, inclusive.

Quantity A	Quantity A
The sum of all the numbers in Set A = 110	150

Therefore, **Quantity B is greater**.

6. **(C)**

Both quantities can be solved straightforwardly with the equations. The first multiple of 7 between 50 and 100 is 56, and the last is 98. Thus: 98 − 56 = 42; $\frac{20}{4} = 5$; and 6 + 1 = 7.

Another way to think about it is that 56 is the 8th multiple of 7, and 98 is the 14th multiple of 7. Now use the counting principle:

14 − 8 = 6 + 1 = 7

There are 7 multiples of 7 between 50 and 100.

Similarly, the first multiple of 9 between 30 and 90 is 36, and the last is 90: 90 − 36 = 54; $\frac{20}{4} = 5$, and 6 + 1 = 7.

Now note that 36 is the 4th multiple of 9, and 90 is the 10th multiple of 9:

10 − 4 = 6 + 1 = 7

There are 7 multiples of 9 between 30 and 90.

Quantity A	**Quantity B**
The number of multiples of 7 between 50 and 100, inclusive = 7	The number of multiples of 9 between 30 and 90, inclusive = 7

Therefore, **the two quantities are equal**.

7. **(A)**

The key here is to notice that Set A is an evenly spaced sequence, and thus you can easily solve for its sum. The median will be equal to

$$\frac{\text{first term} + \text{last term}}{2}:$$

$$\frac{(3x) + (3x - 20)}{2} = \frac{6x - 20}{2} = 3x - 10$$

Although there is a formula to figure out how many terms there are, it is easiest in this case to count. There are six terms in the set.

The sum of the terms in the set is the average value of the terms ($3x - 10$) times the number of terms (6).

$(3x - 10) \times 6 = 18x - 60$

Rewrite the quantities:

Set A is comprised of the following terms: ($3x$), ($3x - 4$), ($3x - 8$), ($3x - 12$), ($3x - 16$), and ($3x - 20$)

Quantity A	**Quantity B**
$18x - 60$	$18x - 70$

Be careful. Quantity A is subtracting a smaller number (60) than is Quantity B (70), and so has a larger value. Therefore, **Quantity A is greater**.

Chapter 21

NUMBER LINES

In This Chapter...

Relative Position & Relative Distance

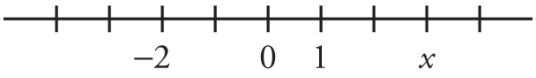
Line Segments

Chapter 21

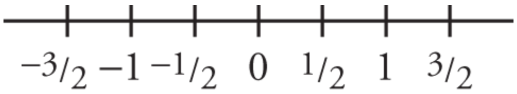
Number Lines

Number lines can appear in a variety of different forms on the GRE. They also provide varying amounts of information.

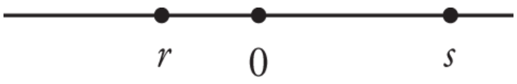
The most structured version of a number line will contain evenly spaced tick marks. These provide the most detail about the position of points on a number line and about the distance between points. For example:



These number lines will almost always contain numbers, and will often contain variables as well. Also note that the distance between tick marks can be an integer amount (like in the number line previously pictured) or a fractional amount (like in the following number line):

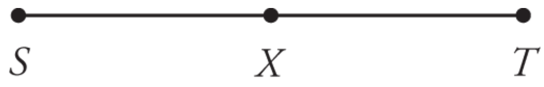


Not all number lines will provide this level of detail. Many number lines will only display a handful of points that are not evenly spaced, like this one:



These number lines are likely to contain fewer actual numbers, and will always contain at least one variable. On these number lines, it is more likely that you won't have specific information about the distance between two points.

Additionally, questions that talk about line segments or points that all lie on a line can be thought of as number lines. For example, a question might state that point *X* is the midpoint of line segment *ST*. This is the picture you would draw:



These number lines will rarely contain any real numbers. Often, the only points on the line will be designated by variables. Questions that require this type of number line may or may not provide information about the specific distance between points, although they may provide proportional information. For instance, in the previous number line, although you don't know the length of line segment *ST*, you do know that *ST* is twice as long as segments *SX* and *XT* (because *X* is the midpoint of *ST*).

Relative Position & Relative Distance

Questions that involve number lines overwhelmingly ask for information about the *position* of a point or points or the *distance* between two points.

POSITION

On any number line you will see, numbers get bigger as they move from left to right. For example:

B is greater than *A*.

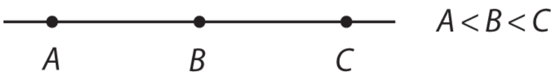
A is less than *B*.

B is more positive than *A* (if both positive).

A is more negative than *B* (if both negative).

The statements shown are true regardless of where 0 is on the number line. Points *A* and *B* could both be positive or both be negative, or *A* could be negative and *B* could be positive.

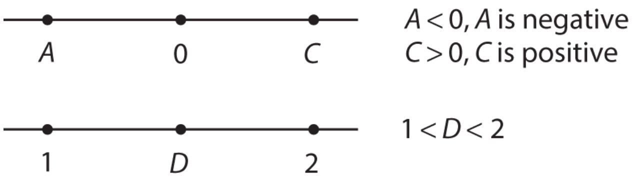
Number lines on the GRE follow rules similar to the rules for geometric shapes. If there is more than one point on a number line, you KNOW the *Relative Position* of each point.



While you do know the relative position of each point, you do not know the *Relative Distance* between points (unless that information is specifically provided).

On the preceding number line, B could be closer to A than to C, closer to C than to A, or equidistant between A and C. Without more information, there is no way to know.

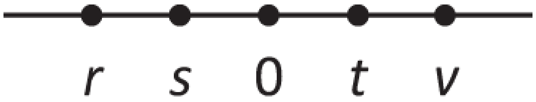
The rules are similar if a number line contains both numbers and variables. For example:



Point D looks like it is halfway between 1 and 2, but that does not mean that it is 1.5. Point D could be 1.5, but it could also be 1.000001, or 1.99999 or, in fact, any number between 1 and 2.

Check Your Skills

Refer to the following number line for questions #1–3.



Which of the following MUST be true?

1. $v > s + t$

2. $v + s > t + r$

3. $rs > v$

DISTANCE

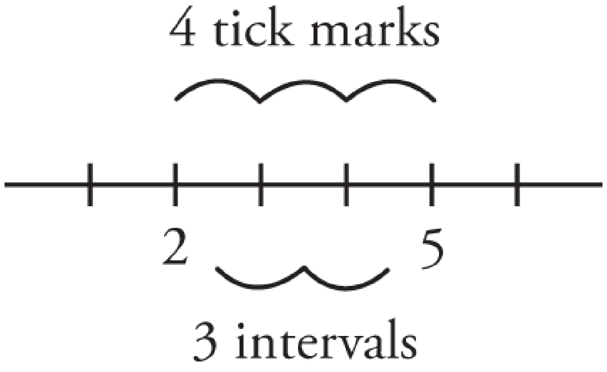
If you know the specific location of two points on a number line, the distance between them is the absolute value of their difference. For example:

Distance $= |8 - (-3)| = |(-3) - 8| = 11$

If a number line contains tick marks and specifically tells you they are evenly spaced, it may be necessary to calculate the distance between tick marks.

On an evenly spaced number line, tick marks represent specific values, and the intervals between tick marks represent the distance between tick marks.

For any specific range, there will always be one more tick mark than interval, for example:



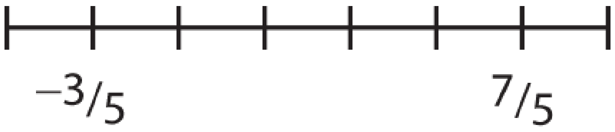
On this number line, there are four tick marks between 2 and 5 (inclusive). There is one fewer interval than tick marks. There are only three intervals between 2 and 5. Now calculate the length of the intervals on this number line. To calculate the distance between any two tick marks (which is the same as the length of the intervals), subtract the lower bound from the upper bound and divide the difference by the number of intervals.

In the previous number line, the lower bound is 2, the upper bound is 5, and there are three intervals between 2 and 5. Use these numbers to calculate the distance between tick marks on the number line:

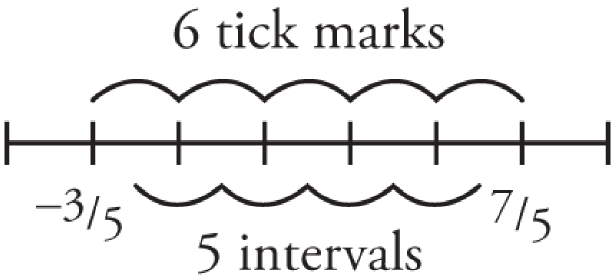
$$\frac{\text{upper} - \text{lower}}{\#\text{ of intervals}} = \frac{5-2}{3} = 1$$

That means that each tick mark in the preceding number line is 1 unit away from each of the two tick marks to which it is adjacent.

Not every number line will have interval lengths with integer values. Note that this method is equally effective if the intervals are fractional amounts. What is the distance between adjacent tick marks on the following number line?

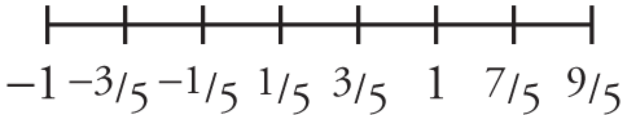


Now the range contains six tick marks and five intervals.

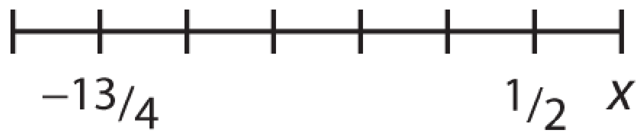


$$\frac{\text{upper} - \text{lower}}{\#\text{ of intervals}} = \frac{7/5 - \left(-3/5\right)}{5} = \frac{10/5}{5} = \frac{10}{25} = \frac{2}{5}$$

Thus, the distance between tick marks is $\frac{1}{2}$:



Check Your Skills



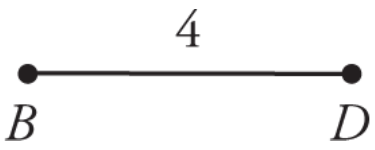
4. On the number line above, what is the value of point *x*?

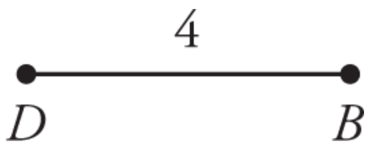
Line Segments

Some questions on the GRE will describe either several points that all lie on a line or line segments that also lie on the same line. To answer these questions correctly, you will need to use the information in the question to construct a number line. Ultimately, position and distance will be of prime importance.

POSITION

To correctly draw number lines, you need to remember one thing: If a question mentions a line segment, there are two possible versions of that segment. Suppose a question tells you that the length of line segment $\overline{BD}$ is 4. These are the two possible versions of $\overline{BD}$:





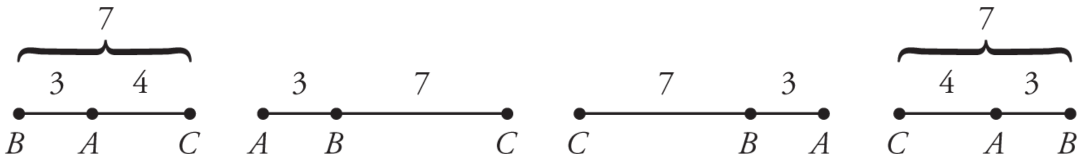
This can be taken even further. Suppose there are three points on a line: *A, B,* and *C*. Without more information, you don't know the order of the three points. Here are some of the possible arrangements:

When questions provide incomplete information about the relative position of points, make sure that you account for the lack of information by drawing multiple number lines.

DISTANCE

Distance on this type of number line can potentially be made more difficult by a lack of complete information about the positions of points on the line.

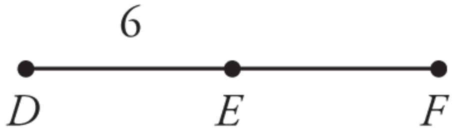
Suppose that *A, B,* and *C* all lie on a number line. Further suppose that $\overline{AB} = 3$ and $\overline{BC} = 7$. Because $\overline{AB}$ is shorter than $\overline{BC}$, there are two possible positions for point *A*: in between *B* and *C*, or on one side of *B*, with *C* on the other side, as shown here:

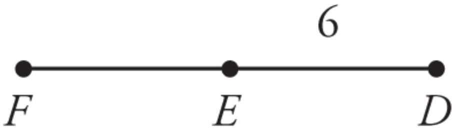


Constructing number lines can be made more difficult by many pieces of information in the question. To construct number lines efficiently and accurately, while remembering to keep track of different possible scenarios, always start with the most restrictive pieces of information first.

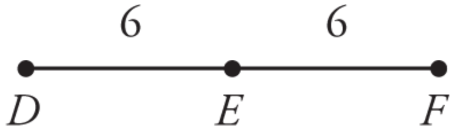
> On a line, E is the midpoint of $\overline{DF}$, and $\overline{BD}$ has a length of 6. Point G does not lie on the line and $\overline{BC} = 7$. What is the range of possible values of $\overline{BC}$?

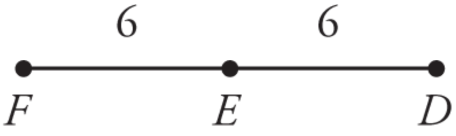
The best way to start this problem is to draw $\overline{DF}$, with *E* in the middle. There are two possible versions. Also note that $\overline{DE} = 6$:



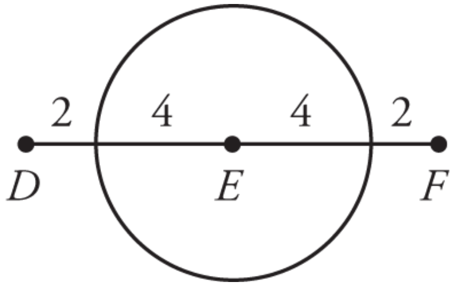


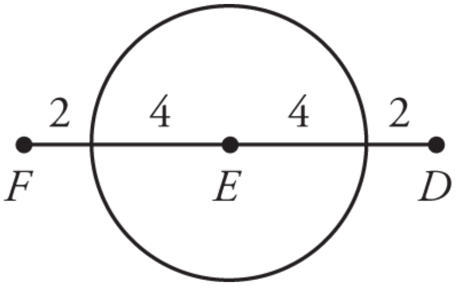
Also, because *E* is the midpoint of $\overline{DF}$, you know that $\overline{BD}$ also has a length of 6:





Now you need to deal with point *G*. Although you do not know the precise position of *G*, you know it is a fixed distance from *E*. The set of all points that are equidistant from a fixed point is actually a circle—in other words, to represent the possible positions of *G*, draw a circle around point *E* with a radius of 4.





As it turns out, both number lines behave the exact same way, so there is no need to continue to look at both.

On this diagram, you can see that *G* would be closest to *F* when it is on the line between *E* and *F.* That point is 2 away from *F*. Similarly, *G* is farthest away from *F* when it is on the line between *D* and *E*. That point is 10 away from *F*.

If *G* could be on the line, the range of possible values of $\overline{BC}$ would be $2 \leq \overline{FG} \leq 10$. Because it can't be on the line, the range is instead $2 < \overline{FG} < 10$.

Check Your Skills

5. *X, Y,* and *Z* all lie on a number line. Segment $\overline{XY}$ has a length of 5 and $\overline{AB}$ has a length of 7. If point *U* is the midpoint of $\overline{BD}$, and $\overline{UZ} > 2$, what is the length of $\overline{AB}$?

Check Your Skills Answer Key

1. **Must Be True**

   Because *v* is already greater than *t*, adding a negative number to *t* will only make it smaller.

2. **Must Be True**

   One way to prove this statement is always true is to add inequalities. You know that $v > t$, and that $s > r$. Thus:

$$\begin{array}{rcl} v & > & t \\ +s & > & r \\ \hline v+s & > & t+r \end{array}$$

3. **Not Always True**

   Point *v* could be any positive number, and *r* and *s* could be any negative number.

   If $r = -3$, $s = -2$, and $v = 4$, then $rs > v$.

   If, however, $r = -2$, $s = -1/2$, and $v = 3$, $v > rs$.

4. $\frac{1}{2}$

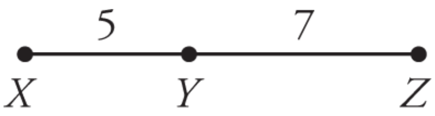
To find *x*, you need to figure out how far apart tick marks are. You can use the two given points ($-\frac{13}{4}, \frac{1}{2}$) to do so. There are five intervals between the two points. Thus:

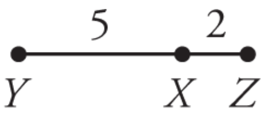
$$\frac{{}^{1}\!/_{2} - \left(-\,{}^{13}\!/_{4}\right)}{5} = \frac{{}^{15}\!/_{4}}{5} = \frac{15}{20} = \frac{3}{4}$$

If the distance between tick marks is $\frac{1}{2}$, then *x* is: $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$.

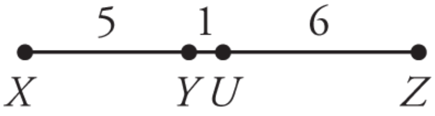
5. **6**

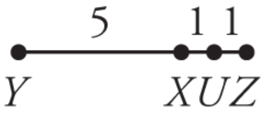
Start with the points *X, Y*, and *Z*. There are two possible arrangements:





Now place *U* on each number line:

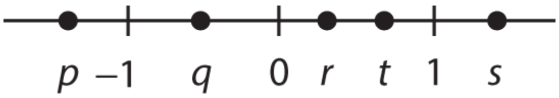




On one number line, $\overline{UZ} > 2$, but the question stated that $\overline{UZ} > 2$, so $\overline{AB}$ must equal 6.

Problem Set

For problems #1–6, refer to the following number line. Decide whether each statement **Must Be True**, **Could Be True**, or will **Never Be True**.



1. $s + q > 0$

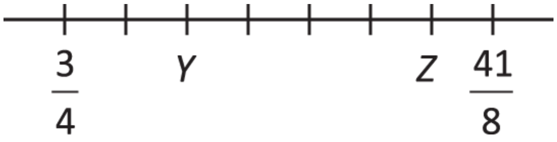
2. $pq > t$

3. $p^2 > s^4$

4. $s - p > r - q$

5. $t - q = 2$

6. $rs > 1$

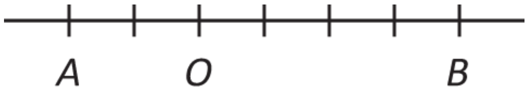


7. If the tick marks on the number line above are evenly spaced, what is the distance between *Y* and *Z*?

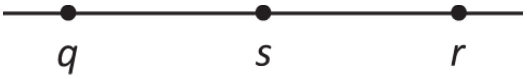
8. Point *A*, *B*, and *C* all lie on a line. Point *D* is the midpoint of $\overline{AB}$ and *E* is the midpoint of *BC*; $\overline{UZ} > 2$ and $\overline{BC} = 10$. Which of the following could be the length of $\overline{BC}$?

(A) 1
(B) 2
(C) 3
(D) 4
(E) 5

9.



Quantity A	**Quantity B**
$(A)(B)$	-1

10. 

s is the midpoint of $\overline{qr}$

$r = -2q$

Quantity A	Quantity B
s	0

11. A, B, C, and D all lie on a number line. C is the midpoint of $\overline{AB}$ and D is the midpoint of $\overline{AC}$.

Quantity A	Quantity B
The ratio of $\overline{BD}$ to $\overline{AB}$	The ratio of $\overline{AC}$ to $\overline{AB}$

Solutions

1. **Must Be True**
   Although you don't have specific values for either s or q, you know that s is greater than 1, and you know that q is between 0 and −1. Even if s was as small as it could be (≈ 1.00001) and q was as negative as it could be (≈ -0.99999), the sum would still be positive.

2. **Could Be True**
   The product pq will be positive since both p and q are negative. You know that t must be between 0 and 1, but the product pq could be either less than 1 or greater than 1, depending on the numbers chosen. If $t = 0.9$, $q = -0.1$, and $p = -2$, then $t > pq$. However, if $t = 0.5$, $q = -0.9$, and $p = -5$, $pq > t$.

3. **Could Be True**
   Both p^2 and s^4 will be positive, but depending on the numbers chosen for p and s, either value could be larger. If $p = -2$ and $s = 3$, then $s^4 > p^2$. If $p = -8$ and $s = 2$, then $s^4 < p^2$.

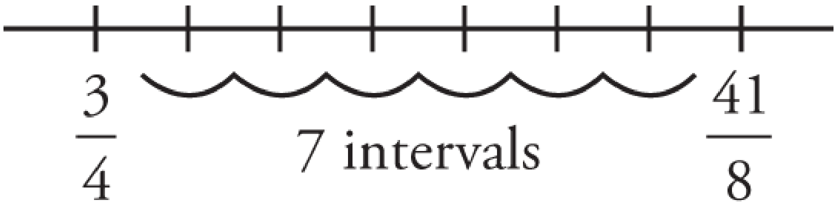
4. **Must Be True**
   You know s is greater than 1 and p is less than −1. The smallest that the difference can be is greater than 2.

   You know r must be between 0 and 1 and q must be between 0 and −1. The greatest the difference can be is less than 2. Thus, $s - p$ will always be greater than $r - q$.

5. **Never Be True**
   Even if t is as large as it can be and q is as small as it can be, the difference will still have to be less than 2. If $t = 0.999999$ and $q = -0.999999$, then $t - q = 1.999998$.

6. **Could Be True**
   If $r = 0.1$ and $s = 2$, then $rs < 1$. If $r = 0.5$ and $s = 3$, then $rs > 1$.

7. **2.5**
   To figure out the distance between Y and Z, you first need to figure out the distance between tick marks. You can use the two points on the number line $\left(\frac{3}{4} \text{ and } \frac{41}{8}\right)$ to find the distance. There are 7 intervals between the two points, as shown here:



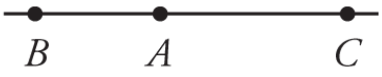
$$\frac{41/8 - 3/4}{7} = \frac{41/8 - 6/8}{7} = \frac{35/8}{7} = \frac{5}{8}$$

You actually do not need to know the positions of Y and Z to find the distance between them. You know that there are 4 intervals between Y and Z, so the distance is:

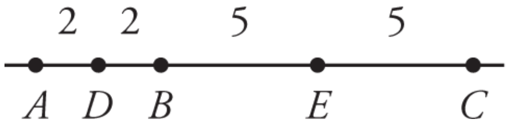
$$\frac{45}{900} = \frac{5}{100} = 0.05$$

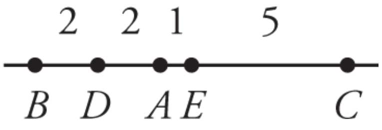
8. **(A)**

The trick to this problem is recognizing that there is more than one possible arrangement for the points on the number line. Because $\overline{BC}$ is longer than $\overline{AB}$, *A* could be either in between *B* and *C* or on one side of *B* with *C* on the other side of *B*, as shown here:



Using the information about the midpoints (*D* and *E*) and the lengths of the line segments, you can fill in all the information for the two number lines:





You can see that $\overline{BC}$ has two possible lengths: 1 and 9, however 1 is the only option that is an answer choice.

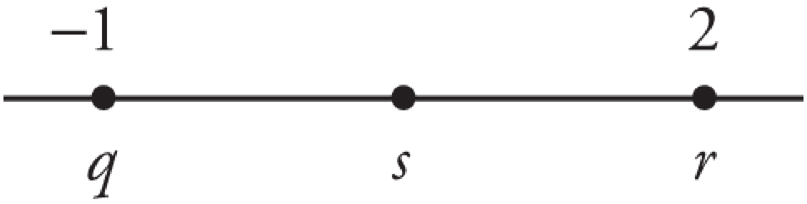
9. **(D)**

With only one actual number displayed on the number line, you have no way of knowing the distance between tick marks. If the tick marks are a small fractional distance away from each other, then $\overline{AB}$ will be greater than −1. For instance, if the distance between tick marks is $\frac{1}{2}$, then *A* is $-\frac{9}{2}$, *B* is $\frac{1}{2}$ and $\overline{AB}$ is $-\frac{9}{2}$, which is greater than −1. If the distance between tick marks is 1, then *A* is −2, *B* is 4, and $\overline{AB}$ is −8, which is less than −1.

Therefore, **the relationship cannot be determined from the information given**.

10. **(A)**

The easiest approach is to pick numbers. Point *q* must be a negative number and *r* must be positive. If *q* = −1, then *r* = 2:



If *s* is the midpoint of *q* and *r*, then *s* must be 0.5. Therefore, *s* > 0.

For any numbers you pick, *s* will be positive. Therefore, **Quantity A is greater**.

11. **(C)**

Visualizing the preceding number line, the ratio of $\overline{BD}$ to $\overline{AB}$ is $\frac{1}{2}$. Similarly, the ratio of $\overline{AC}$ to $\overline{AB}$ is $\frac{1}{2}$. Therefore, **the two quantities are equal**.

Unit Five: Word Problems

This section educates students in the art of translating challenging word problems into organized data, as well as providing structured frameworks for attacking each question type.

In This Unit...

Chapter 22: Algebraic Translations

Chapter 23: Rates & Work

Chapter 24: Ratios

Chapter 25: Statistics

Chapter 26: Combinatorics

Chapter 27: Probability

Chapter 28: Minor Problem Types

Chapter 22

ALGEBRAIC TRANSLATIONS

In This Chapter...

Decoding the GRE Word Problem

Common Word Problem Phrases

Translating Words Correctly

Hidden Constraints

Chapter 22

Algebraic Translations

Decoding the GRE Word Problem

Two thoughts are common to many frustrated students:

> "I don't know where to get started" and "I don't know what they want me to do."

You can attack these frustrations one at a time:

> **"I don't know where to get started."**

A passive thinker takes in information, hopes that it will lead somewhere, waits for a connection to appear, and then … (hopefully) … voilà! In contrast, an active thinker *aggressively* seeks out relationships between the various elements of a problem and looks to write equations that can be solved. You have to be an active thinker on the GRE.

Here's a sample problem:

> A steel rod 50 meters long is cut into two pieces. If one piece is 14 meters longer than the other, what is the length, in meters, of the shorter piece?

The trick to word problems is to not try to do everything all at once. While it's great when the entire process is clear from the start, such clarity about your work is often not the case. That's why **you need to start by identifying unknowns and creating variables to represent the unknowns**. What quantities have you not been given specific values for?

Take a moment to identify those unknown quantities and write them down in the space provided below. Make up letters (variables) to stand for the quantities, and label these letters.

Identify Unknowns and Create Variables

In this question, both the *length of the shorter piece* and the *length of the longer piece* are unknown, so begin by assigning each of those values a variable. You could go with the traditional algebraic variables x and y, but what if you forget which is which while you're busy answering the question? Instead, use letters that can help you remember which variable is assigned to which value:

> S = length of the shorter piece
> L = length of the longer piece

Just like that, you’ve gotten started on this problem. This may seem like a minor accomplishment in terms of the entire question, but it was an important one. You've started to get a handle on things. Often, as soon as you start translating a word problem into *algebra*, the path forward becomes clearer.

Here's another way to think about your progress. In essence, you've started to ask "What do they want?" and "What do they give me?" And you've written down your first answers in the form of algebra.

The first things to focus on writing down are the simple *quantities,* the things you're going to represent with a single letter. So go ahead and name those quantities. One of these quantities will likely be what "they," the test makers, ultimately want.

Now it’s time to deal with the second frustration:

> **“I don’t know what they want me to do.”**

Even now that you’ve identified and labeled your variables, you might still feel confused. That’s fine. Virtually everyone ends up facing a number of problems that are above his or her ability level on the GRE. What distinguishes the higher-performing GRE test-takers in these moments is that they begin spelling out relationships before they know how the equations will prove useful.

It’s similar to untangling a ball of yarn: if you waited until you knew how the entire process would end, you might never get started. Of course, you hope to have a clear vision right from the start, but if you don’t, *dive in and see what you find*—you’ll likely make key realizations along the way. Ironically, often it’s the roadblocks you encounter that point the way. So your next step is to **identify relationships and create equations**.

You're still asking "What do they give me?" but now at a slightly higher level. "They" give you information about *relationships* between quantities. You're going to find and write down those relationships in the form of equations.

Go back to the problem. Look at *one piece of information at a time* and then *translate that information into equations*. Try it first on your own, then read on.

> *A steel rod 50 meters long is cut into two pieces.*

The relationship expressed here is one of the two most common types of relationships found in Word Problems. The original length of the rod was 50 meters, and it was cut into 2 pieces. Therefore, the length of the shorter piece plus the length of the longer piece must equal 50 meters. This common relationship (one you should watch out for in other word problems) is **Parts Add to a Sum**.

In other words, One Part + The Other Part = The Total.

For this problem, you can write down the relationship this way:

$$S + L = 50$$

Now that you’ve translated the first part of the problem, move on to the next part.

> *If one piece is 14 meters longer than the other …*

The relationship expressed here is another common type found in word problems. The longer piece of metal is 14 meters longer than the shorter piece of metal. So if you were to add 14 meters to the shorter piece, it would be the same length as the longer piece.

This relationship (be on the lookout for this one too) is **One Part Can Be Made Equal to the Other**. Either the question will say that two values are equal, or it will tell you exactly *how they differ.* This question told you how they were different, so your equation shows how you could make them equal: One Part + The Difference = The Other Part.

In this case, you would write:

$$S + 14 = L$$

By the way, when constructing equations in which you are making one part equal to the other, it can be very easy to express the relationship backwards. If you mistakenly wrote down $S = L + 14$, you’re not alone. A good habit to get into if you find yourself making this kind of error is to verify your equation with **hypothetical numbers**.

To check if the previous equation is correct, start by imagining that the shorter piece of metal is 20 meters long. If the shorter piece were 20 meters long, then the longer piece would have to be 34 meters long. Now plug those numbers into your equation. Does $20 + 14 = 34$? Yes, it does, so your equation is correct.

Next, move on to the final part of the question:

> *… what is the length, in meters, of the shorter piece?*

This part of the question doesn't describe a relationship that you can use to create an equation, but it does tell you something quite useful: it tells you *what you're solving for!* It clearly answers the question you've had: "What do they want?"

Make sure that you note in some way what value you're actually looking for as you solve a problem—it can help you stay focused on the task at hand. In this problem, you're trying to find S. On your paper, you might even write:

$$S = ?$$

So now that you've **identified unknowns and created variables**, **identified relationships and created equations,** and **identified what the question is asking for**, it's time to put the pieces together and answer the question. Try it on your own first, and then when you have an answer, turn the page and you see the final steps.

First, you should recap, and then review the final steps and answer the question. After reading the question, you can create two equations:

$$S + L = 50$$
$$S + 14 = L$$

You have two variables and two equations, so you need to solve for S:

$$S + L = 50 \rightarrow L = 50 - S$$

$$\begin{array}{l} S + 14 = \ (50 - S) \\ S + 14 = 50 - S \\ \quad -14 - 14 \end{array}$$

$$\begin{array}{l} S = 36 - S \\ +S \qquad + S \\ \dfrac{2S}{2} = \dfrac{36}{2} \\ S = 18 \end{array}$$

If you had trouble getting the correct value for S, then you should probably go back and refresh your algebra skills (see our *Algebra* guide). Knowing how to substitute and solve is absolutely essential if you want to do consistently well on Word Problems. If you’re comfortable with everything you’ve done so far to answer the question, then you’re ready for a tougher problem:

> Jack is 13 years older than Bao. In 8 years, he will be twice as old as Bao. How old is Jack now?

First try this problem on your own. Remember to follow the same steps you followed in the last question. After you're finished, review the explanation on the next page.

Ok, time to get started. The first thing you have to do is **identify unknowns** and **create variables**. In this problem, the two unknown quantities are the ages of Jack and Bao. You can represent them like this:

J = Jack's age *now*
B = Bao's age *now*

Before you move on to the next step, it's important to understand why you want to specify that the variables represent Jack and Bao's ages *now*. As you were solving this problem by yourself, you may have noticed that there was an added wrinkle to this question. You are presented with information that describes two distinct points in time—now and eight years from now. Some WPs on the GRE provide information about two distinct but related situations. When you are dealing with one of those problems, be careful about the reference point for your variables. In this case, you want to say that the variables represent Jack and Bao's ages *now* as opposed to eight years from now. This makes it easier to express their ages at other points in time.

Now that you've created variables, it's time to **identify relationships** and **create equations**. Go through the information presented in the question one piece at a time:

Jack is 13 years older than Bao.

Once again, check that you're putting this together the right way (not putting the +13 on the wrong side of the equation). Your equation should be:

$J = B + 13$, **NOT** $J + 13 = B$

Now, move on to the next piece of information:

In 8 years, he will be twice as old as Bao.

This piece is more challenging to translate than you might otherwise suspect. Remember, the variables represent their ages now, but this statement is talking about their ages 8 years from now. So you can't just write $J = 2B$. This relationship is dependent upon Jack and Bao's ages 8 years from now. You don't want to use new variables to represent these different ages, so adjust the values like this:

$(J + 8)$ = Jack's age 8 years from now
$(B + 8)$ = Bao's age 8 years from now

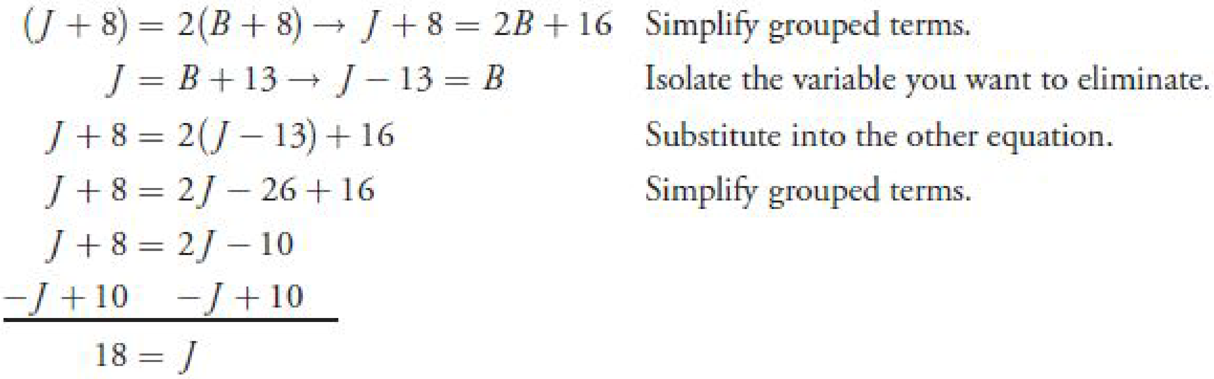
Now you can accurately create equations related to the earlier time *and* to the later time. Plus, if you keep those values in parentheses, then you can avoid potential PEMDAS errors. So your second equation should read:

$(J + 8) = 2(B + 8)$

Only one more piece of the question to go:

How old is Jack now?

This tells you that you're looking for the value of *J*. In other words, *J* = ?. All the pieces are in place and you're ready to solve:



$$\begin{array}{ll}
(J+8)=2(B+8)\rightarrow J+8=2B+16 & \text{Simplify grouped terms.}\\
J=B+13\rightarrow J-13=B & \text{Isolate the variable you want to eliminate.}\\
J+8=2(J-13)+16 & \text{Substitute into the other equation.}\\
J+8=2J-26+16 & \text{Simplify grouped terms.}\\
J+8=2J-10 & \\
\underline{-J+10 \quad -J+10} & \\
18=J &
\end{array}$$

The question asks for Jack's age, so you have your answer. If you're pressed for time on the test and you're confident in all the previous steps, you might just move to the next question. But if you're at all doubtful, it's always a good final move to check your answer against the original facts and make sure everything fits. You're told that Jack is 13 years older than Bao, so Bao must be 18 − 13 = 5 years old. You're also told that in 8 years, Jack will be twice as old as Bao. Does this work? In 8 years, Jack will be 18 + 8 = 26 years old. And Bao will be 5 + 8 = 13 years old. Jack will indeed be twice as old as Bao (2 × 13 = 26).

Now you can review what you know about Word Problems and the steps to take to solve them:

Step 1: Identify unknowns and create variables.

- Don't forget to use descriptive letters (e.g., shorter piece = *S*).

- Be *very specific* when dealing with questions that contain two distinct but related situations (e.g., Jack's age *now* = J versus Jack's age in 8 years = $J + 8$).

Step 2: Identify relationships and create equations.

- Once you have identified how many unknowns (variables) you have, try to find the same number of equations in the text, because you will usually need the same number of equations as you have variables. However, you need to be on the lookout for questions that ask about variable combinations, such as xy or $x + y$, resulting in the need for fewer equations than the number of variables.
- Don't forget to look at one piece of the question at a time. Don't try to do everything at once.
- Use numbers to check that you have set up your equation correctly. For example, if it says that Jack is twice as old as Ben, which is correct: $J = 2B$ or $2J = B$? If Jack were 40, Ben would be 20, so $40 = 2(20)$ or $2(40) = 20$?

Step 3: Identify what the question is asking for.

- Writing down a clear goal (such as $J = ?$) can prevent you from losing track of what you're doing and can help you stay focused on the task at hand.
- By the way, steps 1–3 can happen in any order you want.
- A shorthand for these three steps is this pair of questions: "What do they give me? What do they want?" Keep asking yourself these questions until you're sure you've extracted everything from the words you're given.

Step 4: Solve for the *wanted* element (often by using substitution).

- The ability to perform every step accurately and efficiently is critical to success on the GRE. Make sure to **answer the right question**. A good habit is to write on your paper—before you begin calculating—what the final question is asking. For instance, if the question asks for Mary's age in 5 years, write "M + 5 = ?" on your paper and circle it. In other words, don't let yourself forget the answer to "What do they want?"

Now that you've gone through the basic steps, it's time to practice translating Word Problems into equations. But first, here are some common mathematical relationships found on the GRE, words and phrases you might find used to describe them, and their translations. Use them to help you with the drill sets at the end of this chapter.

Common Word Problem Phrases

Addition

Add, Sum, Total (of parts), More Than: +

The sum of x and y: $x + y$

The sum of the three funds combined: $a + b + c$

When fifty is added to his age: $a + 50$

Six pounds heavier than Dave: $D + 6$

A group of men and women: $m + w$

The cost is marked up: $c + m$

Subtraction

Minus, Difference, Less Than: −

x minus five: $x - 5$

The difference between Quentin's and Rachel's heights (if Quentin is taller): $Q - R$

Four pounds less than expected: $e - 4$

The profit is the revenue minus the cost: $P = R - C$

Multiplication

The product of h and k: $h \times k$

The number of reds times the number of blues: $r \times b$

One-fifth of *y*: (1/5) × *y*

n persons have *x* beads each: total number of beads = *nx*

Go *z* miles per hour for *t* hours: distance = *zt* miles

Ratios and Division

Quotient, Per, Ratio, Proportion: ÷ or /

Five dollars every two weeks: (5 dollars/2 weeks) → 2.5 dollars a week

The ratio of *x* to *y*: *x*/*y*

The proportion of girls to boys: *g*/*b*

Average or Mean

(sum of terms divided by the total number of terms)

The average of *a* and *b*: $\frac{a+b}{2}$

The average salary of the three doctors: $\frac{x+y+z}{3}$

A student's average score on 5 tests was 87: $\frac{\text{sum}}{5} = 87$ or

$\text{Alternative: } \frac{Y}{Z} = \frac{X}{100}$

Translating Words Correctly

Avoid writing relationships backward.

If You See...		Write:		Not:
"*A* is half the size of *B*."	✔	$\frac{10}{22}$ of $\frac{5}{18}$	✘	$\frac{10}{22}$ of $\frac{5}{18}$
"*A* is 5 less than *B*."	✔	$A = B - 5$	✘	$A = 5 - B$
"*A* is less than *B*."	✔	$A < B$	✘	$A > B$
"Jane bought twice as many apples as bananas."	✔	$A = 2B$	✘	$2A = B$

Quickly check your translation with easy numbers.

For the last example in the table, you might think the following:

> "Jane bought twice as many apples as bananas. More apples than bananas. Say she buys five bananas. She buys twice as many apples—that's 10 apples. Makes sense. So the equation is Apples equals two times Bananas, or $A = 2B$, not the other way around."

These numbers do not have to satisfy any other conditions of the problem. Use these "quick picks" only to test the form of your translation.

Write an unknown percent as a variable divided by 100.

If You See...		Write		Not
"P is X percent of Q."	✔	$P = \frac{X}{100}Q$ or $\frac{P}{Q} = \frac{X}{100}$	✘	$P = X\%Q$ (Cannot be manipulated.)

Translate bulk discounts and similar relationships carefully.

If You See...		Write		N
"Pay $10 per CD for the first 2 CDs, then $7 per additional CD."	✔	$n = \#$ of CDs bought $T =$ total amount paid ($) $T = \$10 \times 2 + \$7 \times (n - 2)$ (assuming $n > 2$)	✘	

Always pay attention to the *meaning* of the sentence you are translating. If necessary, take a few extra seconds to make sure you've set up the algebra correctly.

Check Your Skills

Translate the following statements.

1. Lily is two years older than Melissa.

2. A small pizza costs $5 less than a large pizza.

3. Twice A is 5 more than B.

4. R is 45 percent of Q.

5. John has more than twice as many CDs as Ken.

Hidden Constraints

Notice that in some problems, there is a **hidden constraint** on the possible quantities. This would apply, for instance, to the number of apples and bananas that Jane bought. Because each fruit is a physical, countable object, you can only have a **whole number** of each type. Whole numbers are the integers 0, 1, 2, and so on. So you can have 1 apple, 2 apples, 3 apples, and so on, and even 0 apples, but you cannot have fractional apples or negative apples.

As a result of this implied "whole number" constraint, you often have more information than you might think, and you might be able to answer a question with fewer facts.

Consider the following example:

> If Kelly received 1/3 more votes than Micah in a student election, which of the following could have been the total number of votes cast for the two candidates?
>
> 12
> 13
> 14
> 15
> 16

Let M be the number of votes cast for Micah. Then Kelly received $M + (1/3)M$, or $(4/3)M$ votes. The total number of votes cast was therefore “votes for Micah” plus “votes for Kelly,” or $M + (4/3)M$. This quantity equals $(7/3)M$, or $7M/3$.

Because M is a number of votes, it cannot be a fraction—specifically, not a fraction with a 7 in the denominator. Therefore, no matter what M is, the 7 in the expression $7M/3$ will not be canceled out. As a result, the total number of votes cast must be a multiple of 7. Among the answer choices, the only multiple of 7 is 14, so the correct answer is (**C**).

Another way to solve this problem is this: the number of votes cast for Micah (M) must be a multiple of 3, since the total number of votes is a whole number. So $M = 3, 6, 9$, etc. Kelly received 1/3 more votes, so the number of votes she received is 4, 8, 12, etc. Thus, the total number of votes is 7, 14, 21, etc.

Not every unknown value related to “real-world” objects is restricted to whole numbers. Physical measurements such as weights, times, or speeds have to be positive numbers, but do not have to be integers. A few physical measurements can even be negative (e.g., temperatures, x- or y-coordinates). Think about what is being measured or counted, and you will recognize whether a hidden constraint applies.

Check Your Skills

Translate the following statements.

6. In a certain word, the number of consonants is 1/4 more than the number of vowels. Which of the following is a possibility for the number of letters in the word?

(A) 8
(B) 9
(C) 10
(D) 11
(E) 12

Check Your Skills Answer Key

1. $L = M + 2$
2. $S = L - 5$
3. $2A = B + 5$
4. $\frac{60}{13} \times 13 = 60$ or $R = 0.45\,Q$
5. $J > 2K$
6. **(B)**

   There is a hidden constraint in this question. The number of vowels and the number of consonants must both be integers. The number of consonants is 1/4 more than the number of vowels, which means you need to multiply the number of vowels by 1/4 to determine how many more consonants there are. If you label the number of vowels v, then there are $v/4$ more consonants than vowels. The only way that $v/4$ will be an integer is if v is a multiple of 4.

   If $v = 4$, there is $4/4 = 1$ more consonant than there are vowels, so there are $4 + 1 = 5$ consonants. That gives a total of $4 + 5 = 9$ letters in the word. The correct answer is (B).

Problem Set

Solve the following problems using the four-step method outlined in this section.

1. Johan is 20 years older than Brian. 12 years ago, Johan was twice as old as Brian. How old is Brian?
2. Mrs. Miller has two dogs, Jackie and Stella, who weigh a total of 75 pounds. If Stella weighs 15 pounds less than twice Jackie's weight, how much does Stella weigh?
3. Caleb spends $72.50 on 50 hamburgers for the marching band. If single burgers cost $1.00 each and double burgers cost $1.50 each, how many double burgers did he buy?
4. United Telephone charges a base rate of $10.00 for service, plus an additional charge of $0.25 per minute. Atlantic Call charges a base rate of $12.00 for service, plus an additional charge of $0.20 per minute. For what number of minutes would the bills for each telephone company be the same?

5. Carla cuts a 70-inch piece of ribbon into 2 pieces. If the first piece is 5 inches more than one-fourth as long as the second piece, how long is the longer piece of ribbon?

6. Jayla started babysitting when she was 18 years old. Whenever she babysat for a child, that child was no more than half her age at the time. Jayla is currently 32 years old, and she stopped babysitting 10 years ago. What is the current age of the oldest person for whom Jayla could have babysat?

7. Ten years ago, Brian was twice as old as Aubrey.

Quantity A	**Quantity B**
Twice Aubrey's age today	Brian's age today

8. The length of a rectangular room is 8 feet greater than its width. The total area of the room is 240 square feet.

Quantity A	**Quantity B**
The width of the room in feet	12

9. Jaden earns a yearly base salary of $30,000, plus a commission of $500 on every car he sells above his monthly minimum of two cars. Last year, Jaden met or surpassed his minimum sales every month, and earned a total income (salary plus commission) of $60,000.

Quantity A	**Quantity B**
The number of cars Jaden sold last year	90

Solutions

1. **32 years old**
   Brian’s age now = *b*
   Johan’s age now = *j*

   Translate the first sentence:

   (1) *j* = *b* + 20

   Translate and simplify the second sentence:

$$\begin{aligned}(2)(j-12) &= 2\,(b-12)\\ j-12 &= 2b-24\\ j &= 2b-12\end{aligned}$$

   The problem says to solve for *b*, so combine the two equations by substituting the value for *j* in equation (1) into equation (2) to eliminate *j* and solve for *b*.

$$\begin{aligned}-3x+7 &\leq 2x+32\\ -5x &\leq 25\\ x &>=-5\end{aligned}$$

2. **45 pounds**

Let j = Jackie's weight, and let s = Stella's weight. Stella's weight is the Ultimate Unknown: s = ?

(1) The two dogs weigh a total of 75 pounds:

$j + s = 75$

(2) Stella weighs 15 pounds less than twice Jackie's weight:

$s = 2j - 15$

Combine the two equations by substituting the value for s from equation (2) into equation (1):

$$\begin{aligned} j + (2j - 15) &= 75 \\ 3j - 15 &= 75 \\ 3j &= 90 \\ j &= 30 \end{aligned}$$

Find Stella's weight by substituting Jackie's weight into equation (1):

$$\begin{aligned} 30 + s &= 75 \\ s &= 45 \end{aligned}$$

3. **45 double burgers**
   Let s = the number of single burgers purchased.
   Let d = the number of double burgers purchased.

(1) Caleb bought 50 burgers:

$s + d = 50$

(2) Caleb spent $72.50 in all:

$s + 1.5d = 72.50$

Combine the two equations through substitution or by subtracting equation (1) from equation (2).

$$\begin{array}{rcl} s + 1.5d & = & 72.50 \\ -(s + \quad d & = & 50) \\ \hline 0.5d & = & 22.5 \\ d & = & 45 \end{array}$$

4. **40 minutes**

   Let *x* = the number of minutes.
   A call made by United Telephone costs $10.00 plus $0.25 per minute: 10 + 0.25*x*.
   A call made by Atlantic Call costs $12.00 plus $0.20 per minute: 12 + 0.20*x*.

   Set the expressions equal to each other:

$$\begin{aligned} 10 + 0.25x &= 12 + 0.20x \\ 0.05x &= 2 \\ x &= \frac{2}{0.05} = \frac{200}{5} = 40 \end{aligned}$$

5. **52 inches**

   Let *x* = the 1st piece of ribbon.
   Let *y* = the 2nd piece of ribbon.

(1) The ribbon is 70 inches	(2) The 1st piece is 5 inches more than 1/4 as long as

long:	the 2nd:
$x + y = 70$	$y = \frac{x}{2} + 9$

Combine the equations by substituting the value of *x* from equation (2) into equation (1):

$$\begin{aligned} 5 + \frac{y}{4} + y &= 70 \\ 20 + y + 4y &= 280 \\ 5y &= 260 \\ y &= 52 \end{aligned}$$

Now, because $x + y = 70$, $x = 18$. Thus, $x < y$, so *y* is the answer.

6. **23**
You are given actual ages for Jayla, therefore, the easiest way to solve the problem is to think about the extreme scenarios. At one extreme, 18-year-old Jayla could have babysat a child of age 9. Because Jayla is now 32, that child would now be 23. At the other extreme, 22-year-old Jane could have babysat a child of age 11. If Jayla is now 32, that child would be 21. You can see that the first scenario yields the oldest possible current age, 23, of a child that Jayla babysat.

7. **(A)**
Let *A* and *B* denote Aubrey and Brian’s ages today. Then, their ages 10 years ago would be given by $A - 10$ and $B - 10$, respectively. Those ages are related by the problem statement as:

$$B - 10 = 2(A - 10)$$

Expanding and simplifying yields:

$$\frac{7^6}{7^4} = 7^{6-4} = 7^2 = 49$$

Rewrite the quantities in terms of A and B. Twice Aubrey’s age today is $2A$ and Brian’s age today is B.

Ten years ago, Brian was twice as
old as Aubrey.

Quantity A	**Quantity B**
Twice Aubrey’s age today = ***2A***	Brian’s age today = ***B***

According to the equation, B is 10 less than $2A$. Therefore, **Quantity A is greater**.

8. **(C)**
Let L and W stand for the length and width of the room in feet. Then, from the first relation, you can write this equation:

$$(1)\ L = W + 8$$

Moreover, the area of a rectangle is given by length times width, such that:

$$(2)\ LW = 240$$

Taken together, you have two equations with two unknowns, and because the question involves the width rather than the length, you can eliminate the length by substituting from equation (1) into equation (2):

$$(W + 8)W = 240$$

Now expand the product and move everything to the left-hand side, so that you can solve the quadratic equation by factoring it. This gives:

$$\begin{aligned} W^2 + 8W &= 240 \\ W^2 + 8W - 240 &= 0 \\ (W + 20)(W - 12) &= 0 \end{aligned}$$

The two solutions are $W = -20$ and $W = 12$. A negative width does not make sense, so W must equal 12 feet.

It is also possible to arrive at the answer by testing the value in Quantity B as the width of the room. Plug in 12 for W in equation (1):

$$L = 12 + 8 = 20 \text{ feet}$$

If $W = 12$ and $L = 20$, then the area is (20)(12), which equals 240 square feet. Because this agrees with the given fact, you may conclude that 12 feet is indeed the width of the room.

Either method arrives at the conclusion. Therefore, **the two quantities are equal**.

9. **(B)**

   The simplest method for solving a problem like this is to work backwards from the value in Quantity B. Suppose Jaden sold exactly 90 cars. Then, because he met or surpassed his two car minimum each month (which adds up to 24 cars in the entire year), he would have sold another 90 − 24, which equals 66 cars above the minimum.

   The commission he earned on those cars is calculated as follows:

   $$\$500 \times 66 = \$33{,}000$$

   This would put his total yearly income at $30,000 (base salary) + $33,000 (commission), which sums to $63,000. However, you know that Jaden actually earned less than that; therefore, he must have sold fewer than 90 cars.

Quantity A	**Quantity B**
The number of cars Jaden sold last year = **less than 90**	90

   Therefore, **Quantity B is greater**.

   The alternative approach is to translate Jaden’s total earnings into an algebraic expression. Suppose Jaden sold *N* cars. Once again, noting that he met or surpassed his monthly minimum sales, you would need to subtract 24 cars that do not contribute to his bonus from this total, and then solve for *N* as follows:

$$\begin{aligned}
\$60{,}000 &= \$30{,}000 + \$500 \times \ (N - 24) \\
\$30{,}000 &= \$500 \times \ (N - 24) \\
60 &= N - 24 \\
84 &= N
\end{aligned}$$

Chapter 23

RATES & WORK

In This Chapter...

Basic Motion: The RTD Chart

Matching Units in the RTD Chart

Multiple Rates

Average Rate: Don’t Just Add and Divide

Basic Work Problems

Working Together: Add the Rates

Population Problems

Chapter 23

Rates & Work

One common type of Word Problem (WP) on the GRE is the Rate problem. Rate problems come in a variety of forms on the GRE, but all are marked by three primary components: *rate*, *time*, and *distance* or *work*.

These three elements are related by the following equations:

Rate × **Time** = **Distance** OR **Rate** × **Time** = **Work**

These equations can be abbreviated as $RT = D$ or as $RT = W$. Basic Rate problems involve simple manipulation of these equations.

This chapter will discuss the ways in which the GRE makes rate situations more complicated. Often, $RT = D$ problems will involve more than one person or vehicle traveling. Similarly, many $RT = W$ problems will involve more than one worker.

The first step is to review some fundamental properties of Rate problems.

Basic Motion: The RTD Chart

All basic motion problems involve three elements: rate, time, and distance.

Rate is expressed as a ratio of distance and time, with two corresponding units. Some examples of rates include: 30 miles per hour, 10 meters/second, 15 kilometers/day.

Time is expressed using a unit of time. Some examples of times include: 6 hours, 23 seconds, 5 months.

Distance is expressed using a unit of distance. Some examples of distances include: 18 miles, 20 meters, 100 kilometers.

You can make an "RTD chart" to solve a basic motion problem. Read the problem and fill in two of the variables. Then use the *RT* = *D* formula to find the missing variable. For example:

> If a car is traveling at 30 miles per hour, how long does it take to travel 75 miles?

	Rate (miles/hr)	×	*Time* (hr)	=	*Distance* (miles)
Car	30 mi/hr	×		=	75 mi

An RTD chart is shown here Fill in your RTD chart with the given information. Then solve for the time:

$30t = 75$, or $t = 2.5$ hours

Matching Units in the RTD Chart

All the units in your RTD chart must match up with one another. The two units in the rate should match up with the unit of time and the unit of distance. For example:

> It takes an elevator four seconds to go up one floor. How many floors will the elevator rise in two minutes?

The rate is 1 floor/4 seconds, which simplifies to 0.25 floors/second. Note: The rate is NOT 4 seconds per floor! This is an extremely frequent error. **Always express rates as "distance over time,"** not as "time over distance."

The time is 2 minutes. The distance is unknown.

Watch out! There is a problem with this RTD chart. The rate is expressed in floors per second, but the time is expressed in minutes. This will yield an incorrect answer.

	R (floors/sec)	×	*T* (min)	=	*W* (floors)
Elevator	0.25	×	2	=	?

To correct this table, change the time into seconds. Then all the units will match. To convert minutes to seconds, multiply 2 minutes by 60 seconds per minute, yielding 120 seconds.

	R (floors/sec)	×	*T* (sec)	=	*D* (floors)
Elevator	0.25	×	120	=	?

Once the time has been converted from 2 minutes to 120 seconds, the time unit will match the rate unit, and you can solve for the distance using the *RT* = *D* equation:

$$0.25(120) = d \qquad d = 30 \text{ floors}$$

Another example:

> A train travels 90 kilometers/hour. How many hours does it take the train to travel 450,000 meters? (1 kilometer = 1,000 meters)

First, divide 450,000 meters by 1,000 to convert this distance to 450 km. By doing so, you match the distance unit (kilometers) with the rate unit (kilometers per hour).

	R (km/hr)	×	*T* (hr)	=	*D* (km)
Train	90	×	t	=	450

You can now solve for the time: $90t = 450$. Thus, t is 5 hours. Note that this time is the "stopwatch" time: if you started a stopwatch at the start of the trip, what would the stopwatch read at the end of the trip? This is not what a clock on the wall would read, but if you take the *difference* of the start and end clock times (say, 1pm and 6pm), you will get the stopwatch time of 5 hours.

The RTD chart may seem like overkill for relatively simple problems such as these. In fact, for such problems, you can simply set up the equation $RT = D$ or $RT = W$ and then substitute. However, the RTD chart comes into its own when you have more complicated scenarios that contain more than one RTD relationship, as you'll see in the next section.

Check Your Skills

1. Convert 10 meters per second to meters per hour.

2. It takes an inlet pipe 2 minutes to supply 30 gallons of water to a pool. How many hours will it take to fill a 27,000 gallon pool that starts out empty?

Multiple Rates

Some rate questions on the GRE will involve *more than one trip or traveler*. To deal with this, you will need to deal with multiple $RT = D$ relationships. You can still use the RTD chart. Just add rows.

For example:

> Harvey runs a 30-mile course at a constant rate of 4 miles per hour. If Clyde runs the same track at a constant rate and completes the course in 90 fewer minutes, how fast did Clyde run?

An RTD chart for this question would have two rows: one for Harvey and one for Clyde.

	R (miles/hr)	×	*T* (hr)	=	*D* (miles)
Harvey					
Clyde					

To answer these questions correctly, you need to pay attention to the relationships between these two equations. By doing so, you can reduce the total number of variables you need and can solve for the desired value with the number of equations you have.

For instance, both Harvey and Clyde ran the same course, so the distance they both ran was 30 miles. Additionally, you know Clyde ran for 90 fewer minutes. To make units match, you can convert 90 minutes to 1.5 hours. If Harvey ran *t* hours, then Clyde ran (*t* − 1.5) hours:

	R (miles/hr)	×	*T* (hr)	=	*D* (miles)
Harvey	4		*t*		30
Clyde	*r*		*t* − 1.5		30

Now you can solve for *t*:

$$\begin{aligned} 4t &= 30 \\ t &= 7.5 \end{aligned}$$

If *t* = 7.5, then Clyde ran for 7.5 − 1.5 = 6 hours. You can now solve for Clyde's rate. Let *r* equal Clyde's rate:

$$\begin{aligned} r \times 6 &= 30 \\ r &= 5 \end{aligned}$$

For questions that involve multiple rates, remember to set up multiple *RT* = *D* equations and look for relationships between the equations. These relationships will help you reduce the number of variables you need and allow you to solve for the desired value.

Check Your Skills

3. One hour after Adrienne started walking the 60 miles from Town X to Town Y, James started walking from X to Y as well. Adrienne walks 3 miles per hour, and James walks 1 mile per hour faster than Adrienne. How far from X will James be when he catches up to Adrienne?

(A) 8 miles
(B) 9 miles
(C) 10 miles
(D) 11 miles
(E) 12 miles

4. Nicky and Cristina are running a 1,000 meter race. Because Cristina is faster than Nicky, she gives him a 12-second head start. If Cristina runs at a pace of 5 meters per second and Nicky runs at a pace of only 3 meters per second, how many seconds will Nicky have run before Cristina catches up to him?

(A) 15 seconds
(B) 18 seconds
(C) 25 seconds
(D) 30 seconds
(E) 45 seconds

Average Rate: Don’t Just Add and Divide

Consider the following problem:

> If Lucia walks to work at a rate of 4 miles per hour, but she walks home by the same route at a rate of 6 miles per hour, what is Lucia’s average walking rate for the round trip?

It is very tempting to find an average rate as you would find any other average: add and divide. Thus, you might say that Lucia’s average rate is 5 miles per hour (4 + 6 = 10 and 10 ÷ 2 = 5). However, this is INCORRECT!

If an object moves the same distance twice, but at different rates, then ***the average rate will NEVER be the average of the two rates given for the two legs of the journey***. In fact, because the object spends *more time* traveling at the slower rate, *the average rate will be closer to the slower of the two rates than to the faster*.

To find the average rate, you must first find the *total* combined time for the trips and the *total* combined distance for the trips.

First, you need a value for the distance. All you need to know to determine the average rate is the *total time* and *total distance*, so you can actually pick any number for the distance. The portion of the total distance represented by each part of the trip (“Going” and “Return”) will dictate the time.

Pick a Smart Number for the distance. Because you would like to choose a multiple of the two rates in the problem, 4 and 6, 12 is an ideal choice.

Set up a Multiple RTD chart:

	Rate (mi/hr)	×	*Time* (hr)	–	*Distance* (mi)
Going	4	×	t_1	=	12
Return	6	×	t_2	=	12
Total	r	×	t_3	=	24

The times can be found using the RTD equation. For the Going trip, $4t_1 = 12$, so t_1 is 3 hours. For the Return trip, $6t_2 = 12$, so t_2 is 2 hours. Thus, the total time is 5 hours. Now plug in these numbers:

	Rate (mi/hr)	×	*Time* (hr)	=	*Distance* (mi)
Going	4	×	**3**	=	12
Return	6	×	**2**	=	12
Total	r	×	**5**	=	24

Now that you have the total time and the total distance, you can find the average rate using the RTD equation:

$$\begin{aligned} RT &= D \\ r(5) &= 24 \\ r &= 4.8 \text{ miles per hour} \end{aligned}$$

Again, 4.8 miles per hour is *not* the simple average of 4 miles per hour and 6 miles per hour. In fact, it is the weighted average of the two rates, with the *times* as the weights. Because of that, the average rate is closer to the slower of the two rates.

You can test different numbers for the distance (try 24 or 36) to prove that you will get the same answer, regardless of the number you choose for the distance.

Check Your Skills

5. Juan bikes halfway to school at 9 miles per hour, and walks the rest of the distance at 3 miles per hour. What is Juan's average speed for the whole trip?

Basic Work Problems

Work problems are just another type of Rate problem. Instead of distances, however, these questions are concerned with the amount of “work” done.

Work takes the place of distance. Instead of *RT* = *D*, use the equation *RT* = *W*. The amount of work done is often a number of jobs completed or a number of items produced.

Time is the time spent working.

Rate expresses the amount of work done in a given amount of time. Rearrange the equation to isolate the rate:

$$\frac{4}{13} \text{ or } \frac{1}{3}$$

Be sure to express a rate as work per time (*W*/*T*), NOT time per work (*T*/*W*). For example, if a machine produces pencils at a constant rate of 120 pencils every 30 seconds, the rate at which the machine works is

$$\frac{120 \text{ pencils}}{30 \text{ seconds}} = 4 \text{ pencils}/\text{second}.$$

Many Work problems will require you to calculate a rate. Try the following problem:

> Martha can paint $\frac{1}{2}$ of a room in $1\frac{1}{4}$ hours. If Martha finishes painting the room at the same rate, how long will it have taken Martha to paint the room?

(A) $1\frac{1}{4}$ hours

(B) 9 hours

(C) $1\frac{1}{4}$ hours

(D) $-\frac{B}{A}$ hours

(E) $-\frac{B}{A}$ hours

Your first step in this problem is to calculate the rate at which Martha paints the room. You can say that painting the entire room is completing 1 unit of work. Set up an *RTW* chart:

	R (rooms/hr)	×	*T* (hr)	=	*W* (rooms)
Martha	*r*				

			$\frac{1}{2}$		$\frac{1}{2}$

Now solve for the rate:

$$r \times \frac{9}{2} = \frac{3}{7}$$
$$r = \frac{3}{7} \times \frac{2}{9} = \frac{2}{21}$$

The division would be messy, so leave it as a fraction. Martha paints $\frac{23}{7}$ of the room every hour. Now you have what you need to answer the question. Remember, painting the whole room is the same as doing 1 unit of work. Set up another *RTW* chart:

	R (rooms/hr)	×	*T* (hr)	=	*W* (rooms)
Martha	$\frac{23}{7}$		*t*		1

Now solve for the time:

$$\left(\frac{2}{21}\right)t = 1$$
$$t = \frac{21}{2} = 10\,\frac{1}{2}$$

The correct answer is (D). Notice that the rate and the time in this case were reciprocals of each other. This will always be true when the amount of work done is 1 unit (because reciprocals are defined as having a product of 1).

Working Together: Add the Rates

More often than not, Work problems will involve more than one worker. When two or more workers are performing the same task, their rates can be added together. For instance, if Machine A can make 5 boxes in an hour, and Machine B can make 12 boxes in an hour, then working together the two machines can make 5 + 12 = 17 boxes per hour.

Likewise, if Lucas can complete 1/3 of a task in an hour and Serena can complete 1/2 of that task in an hour, then working together they can complete 1/3 + 1/2 = 5/6 of the task every hour.

If, on the other hand, one worker is undoing the work of the other, subtract the rates. For instance, if one hose is filling a pool at a rate of 3 gallons per minute, and another hose is draining the pool at a rate of 1 gallon per minute, the pool is being filled at a rate of 3 − 1 = 2 gallons per minute.

Try the following problem:

> Machine A fills soda bottles at a constant rate of 60 bottles every 12 minutes and Machine B fills soda bottles at a constant rate of 120 bottles every 8 minutes. How many bottles can both machines working together at their respective rates fill in 25 minutes?

To answer these questions quickly and accurately, it is a good idea to begin by expressing rates in equivalent units:

$$\text{Rate}_{\text{MachineA}} = \frac{60 \text{ bottles}}{12 \text{ minutes}} = 5 \text{ bottles/minute}$$

$$\text{Rate}_{\text{MachineB}} = \frac{120 \text{ bottles}}{8 \text{ minutes}} = 15 \text{ bottles/minute}$$

That means that working together they fill 5 + 15 = 20 bottles every minute. Now you can fill out an RTW chart. Let *b* be the number of bottles filled:

	R (bottles/min)	×	T (min)	=	W (bottles)
$A + B$	20		25		b

Now solve for the *b*:

$b = 20 \times 25 = 500$ bottles

Remember that, even as Work problems become more complex, there are still only a few relevant relationships: $RT = W$ and $R_A + R_B = R_{A+B}$.

> Alejandro, working alone, can build a doghouse in 4 hours. Betty can build the same doghouse in 3 hours. If Betty and Carmelo, working together, can build the doghouse twice as fast as Alejandro, how long would it take Carmelo, working alone, to build the doghouse?

Begin by solving for the rate that each person works. Let *c* represent the number of hours it takes Carmelo to build the doghouse.

Alejandro can build $\frac{1}{2}$ of the doghouse every hour, Betty can build $\frac{1}{2}$ of the doghouse every hour, and Carmelo can build $\frac{1}{2}$ of the doghouse every hour.

The problem states that Betty and Carmelo, working together, can work twice as fast as Alejandro. That means that their rate is twice Alejandro's rate:

$$\begin{aligned}\text{Rate}_B + \text{Rate}_C &= 2\left(\text{Rate}_A\right)\\ \frac{1}{3}+\frac{1}{c} &= 2\left(\frac{1}{4}\right)\\ \frac{1}{c} &= \frac{1}{2}-\frac{1}{3}=\frac{1}{6}\\ c &= 6\end{aligned}$$

It takes Carmelo 6 hours working by himself to build the doghouse.

When dealing with multiple rates, be sure to express rates in equivalent units. When the the work involves completing a task, remember to treat completing the task as doing one "unit" of work. Once you know the rates of every worker, add the rates of workers who work together on a task.

Check Your Skills

6. Sophie can address 20 envelopes in one hour. How long will it take her to address 50 envelopes?

7. If a steel mill can produce 1,500 feet of I-beams every 20 minutes, how many feet of I-beams can it produce in 50 minutes?

8. Tarik can complete a job in 12 minutes. If Andy helps Tarik, they can complete the job in 4 minutes. How long would it take for Andy to complete the job on his own?

Population Problems

The final type of Rate problem on the GRE is the Population problem. In such problems, some population typically increases *by a common factor* every time period. These can be solved with a Population chart.

Consider the following example:

> The population of a certain type of bacterium triples every 10 minutes. If the population of a colony 20 minutes ago was 100, in approximately how many minutes from now will the bacteria population reach 24,000?

You can solve simple Population problems, such as this one, by using a Population chart. Make a table with a few rows, labeling one of the middle rows as “NOW.” Work forward, backward, or both (as necessary in the problem), obeying any conditions given in the problem statement about the rate of growth or decay. In this case, simply triple each population number as you move down a row. Notice that while the population increases by a constant *factor,* it does *not* increase by a constant *amount* each time period.

For this problem, the Population chart below shows that the bacterial population will reach 24,000 about 30 minutes from now.

In some cases, you might pick a Smart Number for a starting point in your Population chart. If you do so, pick a number that makes the computations as simple as possible.

Time Elapsed	Population
20 minutes ago	100
10 minutes ago	300
NOW	900
in 10 minutes	2,700
in 20 minutes	8,100
in 30 minutes	24,300

Check Your Skills

9. The population of amoebas in a colony doubles every two days. If there were 200 amoebas in the colony six days ago, how many amoebas will there be four days from now?

Check Your Skills Answer Key

1. **36,000 meters/hour**

   First convert seconds to minutes. There are 60 seconds in a minute, so 10 m/sec × 60 sec/min = 600 m/min.

   Now convert minutes to hours. There are 60 minutes in 1 hour, so 600 m/min × 60 min/hr = 36,000 m/hr.

2. **30 hours**

   First simplify the rate: $R = \frac{30\,\text{gal}}{2\,\text{min}} = \frac{15\,\text{gal}}{1\ \text{min}}$, which is the same as 15 gal/min. The question asks for the number of hours it will take to fill the pool, so convert minutes to hours. There are 60 minutes in an hour, so the rate is 15 gal/min × 60 = 900 gal/hr. Now you can set up an RTW chart. Let *t* be the time it takes to fill the pool:

	R (gal/hr)	×	*T* (hr)	=	*W* (gallons)
inlet pipe	900	×	*t*	=	27,000

   Now solve for *t*:

$$\begin{aligned} 900t &= 27,000 \\ t &= 30\ \text{hours} \end{aligned}$$

3. **(E)**

Organize this information in an RTD chart as follows:

	R (mi/hr)	×	T (hr)	=	D (mi)
Adrienne	3	×	$t + 1$	=	d
James	4	×	t	=	d
Total	—		—		$2d$

Set up algebraic equations to relate the information in the chart, using the $RT = D$ equation.

Adrienne: $3(t + 1) = d$

James: $4t = \mathrm{d}$

Notice that the *d* here is not 60 miles, since neither Adrienne nor James have walked all the way from X to Y. Rather, *d* represents how far each person has walked when James catches up to Adrienne. At that point in time, they will have both walked the same distance from Town X *toward* Town Y.

Substitute 4*t* for *d* in the first equation:

$$\begin{aligned} 3(t+1) &= 4t \\ 3t+3 &= 4t \\ t &= 3 \end{aligned}$$

Therefore, $d = 4(3)$, which equals 12 miles.

4. **(D)**

Fill in the RTD chart. Nicky starts 12 seconds before Cristina, so Nicky's time is $t + 12$.

	R (m/s)	×	T (second)	=	D (meter)
Cristina	5	×	t	=	$5t$
Nicky	3	×	$t + 12$	=	$3(t + 12)$

Write expressions for the total distance, and then set these two distances equal to each other.

$$\begin{aligned}
&\text{Cristina:} && 5t = \text{distance} \\
&\text{Nicky:} && 3(t + 12) = \text{distance} \\
&\text{Combine:} && 5t = 3(t + 12) \\
& && 5t = 3t + 36 \\
& && 2t = 36 \\
& && t = 18
\end{aligned}$$

Therefore, Nicky will have run for 18 + 12 = 30 seconds before Cristina catches up to him.

5. **4.5 mph**

Assume a Smart Number for the distance to school. The Smart Number should be divisible by 9 and 3. The simplest choice is 18 miles for this distance. Now solve:

Time biking: $T = D/R = 9/9 = 1$ hr

Time walking: $T = D/R = 9/3 = 3$ hr
Total time: 1 hr + 3 hr = 4 hr

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{18 \text{ miles}}{4 \text{ hours}} = 4.5 \text{ mph}$$

6. **2.5 hours**
   If Sophie addresses 20 envelopes in 1 hour, then the rate at which she addresses is 20 envelopes/hr. Set up an *RTW* equation:

   20 envelopes/hr × T = 50 envelopes

   $T = 50/20 = 2.5$ hr

7. **3,750 feet**
   The rate at which the steel mill produces I-beams is $\frac{\textbf{Part}}{\textbf{Whole}} = \frac{\textbf{Percent}}{\textbf{100}}$ Next, set up an *RTW* equation. Let w represent the number of feet of I-beam produced:

$$\begin{aligned} 75 \text{ ft/min} \times 50 \text{ min} &= w \\ 3{,}750 \text{ ft } &= w \end{aligned}$$

8. **6 minutes**
   Always express work rates as jobs per unit of time. Remember that the combined rates for Tarik and Andy are additive:

$$\underbrace{\frac{1\text{ job}}{12\text{ minutes}}}_{\text{Tarik}} + \underbrace{\frac{1\text{ job}}{a\text{ minutes}}}_{\text{Andy}} = \frac{1\text{ job}}{4\text{ minutes}}$$

$$\frac{1}{12} + \frac{1}{a} = \frac{1}{4}$$

$$\frac{1}{a} = \frac{1}{4} - \frac{1}{12}$$

$$\frac{1}{a} = \frac{3}{12} - \frac{1}{12} = \frac{1}{6} \qquad a = 6\,\text{minutes}$$

9. **6,400**

Time	**Population**
6 days ago	200
4 days ago *(Careful! Count by two days.)*	400
2 days ago	800
NOW	1,600
2 days from now	3,200
4 days from now	6,400

Problem Set

Solve the following problems using the strategies you have learned in this section. Use RTD or RTW charts as appropriate to organize information.

1. A cat travels at a speed of 60 inches/second. How long will it take this cat to travel 300 feet? (12 inches = 1 foot)

2. Water is being poured into a tank at the rate of approximately 4 cubic feet per hour. If the tank is 6 feet long, 4 feet wide, and 8 feet deep, how many hours will it take to fill up the tank?

3. The population of grasshoppers doubles in a particular field every year. Approximately, how many years will it take the population to grow from 2,000 grasshoppers to 1,000,000 or more?

4. An empty bucket being filled with paint at a constant rate takes 6 minutes to be filled to 7/10 of its capacity. How much

more time will it take to fill the bucket to full capacity?

5. Four years from now, the population of a colony of bees will reach 1.6×10^8. If the population of the colony doubles every 2 years, what was the population 4 years ago?

6. The Technotronic can produce 5 bad songs per hour. Wanting to produce bad songs more quickly, the record label also buys a Wonder Wheel, which works as fast as the Technotronic. Working together, how many bad songs can the two produce in 72 minutes?

7. Jack is putting together gift boxes at a rate of 3 per hour in the first hour. Then Jill comes over and yells, “Work faster!” Jack, now nervous, works at the rate of only 2 gift boxes per hour for the next 2 hours. Then Alexandra comes to Jack and whispers, “The steadiest hand is capable of the divine.” Jack, calmer, then puts together 5 gift boxes in the fourth hour. What is the average rate at which Jack puts together gift boxes over the entire period?

8. A bullet train leaves Kyoto for Tokyo traveling 240 miles per hour at 12 noon. Ten minutes later, a train leaves Tokyo for

Kyoto traveling 160 miles per hour. If Tokyo and Kyoto are 300 miles apart, at what time will the trains pass each other?

(A) 12:40pm
(B) 12:49pm
(C) 12:55pm
(D) 1:00pm
(E) 1:05pm

9. Andrew drove from A to B at 60 miles per hour. Then he realized that he forgot something at A, and drove back at 80 miles per hour. He then zipped back to B at 90 mph. What was his approximate average speed in miles per hour for the entire trip?

10. A car travels from Town A to Town B at an average speed of 40 miles per hour, and returns immediately along the same route at an average speed of 50 miles per hour. What is the average speed in miles per hour for the round-trip?

11. Two hoses are pouring water into an empty pool. Hose 1 alone would fill up the pool in 6 hours. Hose 2 alone would fill up the pool in 4 hours. How long would it take for both hoses to fill up two-thirds of the pool?

12. Aimee takes 6 minutes to pack a box and Brianna takes 5 minutes to pack a box. How many hours will it take them to pack 110 boxes?

13. Hector can solve one word problem every 4 minutes before noon, and one word problem every 10 minutes after noon.

Quantity A	**Quantity B**
The number of word problems Hector can solve between 11:40am and noon	The number of word problems Hector can solve between noon and 12:40pm

14. The number of users (non-zero) of a social networking website doubles every 4 months.

Quantity A	**Quantity B**
Ten times the number of users one year ago	The number of users today

15. A bullet train can cover the 420 kilometers between Xenia and York at a rate of 240 kilometers per hour.

Quantity A	**Quantity B**
The number of minutes it will take the train to travel from Xenia	110

to York

Solutions

1. **1 minute**

   This is a simple application of the *RT* = *D* equation, involving one unit conversion. First convert the rate, 60 inches/second, into 5 feet/second (given that 12 inches = 1 foot). Substitute this value for *R*. Substitute the distance, 300 feet, for *D*. Then solve:

R (ft/sec)	×	*T* (sec)	=	*D* (ft)
5	×	*t*	=	300

$$(5\,\text{ft}/\text{s})(t) = 300\,\text{ft}$$
$$t = \frac{300\,\text{ft}}{5\,\text{ft}/\text{s}} = 60\,\text{seconds} = 1\,\text{minute}$$

2. **48 hours**

   The capacity of the tank is 6 × 4 × 8, or 192 cubic feet. Use the *RT* = *W* equation, substituting the rate, 4 ft^3/hour, for *R*, and the capacity, 192 cubic feet, for *W*:

R (ft^3/hr)	×	*T* (hr)	=	*W* (ft)
4	×	*t*	=	192

$$(4\text{ cubic feet}/\text{hr})(t) = 192\text{ cubic feet}$$
$$t = \frac{192\,\text{cubic feet}}{4\,\text{cubic feet}/\text{hr}} = 48\,\text{hours}$$

3. **9 years**

Organize the information given in a population chart. Notice that because the population is increasing exponentially, it does not take very long for the population to top 1,000,000.

Time Elapsed	Population
NOW	2,000
1 year	4,000
2 years	8,000
3 years	16,000
4 years	32,000
5 years	64,000
6 years	128,000
7 years	256,000
8 years	512,000
9 years	1,024,000

4. $1\frac{1}{4}$ minutes

Use the *RT* = *W* equation to solve for the rate, with *t* = 6 minutes and *w* = 7/10:

R (bkt/min)	×	*T* (min)	=	*W* (bucket)
r	×	6	=	7/10

$$r(6\text{minutes}) = 7/10$$
$$r = \frac{7}{10} \div 6 = \frac{7}{10} \times \frac{1}{6} = \frac{7}{60}\ \text{buckets per minute}$$

Then, substitute this rate into the equation again, using 3/10 for *w* (the remaining work to be done):

R Regular	×	*T* Regular	=	*W* Regular
7/60	×	*t*	=	3/10

$$\left(\frac{7}{60}\right)t = \frac{3}{10}$$
$$t = \frac{3}{10} \div \frac{7}{60} = \frac{3}{10} \times \frac{60}{7} = \frac{18}{7} = 2\,\frac{4}{7}\ \text{minutes}$$

5. **1 × 10^7**

Organize the information given in a population chart.

Time Elapsed	Population
4 years ago	0.1×10^8
2 years ago	0.2×10^8
NOW	0.4×10^8
in 2 years	0.8×10^8
in 4 years	1.6×10^8

Then, convert:

$0.1 \times 10^8 = 10{,}000{,}000 = 1 \times 10^7$ bees

6. **12 songs**

   This is a “working together” problem, so add the individual rates: 5 + 5 = 10 songs per hour.

   The two machines together can produce 10 bad songs in 1 hour. Convert the given time into hours:

$$(72 \text{ minutes})\left(\frac{1 \text{ hour}}{60 \text{ minutes}}\right) = \frac{72}{60} = 1.2 \text{ hours}$$

   Then, use the *RT* = *W* equation to find the total work done:

R (songs/hr)	×	*T* (hr)	=	*W* (songs)
10	×	1.2	=	*w*

   $(10)(1.2 \text{ hours}) = w$
   $w = 12$ bad songs

7. **3 boxes per hour**

   The average rate is equal to the total work done divided by the time in which the work was done. Remember that you cannot simply average the rates. You must find the total work and total time. The total time is 4 hours. To find the total work, add up the boxes Jack put together in each hour: 3 + 2 + 2 + 5 = 12. Therefore, the average rate is $\frac{23}{7}$, or 3 boxes per hour. The completed chart looks like this:

	R (box/hr)	×	*T* (hr)	=	*W* (box)

Phase 1	3	×	1	=	3
Phase 2	2	×	2	=	4
Phase 3	5	×	1	=	5
Total	3 = 12/4		4 Sum		12 Sum

8. **(B)**

	R (mi/hr)	×	*T* (hr)	=	*D* (mi)
Train K to T	240	×	$t + 1/6$	=	$240(t + 1/6)$
Train T to K	160	×	t	=	$160t$
Total	—		—		300

Solve this problem by filling in the RTD chart. Note that the train going from Kyoto to Tokyo leaves first, so its time is longer by 10 minutes, which is 1/6 hour.

Next, write the expressions for the distance that each train travels, in terms of t. Now, sum those distances and set that total equal to 300 miles:

$$\begin{aligned}
240\left(t+\frac{1}{6}\right)+160t &= 300\\
240t+40+160t &= 300\\
400t &= 260\\
20t &= 13\\
t &= \frac{13}{20}\text{ hour} = \frac{39}{60}\text{ hour} = 39\text{ minutes}
\end{aligned}$$

The first train leaves at 12 noon. The second train leaves at 12:10pm. Thirty-nine minutes after the second train has left, at 12:49pm, the trains pass each other.

9. **Approximately 74.5 mph**
Use a Multiple RTD chart to solve this problem. Start by selecting a Smart Number for *d*, such as 720 miles. (This is a common multiple of the 3 rates in the problem.) Then work backward to find the time for each trip and the total time:

$$\begin{aligned}
&t_A = \frac{720}{60} = 12\,\text{hrs}\\
&t_B = \frac{720}{80} = 9\,\text{hrs}\\
&t_C = \frac{720}{90} = 8\,\text{hrs}\\
&t = 12+9+8 = 29\,\text{hours}\\
&\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{2{,}160}{29} \approx 74.5\text{ miles per hour}
\end{aligned}$$

	R (mi/hr)	×	T (hr)	=	D (mi)
A to B	60	×	t_A	=	720

B to A	80	×	t_B	=	720
A to B	90	×	t_C	=	720
Total	—		t		2,160

10. $-\frac{B}{A}$ miles per hour

Use a Multiple RTD chart to solve this problem. Start by selecting a Smart Number for *d*, such as 200 miles. (This is a common multiple of the two rates in the problem.) Then work backward to find the time for each trip and the total time:

$$t_1 = \frac{200}{40} = 5 \text{ hrs} \qquad t_2 = \frac{200}{50} = 4 \text{ hrs} \qquad t = 4 + 5 = 9$$

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{400}{9} = 44\frac{4}{9} \text{ miles per hour}$$

Do NOT simply average 40 miles per hour and 50 miles per hour to get 45 miles per hour. The fact that the right answer is very close to this wrong result makes this error especially pernicious: avoid it!

11. $\frac{1}{10}, 2\frac{16}{17}$

If Hose 1 can fill the pool in 6 hours, its rate is 1/6 pool per hour, or the fraction of the job it can do in 1 hour. Likewise, if Hose 2 can fill the pool in 4 hours, its rate is 1/4 pool per hour. Therefore, the combined rate is 5/12 pool per hour (1/4 + 1/6 = 5/12).

$$RT = W$$

$$(5/12)t = 2/3$$

$$t = \frac{2}{\underset{1}{\cancel{3}}} \times \frac{\overset{4}{\cancel{12}}}{5} = \frac{8}{5} = 1\,\frac{3}{5}$$

	R (pool/hr)	×	*T* (hr)	=	*W* (pool)
	5/12	×	*t*	=	2/3

12. **5 hours**

    Working together, Aimee and Brianna pack $\frac{1}{4} \times \frac{3}{3} = \frac{3}{12}$ boxes per minute. Next use a proportion:

$$\frac{11\,\text{boxes}}{30\,\text{minutes}} = \frac{110\,\text{boxes}}{x\,\text{minutes}}$$

$$x = \frac{(110)(30)}{(11)} = 300\,\text{minutes, or 5 hours}$$

13. **(A)**

    This problem can be solved using an RTW chart or by a proportion. There are 20 minutes between 11:40am and noon, and 40 minutes between noon and 12:40pm. Hector’s work rate is different for the two time periods. For the work period before noon, this is the proportion:

Let *b* represent the number of problems Hector solves *before* noon:

$$\begin{aligned} \frac{b}{20\text{ min}} &= \frac{1\text{ problem}}{4\text{ min}} \\ 4b &= 20 \\ b &= 5 \end{aligned}$$

Let *a* represent the number of problems Hector solves *after* noon:

$$\begin{aligned} \frac{b}{20\text{ min}} &= \frac{1\text{ problem}}{4\text{ min}} \\ 4b &= 20 \\ b &= 5 \end{aligned}$$

Rewrite the quantities:

Quantity A	**Quantity B**
The number of word problems Hector can solve between 11:40am and noon = 5	The number of word problems Hector can solve between noon and 12:40pm = 4

Therefore, **Quantity A is greater**.

14. **(A)**

Set up a Population chart, letting *X* denote the number of users one year ago:

Time	Number of Users

Time	Number of Users
12 months ago	X
8 months ago	$2X$
4 months ago	$4X$
NOW	$8X$

Ten times the number of users one year ago is $10X$, while the number of users today is $8X$. Rewrite the quantities:

Quantity A	**Quantity B**
Ten times the number of users one year ago = $10X$	The number of users today = $8X$

Therefore, $10X$ is greater than $8X$ because X must be a positive number. Thus, **Quantity A is greater**.

15. **(B)**

You can use the rate equation to solve for the time it will take the train to cover the distance. Your answer will be in hours because the given rate is in kilometers per hour. Let t stand for the total time of the trip:

$$R \times T = D$$
$$(240) \times t = (420)$$
$$t = \frac{420}{240} = \frac{7}{4}$$

(Note that you can omit the units in your calculation if you verify ahead of time that you are dealing with a consistent system of units.) Finally, convert the time from hours into minutes:

$$\frac{7}{4} \times 60 = \frac{7}{\underset{1}{\cancel{4}}} \times \overset{15}{\cancel{60}} = 105\,\text{minutes}$$

Rewrite the quantities:

Quantity A	**Quantity B**
The number of minutes it will take the train to travel from Xenia to York	110

An efficient way to solve this problem is to use the value in Quantity B to “cheat.” Assume the train traveled for 110 minutes. Convert 110 minutes to hours:

$$\frac{110}{60} = \frac{11}{6}\ \text{hours}$$

Now multiply the time ($\frac{11}{6}$ hours) by the rate (240 kilometers per hour) to calculate the distance:

$$D = \frac{11}{6} \times 240 = \frac{11}{\underset{1}{\cancel{6}}} \times \overset{40}{\cancel{240}} = 440\,\text{kilometers}$$

The train can travel 440 kilometers in 110 minutes, but the distance between the cities is 420 kilometers. Therefore, the train must have traveled less than 110 minutes to reach its destination. Thus, **Quantity B is greater**.

Chapter 24

RATIOS

In This Chapter…

Label Each Part of the Ratio with Units
Proportions
The Unknown Multiplier
Multiple Ratios: Make a Common Term

Chapter 24

Ratios

A ratio expresses a particular relationship between two or more quantities. Here are some examples of ratios:

> The two partners spend time working in the ratio of 1 to 3. (For every 1 hour the first partner works, the second partner works 3 hours.)
>
> Three sisters invest in a certain stock in the ratio of 2 to 3 to 8. (For every \$2 the first sister invests, the second sister invests \$3, and the third sister invests \$8.)
>
> The ratio of men to women in the room is 3 to 4. (For every 3 men, there are 4 women.)

Here are some key points about ratios:

Ratios can be expressed in a few different ways:

Using the word “to”, as in 3 to 4
Using a colon, as in 3 : 4

By writing a fraction, as in $\frac{1}{2}$ (note that this only works for ratios of exactly *two* quantities)

Ratios can express a part–part relationship or a part–whole relationship:

A part–part relationship: The ratio of men to women in the office is 3 : 4.
A part–whole relationship: There are 3 men for every 7 employees.

Notice that if there are only two parts in the whole, you can figure out a part–whole ratio from a part–part ratio, and vice versa.

The relationship that ratios express is division:

If the ratio of men to women in the office is 3 : 4, then the number of men divided by the number of women equals $\frac{1}{2}$ or 0.75.

So you can often write: or $\mathrm{Ratio} = \frac{\mathrm{Part}}{\mathrm{Whole}}$.

Remember that ratios only express a *relationship* between two or more items; they do not provide enough information, on their own, to determine the exact quantity for each item. For example, knowing that the ratio of men to women in an office is 3 to 4 does NOT tell you exactly how many men and how many women are in the office. All you know is that the number of men is $\frac{1}{2}$ the number of women.

This idea is also frequently tested in Data Interpretation questions, which often contain charts that show part–part and part–whole ratios.

If two quantities have a constant ratio, they are directly proportional to each other. For example:

> If the ratio of men to women in the office is 3 : 4, then $\frac{\#\text{ of men}}{\#\text{ of women}} = \frac{3}{4}$.

If the number of men is directly proportional to the number of women, then the number of men divided by the number of women is always the same. So if the ratio of men to women is 3 : 4, there could be 3 men and 4 women, 9 men and 12 women, or even 600 men and 800 women, but there could not be 4 men and 3 women because then the number of men divided by the number of women would NOT equal $\frac{1}{2}$.

Label Each Part of the Ratio with Units

The order in which a ratio is given is vital. For example, “the ratio of dogs to cats is 2 : 3” is very different from “the ratio of dogs to cats is 3 : 2.” The first ratio says that for every 2 dogs, there are 3 cats. The second ratio says that for every 3 dogs, there are 2 cats.

It is very easy to accidentally reverse the order of a ratio—especially on a timed test like the GRE. Therefore, to avoid these reversals, always write units on either the ratio itself or on the variables you create, or both.

Thus, if the ratio of dogs to cats is 2 : 3, you can write $\frac{x\,\text{dogs}}{y\,\text{cats}} = \frac{2\,\text{dogs}}{3\,\text{cats}}$, or simply $\frac{x\,\text{dogs}}{y\,\text{cats}} = \frac{2}{3}$, or even $\frac{D}{C} = \frac{2\,\text{dogs}}{3\,\text{cats}}$, where *D* and *C* are variables standing for the number of dogs and cats, respectively.

However, do not just write $\frac{7}{10{,}000}$. You could easily forget which variable stands for cats and which for dogs.

Also, NEVER write $\frac{2d}{3c}$. The reason is that you might think that *d* and *c* stand for *variables*—that is, numbers in their own right. Always write out the full unit.

Proportions

Simple Ratio problems can be solved with a proportion. For example:

> The ratio of girls to boys in the class is 4 to 7. If there are 35 boys in the class, how many girls are there?

Step 1: Set up a labeled proportion:

$$\frac{4\,\text{girls}}{7\,\text{boys}} = \frac{x\,\text{girls}}{35\,\text{boys}}$$

Step 2: Cross-multiply to solve:

$$\begin{aligned} 7x &= 4 \times 35 \\ x &= \frac{4 \times 35}{7} = 20 \end{aligned}$$

Remember that you can cancel fractions to simplify calculations as you go ($\frac{23}{7} = \frac{1}{2}$), although you can usually avoid this extra step by using the calculator. For example:

$$\frac{4\,\text{girls}}{7\,\text{boys}} = \frac{x\,\text{girls}}{35\,\text{boys}} \qquad \frac{4\,\text{girls}}{\cancel{7}\,1\,\text{boy}} = \frac{x\,\text{girls}}{\cancel{35}\,5\,\text{boys}} \qquad \frac{4}{1} = \frac{x}{5} \qquad x = 20$$

Note: Never cancel factors diagonally across an equals sign. That would change the values incorrectly.

Check Your Skills

1. The ratio of apples to oranges in a fruit basket is 3 : 5. If there are 15 apples, how many oranges are there?

2. Miki has 7 jazz CDs for every 12 classical CDs in his collection. If he has 60 classical CDs, how many jazz CDs does he have?

The Unknown Multiplier

For more complicated Ratio problems, in which the total of all items is given, the “Unknown Multiplier” technique is useful:

> The ratio of men to women in a room is 3 : 4. If there are 56 people in the room, how many of the people are men?

Using the methods from the previous page, you can write the ratio relationship as $\frac{M \text{ men}}{W \text{ women}} = \frac{3}{4}$.

Together with $M + W = 56$, you can solve for M (and W, for that matter). The standard algebra techniques used for solving this kind of “two equations and two unknowns” problem are substitution and elimination, which is noted in the Algebraic Translations chapter of this book.

However, there is even an easier way. It requires a slight shift in your thinking, but if you can make this shift, *you can save yourself a lot of work on some problems*. Instead of representing the number of men as M, represent it as $3x$, where x is some unknown (positive) number. Likewise, instead of representing the number of women as W, represent it as $4x$, where x is the same unknown number. In this case (as in many others), x has to be a whole number. This is another example of a *hidden constraint*.

What does this seemingly odd step accomplish? It guarantees that the ratio of men to women is 3 : 4. The ratio of men to women can now be expressed as $3\frac{1}{2}$, which reduces to $\frac{1}{2}$, the desired ratio. (Note that you can cancel the *x*'s because you know that *x* is not zero.) This variable *x* is known as the **Unknown Multiplier**. The Unknown Multiplier allows you to reduce the number of variables, making the algebra easier.

Now determine the value of the Unknown Multiplier using the other equation:

$$\begin{aligned} \text{Men} + \text{Women} &= 56 \\ 3x + 4x &= 56 \\ 7x &= 56 \\ x &= 8 \end{aligned}$$

Now you know that the value of *x*, the Unknown Multiplier, is 8. Therefore, you can determine the exact number of men and women in the room:

The number of men equals: $3x = 3(8) = 24$. The number of women equals: $4x = 4(8) = 32$.

When *should* you use the Unknown Multiplier? You should use it when (1) the *total* items is given, or (2) neither quantity in the ratio is already equal to a number or a variable expression. Generally, the first ratio in a problem can be set up with an Unknown Multiplier. In the "girls and boys" problem on the previous page, however, you can glance ahead and see that the number of boys is given as 35. This means that you can just set up a simple proportion to solve the problem.

The Unknown Multiplier is particularly useful with three-part ratios:

> A recipe calls for amounts of lemon juice, wine, and water in the ratio of 2 : 5 : 7. If all three combined yield 35 milliliters of liquid, how much wine was included?

Make a quick table:	Lemon Juice	+	Wine	+	Water	=	Total
	$2x$	+	$5x$	+	$7x$	=	$14x$

Now solve for x: $14x = 35$, or $x = 2.5$. Thus, the amount of wine is $5x = 5(2.5) = 12.5$ milliliters.

In this problem, the Unknown Multiplier turns out not to be an integer. This result is fine, because the problem deals with continuous quantities (milliliters of liquids). In problems like the first one, which deals with integer quantities (men and women), the Unknown Multipier must be a positive integer. In that specific problem, the multiplier is literally the number of “complete sets” of three men and four women each.

You can only use the Unknown Multiplier only once per problem to solve though. So if the dogs to cats is 2 : 3 and cats to mice is 5 : 4, you shouldn’t write $2x : 3x$ and $5y : 4y$. Instead, you need to make a common term.

Check Your Skills

3. The ratio of apples to oranges in a fruit basket is 3 : 5. If there are a total of 48 pieces of fruit, how many oranges are there?

4. Steve has nuts, bolts, and washers in the ratio 5 : 4 : 6. If he has a total of 180 pieces of hardware, how many bolts does he have?

5. A dry mixture consists of 3 cups of flour for every 2 cups of sugar. How much sugar is in 4 cups of the mixture?

Multiple Ratios: Make a Common Term

You may encounter two ratios containing a common element. To combine the ratios, you can use a process similar to creating a common denominator for fractions.

Because ratios act like fractions, you can multiply both sides of a ratio (or all sides, if there are more than two) by the same number, just as you can multiply the numerator and denominator of a fraction by the same number. You can change *fractions* to have common *denominators*. Likewise, you can change *ratios* to have common *terms* corresponding to the same quantity. Consider the following problem:

> In a box containing action figures of the three Fates from Greek mythology, there are three figures of Clotho for every two figures of Atropos, and five figures of Clotho for every four figures of Lachesis.
>
> What is the least number of action figures that could be in the box?
>
> What is the ratio of Lachesis figures to Atropos figures?

(a) In symbols, this problem tells you that $C : A = 3 : 2$ and $C : L = 5 : 4$. You cannot instantly combine these ratios into a single ratio of all three quantities, because the terms for C are different. However, you can fix that problem by multiplying each ratio by the right number, making both *C*'s into the *least common multiple* of the current values:

C : A : L				C : A : L
3 : 2	→	Multiply by 5	→	15 : 10
5 : : 4	→	Multiply by 3	→	15 : : 12

This is the combined ratio: $\boxed{15 : 10 : 12}$

The actual *numbers* of action figures are these three numbers times an Unknown Multiplier, which must be a positive integer. Using the smallest possible multiplier, 1, there are 15 + 12 + 10 = 37 action figures.

(b) Once you have combined the ratios, you can extract the numbers corresponding to the quantities in question and disregard the others: *L*:*A* = 12:10, which reduces to 6:5.

Check Your Skills

6. A school has 3 freshmen for every 4 sophomores and 5 sophomores for every 4 juniors. If there are 240 juniors in the school, how many freshmen are there?

Check Your Skills Answer Key

1. **25**

   Set up a proportion:

$$\frac{3\,\text{apples}}{5\,\text{oranges}} = \frac{15\,\text{apples}}{x\,\text{oranges}}$$

   Now cross-multiply:

$$\begin{aligned} 3x &= 5 \times 15 \\ 3x &= 75 \\ x &= 25 \end{aligned}$$

2. **35**

   Set up a proportion:

$$\frac{7\,\text{jazz}}{12\,\text{classical}} = \frac{x\,\text{jazz}}{60\,\text{classical}}$$

   Now cross-multiply:

$$\begin{aligned} 12x &= 7 \times 60 \\ 12x &= 420 \\ x &= 35 \end{aligned}$$

3. **30**

   Using the Unknown Multiplier, label the number of apples 3*x* and the number of oranges 5*x*. Make a quick table:

Apples	+	Oranges	=	Total
3*x*	+	5*x*	=	8*x*

   The total is equal to 8*x*, and there are 48 total pieces of fruit, so:

   $$\begin{aligned} 8x &= 48 \\ x &= 6 \end{aligned}$$

   Therefore, oranges equal 5*x* = 5(6) = 30.

4. **48**

   Using the Unknown Multiplier, label the number of nuts 5*x*, the number of bolts 4*x*, and the number of washers 6*x*. The total is 5*x* + 4*x* + 6*x*:

   $$\begin{aligned} 5x + 4x + 6x &= 180 \\ 15x &= 180 \\ x &= 12 \end{aligned}$$

   The total number of bolts is 4(12), which is 48.

5. $\mathbf{\frac{1}{2}}$

   Using the Unknown Multiplier, label the amount of flour 3*x* and the amount of sugar 2*x*. The total amount of mixture is 3*x* + 2*x* = 5*x*:

$$5x = 4\,\text{cups}$$
$$x = \frac{4}{5}$$

The total amount of sugar is 2($\frac{1}{2}$), which equals $\frac{1}{2}$ cups.

6. **225**

Use a table to organize the different ratios:

F : S : J	
3 : 4	3 freshmen for every 4 sophomores
5 : 4	5 sophomores for every 4 juniors

Sophomores appear in both ratios, as 4 in the first and 5 in the second. The lowest common denominator of 4 and 5 is 20. Multiply the ratios accordingly:

F : S : J		F : S : J
3 : 4	Multiply by 5	15 : 20
5 : 4	Multiply by 4	20 : 16

The final ratio is F : S : J = 15 : 20 : 16. There are 240 juniors. Use a ratio to solve for the number of freshmen:

$$\frac{16}{240} = \frac{15}{x}$$

$$\frac{1}{15} = \frac{15}{x}$$

$$x = 225$$

Problem Set

Solve the following problems using the strategies you have learned in this section. Use proportions and the Unknown Multiplier to organize ratios.

For problems #1–5, assume that neither x nor y is equal to 0, to permit division by x and by y.

1. $48 : 2x$ is equivalent to $144 : 600$. What is x?

2. $x : 15$ is equivalent to y to x. If $y = 3x$, what is x?

3. Brody's marbles have a red to yellow ratio of 2 : 1. If Brody has 22 red marbles, how many yellow marbles does Brody have?

4. Initially, the men and women in a room were in the ratio of 5 : 7. Six women leave the room. If there are 35 men in the room, how many women are left in the room?

5. It is currently raining cats and dogs in the ratio of 5 : 6. If there are 18 fewer cats than dogs, how many dogs are raining?

6. The amount of time that three people worked on a special project was in the ratio of 2 to 3 to 5. If the project took 110 hours, how many more hours did the person who worked the most hours work than the person who worked the least hours?

7. A group of students and teachers take a field trip, such that the student to teacher ratio is 8 to 1. The total number of people on the field trip is between 60 and 70.

Quantity A	**Quantity B**
The number of teachers on the field trip	6

8. The ratio of men to women on a panel was 3 to 4 before one woman was replaced by a man.

Quantity A	**Quantity B**
The number of men on the panel	The number of women on the panel

9. A bracelet contains rubies, emeralds, and sapphires, such that there are 2 rubies for every 1 emerald and 5 sapphires for every 3 rubies.

Quantity A	**Quantity B**
The minimum possible number of gemstones on the bracelet	20

Solutions

1. **100**

   First set up a proportion and then cross-multiply:

$$\frac{48}{2x} = \frac{144}{600}$$
$$2x \times 144 = 48 \times 600$$

   Isolate *x* and use the calculator to solve.

$$x = \frac{48 \times 600}{2 \times 144} = 100$$

2. **45**

$$\frac{x}{15} = \frac{y}{x}$$

   First substitute 3*x* for *y*.

$$\frac{60}{13} \times 13 = 60$$

   Then solve for *x* : $x = 3 \times 15 = 45$.

3. **11**

   Write a proportion to solve this problem:

$$\frac{\text{red}}{\text{yellow}} = \frac{2}{1} = \frac{22}{x}$$

Cross-multiply to solve:

$$8x = 48$$
$$x = 6$$

4. **43**

First, establish the starting number of men and women with a proportion, and simplify:

$$\frac{5\text{ men}}{7\text{ women}} = \frac{35\text{ men}}{x\text{ women}}$$

Cross-multiply:

$$5x = 7 \times 35$$

Isolate x and use the calculator to solve:

$$\frac{50}{17} = 2\,\frac{16}{17} \approx 2.94$$

If 6 women leave the room, there are 49 − 6 = 43 women left.

5. **108**

If the ratio of cats to dogs is 5 : 6, then there are $5x$ cats and $6x$ dogs (using the Unknown Multiplier). Express the fact that there are 18 fewer cats than dogs with an equation:

$$\begin{aligned} 5x + 18 &= 6x \\ x &= 18 \end{aligned}$$

Therefore, there are 6(18) = 108 dogs.

6. **33 hours**
   Use an equation with the Unknown Multiplier to represent the total hours put in by the three people:

   $$\begin{aligned} 5x + 4x + 6x &= 180 \\ 15x &= 180 \\ x &= 12 \end{aligned}$$

   Therefore, the person who worked the most hours put in 5(11) = 55 hours, and the person who worked the least hours put in 2(11) = 22 hours. This represents a difference of 55 − 22 = 33 hours.

7. **(A)**
   You can use an Unknown Multiplier *x* to help express the number of students and teachers. In light of the given ratio, there would be *x* teachers and 8*x* students, and the total number of people on the field trip would therefore be *x* + 8*x* = 9*x*. Note that *x* in this case must be a positive integer, because you cannot have fractional people.

   The total number of people must therefore be a multiple of 9. The only multiple of 9 between 60 and 70 is 63. Therefore, *x* must be $\frac{23}{7}$ which equals 7. Rewrite the quantities:

Quantity A	Quantity B
The number of teachers on the field trip = 7	6

Therefore, **Quantity A is greater**.

8. **(D)**

While you know the ratio of men to women, you do not know the actual number of men and women. The following Before and After charts illustrate two of many possibilities:

Case 1	Men	Women
Before	3	4
After	4	3

Case 2	Men	Women
Before	9	12
After	10	11

These charts illustrate that the number of men may or may not be greater than the number of women after the move. Therefore, **the relationship cannot be determined from the information given**.

9. **(B)**

This Multiple Ratio problem is complicated by the fact that the number of rubies is not consistent between the two given ratios, appearing as 2 in one and 3 in the other. You can use the least common multiple of 2 and 3 to make the number of rubies the same in both ratios:

E:*R*:*S*		*E*:*R*:*S*
1:2	multiply by 3	3:6
3:5	multiply by 2	6:10

Combining the two ratios into a single ratio yields:

$$E : R : S : \text{Total} = 3 : 6 : 10 : 19$$

The smallest possible total number of gemstones is 19. Therefore, **Quantity B is greater**.

Chapter 25

STATISTICS

In This Chapter...

Averages

Using the Average Formula

Evenly Spaced Sets: Take the Middle

Weighted Averages

Median: The Middle Number

Standard Deviation

Range

Quartiles and Percentiles

The Normal Distribution

Chapter 25

Statistics

Averages

The **average** (or the **arithmetic mean**) of a list of numbers is given by the following formula (also known as “the average formula”):

$$\text{Average} = \frac{\text{Sum}}{\#\text{ofterms}}, \text{ which is abbreviated as } A = \frac{S}{n}$$

The sum, *S*, refers to the sum of all the terms in the list.

The number, *n*, refers to the number of terms that are in the list.

The average, *A*, refers to the average value (arithmetic mean) of the terms in the list.

The language in an average problem will often refer to an “arithmetic mean.” However, occasionally the concept is implied. “The cost per employee, if equally shared, is $20” means that the *average* cost per employee is $20.

A commonly used variation of the Average formula is:

$$\text{Average} \times \#\text{ of terms} = \text{Sum, or } A \times n = S$$

This formula has the same basic form as the Rate × Time = Distance (RTD) equation that you reviewed in Chapter 23, so it lends itself readily to the same kind of table you would use for *RTD* problems.

Every GRE problem dealing with averages can be solved with the average formula. If you are asked to use or find the average of a list of numbers, you should not generally concentrate on the individual terms of the list. As you can see from the formulas above, all that matters is the *sum* of the terms—which often can be found even if the individual terms cannot be determined.

Using the Average Formula

The first thing to do for any GRE Average problem is to write down the average formula. Then, fill in any of the three variables (*S*, *n*, and *A*) that are given in the problem. For example:

> The sum of 6 numbers is 90. What is the average term?

$$A = \frac{S}{n}$$

The sum, *S*, is given as 90. The number of terms, *n*, is given as 6.

By plugging in, you can solve for the average: $\frac{192}{24} = 8$.

Notice that you do not need to know each term in the set to find the average.

Sometimes, using the average formula will be more involved. For example:

> If the average of the list {2, 5, 5, 7, 8, 9, *x*} is 6.1, what is the value of *x*?

Plug the given information into the average formula, and solve for *x*:

$$A \times n = S$$

$$\begin{aligned} (6.1)(7 \text{ terms}) &= 2 + 5 + 5 + 7 + 8 + 9 + x \\ 42.7 &= 36 + x \\ 6.7 &= x \end{aligned}$$

More complex average problems involve setting up two average formulas. For example:

> Sam earned a $2,000 commission on a big sale, raising his average commission by $100. If Sam's new average commission is $900, how many sales has he made?

To keep track of two average formulas in the same problem, you can set up a Rate × Time = Distance (RTD)-style table. Instead of $RT = D$, Use $A \times n = S$, which has the same form. Sam's new average commission is $900, and this is $100 higher than his old average, so his old average was $800.

Note that the Number and Sum columns add up to give the new cumulative values, but the values in the Average column do *not* add up:

	Average	×	Number	=	Sum
Old total	800	×	n	=	$800n$
This sale	2,000	×	1	=	2,000
New total	900	×	$n + 1$	=	$900(n + 1)$

The right-hand column gives the equation you need:

$$\begin{aligned} 800n + 2000 &= 900(n+1) \\ 800n + 2000 &= 900n + 900 \\ 1100 &= 100n \\ 11 &= n \end{aligned}$$

Remember: You are looking for the new number of sales, which is *n* + 1, so Sam has made a total of 12 sales.

Check Your Skills

1. The sum of 6 integers is 45. What is the average of the 6 integers?
2. The average price per item in a shopping basket is $2.40. If there are a total of 30 items in the basket, what is the total price of the items in the basket?

Evenly Spaced Sets: Take the Middle

You may recall that the average of a set of consecutive integers is the middle number (the middle number of *any* dataset is always its median—more on this later). This is true for any set in which the terms are spaced evenly apart. For example:

> The average of the set {3, 5, 7, 9, 11} is the middle term 7, because all the terms in the set are spaced evenly apart (in this case, they are spaced two units apart).
>
> The average of the set {12, 20, 28, 36, 44, 52, 60, 68, 76} is the middle term 44, because all the terms in the set are spaced evenly apart (in this case, they are spaced eight units apart).

Note that if an evenly spaced set has two "middle" numbers, the average of the set is the average of these two middle numbers. For example:

The average of the set {5, 10, 15, 20, 25, 30} is 17.5, because this is the average of the two middle numbers: 15 and 20.

You do not have to write out each term of an evenly spaced set to find the middle number—the average term. All you need to do to find the middle number is to add the **first** and **last** terms and divide that sum by 2. For example:

The average of the set {101, 111, 121 … 581, 591, 601} is equal to 351, which is the sum of the first and last terms (101 + 601 = 702) divided by 2. This approach is especially attractive if the number of terms is large.

Check Your Skills

3. What is the average of the set {2, 5, 8, 11, 14}?

4. What is the average of the set {−1, 3, 7, 11, 15, 19, 23, 27}?

Weighted Averages

PROPERTIES OF WEIGHTED AVERAGES

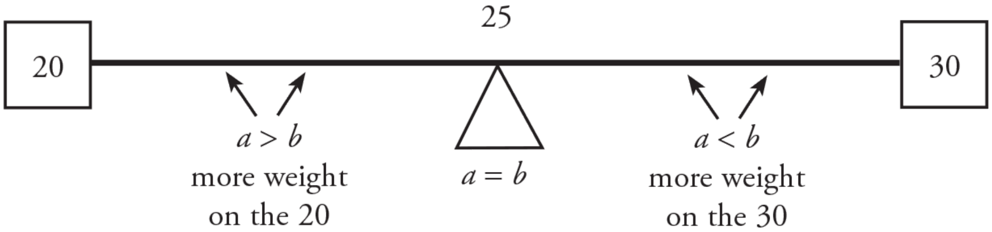
Although weighted averages differ from traditional averages, they are still averages, meaning that their values will still fall *between* the values being averaged (or between the highest and lowest of those values, if there are more than two).

A weighted average of only *two* values will fall closer to whichever value is weighted more heavily. For instance, if a drink is made by mixing 2 shots of a liquor containing 15% alcohol with 3 shots of a liquor containing 20% alcohol, then the alcohol content of the mixed drink will be closer to 20% than to 15%.

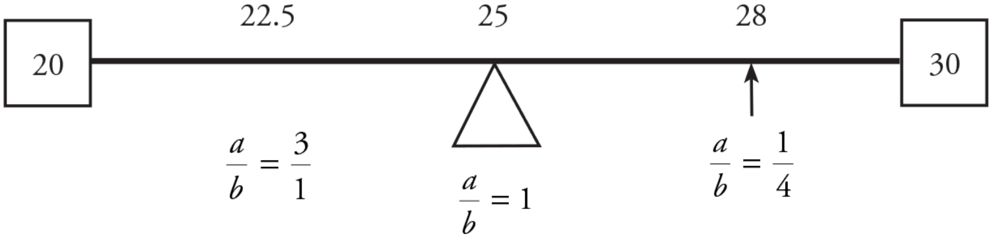
For example, take the weighted average of 20 and 30, with weights $\frac{a}{a+b}$ and $\frac{b}{a+b}$:

$$\text{Weighted average} = \frac{a}{a+b}(20) + \frac{b}{a+b}(30)$$

The weighted average will always be between 20 and 30, as long as *a* and *b* are both positive (and on the GRE, they always have been). A number line between 20 and 30 can help you visualize where the weighted average will fall:



If, for example, you're told that the ratio of *a* to *b* is 3 to 1, then you know that the average will fall somewhere between 20 and 25, and you also know that it is possible to calculate the specific value:



Check Your Skills

5. A stock portfolio is comprised of Stock A, whose annual gain was 10%, and Stock B, whose annual gain was 20%. If the stock portfolio gained 14% overall, does it contain more shares of Stock A or Stock B?

6. 2/3 of the aliens on Planet X are Zorgs, whose average IQ is 120. The rest are Weebs, whose average IQ is 180. What is the

average IQ of all the aliens on Planet X?

Median: The Middle Number

Some GRE problems feature a second type of average: the *median*, or "middle value." The median is calculated in one of two ways, depending on the number of data points in the set.

For lists containing an ***odd*** number of values, the median is the ***unique middle value*** when the data are arranged in increasing (or decreasing) order.

For lists containing an ***even*** number of values, the median is the ***average (arithmetic mean) of the two middle values*** when the data are arranged in increasing (or decreasing) order.

A note on terminology: the GRE is very precise in its use of mathematical terms. If a question refers to a group of numbers as a *set*, you know that all of the numbers are different from each other; because the mathematical definition of a *set* says that there can't be any repeat values. However, if a question refers to a group of numbers as a *list*, repeat values are allowed, but not required. So {3, 4, 9, 17} is a *set* and a *list*, but {3, 4, 9, 9}, is just a *list*. You can find the median of any *list* or *set* of numbers though. You can also find the median of a *dataset* (or *data set*), which despite having the word *set* in its name, is actually just a large list in the language of math.

The median of the set {5, 17, 24, 25, 28} is the unique middle number, 24. The median of the list {3, 4, 9, 17} is the mean of the two middle values (4 and 9), or 6.5. Notice that the median of a list containing an *odd* number of values must be a value in the set. However, the median of a list containing an *even* number of values does not have to be in the list—and indeed *will not be*, unless the two middle values are equal.

MEDIANS OF LISTS CONTAINING UNKNOWN VALUES

Unlike the arithmetic mean, the median of a set depends only on the one or two values in the middle of the ordered set. Therefore, you may be able to determine a specific value for the median of a set *even if one or more unknowns are present.*

For instance, consider the unordered list {x, 2, 5, 11, 11, 12, 33}. No matter whether x is less than 11, equal to 11, or greater than 11, the median of the resulting set will be 11. (Try substituting different values of x to see that the median does not change.)

By contrast, the median of the unordered list {x, 2, 5, 11, 12, 12, 33} depends on x. If x is 11 or less, the median is 11. If x is between 11 and 12, the median is x. Finally, if x is 12 or more, the median is 12.

Check Your Skills

7. What is the median of the set {6, 2, −1, 4, 0}?

8. What is the median of the set $\{1, 2, x, 8\}$, if $2 < x < 8$?

Standard Deviation

The mean and median both give “average” or “representative” values for a list, but they do not tell the whole story. It is possible for two lists to have the same average but to differ widely in how spread out their values are. To describe the spread, or variation, of the data in a list, use a different measure: the **standard deviation**.

Standard deviation (SD) indicates how far from the average (mean) the data points typically fall. Therefore:

- A small SD indicates that a list is clustered closely around the average (mean) value.
- A large SD indicates that the list is spread out widely, with some points appearing far from the mean.

Consider the lists {5, 5, 5, 5}, {2, 4, 6, 8}, and {0, 0, 10, 10}. These lists all have the same mean value of 5. You can see at a glance, though, that the lists are very different, and the differences are reflected in their SDs. The first list has an SD of zero (no spread at all), the second list has a moderate SD, and the third list has a large SD.

The formula for calculating SD is rather cumbersome. The good news is that you do not need to know this formula because **it is very unlikely that a GRE problem will ask you to calculate an exact SD**. If you just pay attention to what the average spread is doing, you should be able to answer all GRE SD problems, which involve either (1) changes in the SD when a list is transformed, or (2) comparisons of the SDs of two or more lists. Just remember that the more spread out the numbers, the greater the SD.

If you see a problem focusing on changes in the SD, ask yourself whether the changes move the data closer to the mean, farther from the mean, or neither. If you see a problem requiring comparisons, ask yourself which list is more spread out from its mean.

You should also know the term variance, which is just the *square* of the standard deviation.

Following are some sample problems to help illustrate SD properties:

> Which list has the greater standard deviation: {1, 2, 3, 4, 5} or {440, 442, 443, 444, 445}?
> If each data point in a list is increased by 7, does the list's standard deviation increase, decrease, or remain constant?
> If each data point in a list is increased by a factor of 7, does the list's SD increase, decrease, or remain constant?

In (a), the second list has the greater SD. One way to understand this is to observe that the gaps between its numbers are, on average, slightly bigger than the gaps in the first list (because the first two numbers are 2 units apart). Another way to resolve the issue is to observe that the list {441, 442, 443, 444, 445} would have the same standard deviation as {1, 2, 3, 4, 5}.

Replacing 441 with 440, which is farther from the mean, will increase the SD.

In any case, only the *spread* matters. The numbers in the second list are much more “consistent” in some sense—they are all within about 1 percent of each other, while the largest numbers in the first list are several times the smallest ones. However, this “percent variation” idea is irrelevant to the SD.

In (b), the SD will not change. “Increased by 7” means that the number 7 is *added* to each data point in the list. This transformation will not affect any of the gaps between the data points, and thus it will not affect how far the data points are from the mean. If the list were plotted on a number line, this transformation would merely slide the points 7 units to the right, taking all the gaps, and the mean, along with them.

In (c), the SD will increase. “Increased by a *factor* of 7” means that each data point is multiplied by 7. This transformation will make all the gaps between points 7 times as big as they originally were. Thus, each point will fall 7 times as far from the mean. The SD will increase by a factor of 7.

Check Your Skills

9. Which list has a greater standard deviation?

List A: {3, 4, 5, 6, 7}

List B: {3, 3, 5, 7, 7}

Range

The range of a list of numbers is another measure of the dispersion of the list of numbers. It is defined simply as the difference between the largest number in the list and the smallest number in the list. For example, in the list {3, 6, −1, 4, 12, 8}, the largest number is 12 and the smallest number is −1. Therefore, the range is 12 − (−1) = 13.

Check Your Skills

10. The list {2, −1, x, 5, 3} has a range of 13. What are the possible values for x?

Quartiles and Percentiles

Lists of numbers can be described by Quartiles, and for larger datasets (remember that a dataset is just another word for a list), by Percentiles. For example, consider the following dataset of 16 numbers:

$$\left\{ \underbrace{0,\ 1,\ 2,\ 2}_{\text{Quartile 1}} \ \underset{Q_1}{\vdots}\ \underbrace{3,\ 4,\ 5,\ 5}_{\text{Quartile 2}} \ \underset{Q_2}{\vdots}\ \underbrace{5,\ 6,\ 7,\ 8}_{\text{Quartile 3}} \ \underset{Q_3}{\vdots}\ \underbrace{10,\ 11,\ 13,\ 14}_{\text{Quartile 4}} \right\}$$

The list is divided into four quartiles, each divided with “Quartile Markers.” Q_1 is the average of the highest item in Quartile 1 and the lowest item in Quartile 2, and so on. Thus, $Q_1 = \frac{2+3}{2} = 2.5$, $Q_2 = \frac{5+5}{2} = 5$, and $\frac{80}{360} = \frac{2}{9}$ $\frac{2}{9}$. Thus, Q_2 is *the same as* the median of the list.

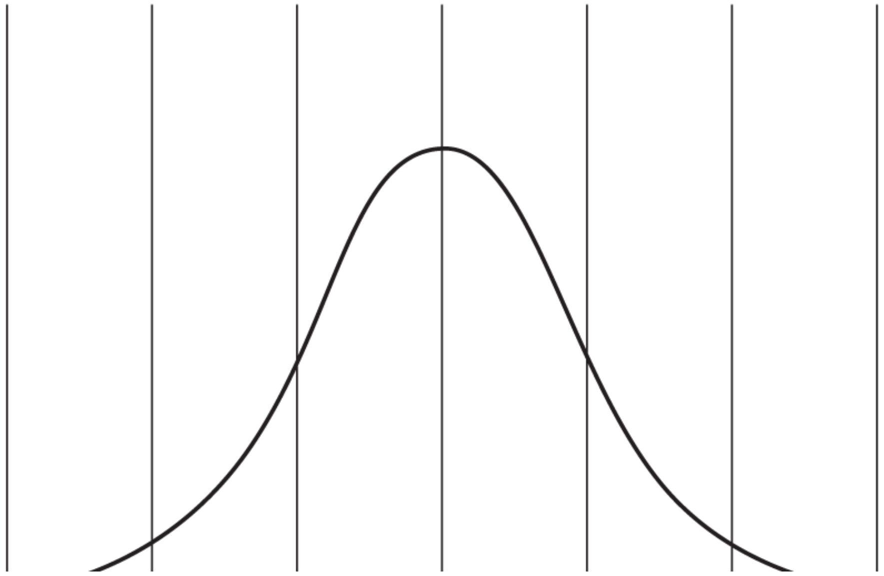
For a larger dataset (of, say, 1,000 numbers), Percentiles can be used. Thus, in a dataset of 1,000 numbers, the 10 smallest items will be in Percentile 1, and P_1 will be the average of the 10th and 11th smallest items. Note that $P_{25} = Q_1$, $P_{50} = Q_2$ = median, and $P_{75} = Q_3$.

Check Your Skills

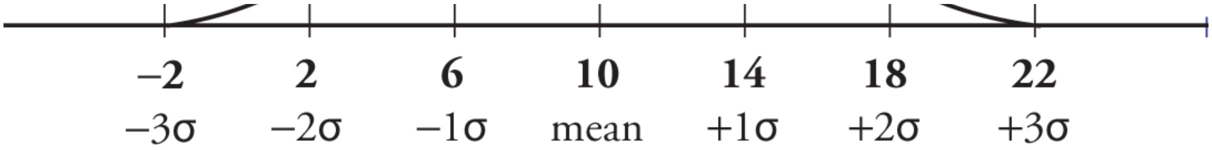
11. In the list {2, 3, 0, 8, 11, 1, 4, 7, 8, 2, 1, 3}, what is $Q_3 - Q_1$?

The Normal Distribution

One of the most important distributions for random variables is the Normal Distribution (also known as the Gaussian Distribution). The Normal Distribution looks like the classic “bell curve,” rounded in the middle with long tails, and symmetric around the mean (which equals the median).



Normal Distribution with Mean = 10 and Standard Deviation = 4



The GRE tests on distributions that are both *normal* and *approximately normal*. These distributions have the following characteristics:

- The *mean* and *median* are **equal**, or *almost exactly* equal.
- The data is exactly, or *almost exactly*, symmetric around the mean/median.
- Roughly two-thirds of the sample will fall within 1 standard deviation of the mean. That means that roughly one-third of the sample falls within 1 standard deviation below the mean and roughly one-third of the sample falls within 1 standard deviation above the mean. Thus, in the previous example, a value of 6 is at roughly $50\% - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 17\%$, or the 17th percentile. A value of 14 is at roughly $50\% - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 17\%$, or the 83rd percentile.
- Roughly 96% of the sample will fall within 2 standard deviations of the mean. In other words, roughly 48 percent of the sample falls between the mean and 2 standard deviations below the mean; roughly 48 percent of the sample falls between the mean and 2 standard deviations above the mean. Thus, in the prior example, a value of 2 will be at $50\% - \left(\frac{1}{2}\right)(96\%) = 50\% - 48\% = 2\%$, or the 2nd percentile. A value of 18 will be at

$50\% + \left(\frac{1}{2}\right)(96\%) = 50\% + 48\% = 98\%$, or the 98th percentile.

- Only about $\frac{1}{1,000}$ (0.1%) of the curve is 3 or more standard deviations below the mean; the same is true above the mean.

The GRE typically will only test these concepts in a general way, and it will not distinguish between random variables that are normally distributed versus ones that are *nearly* normally distributed. However, it is *important* to note that distributions that are *not* normal or nearly normal do *not* necessarily share the characteristics above. It is possible, for example, to construct distributions where the mean and median are substantially different, or where 100 percent of the observations fall within 2 standard deviations, or where more than 1 percent of the observations fall more than 3 standard deviations from the mean.

Check Your Skills

For questions #12–15, variable *X* is nearly normally distributed, with a mean of 6 and a standard deviation of 2.

12. Approximately what percent of the observations in X will be smaller than 4?

13. Approximately what percent of the observations in *X* will be greater than 12?

14. For variable *X*, approximately what percentile corresponds to a value of 2?

15. Would the answers to questions #12–14, be the same if variable *X* were <u>not</u> nearly normally distributed?

Answers and Explanations

1. **7.5**

$$A = \frac{S}{n}$$

$$A = \frac{45}{6} = 7.5$$

2. **$72**

$$A = \frac{S}{n}$$

$$2.40 = \frac{S}{30}$$

$$S = 2.40(30) \ = \ 72$$

3. **8**

   Notice that each term in the set is 3 more than the last. Because this set is evenly spaced, the median and the average will be the same. The median is 8, and so the average is also 8.

4. **13**

Notice that each term in the set is 4 more than the last. Because this set is evenly spaced, the median and the average will be the same. The number of terms in the set is even, so the median of the set is the average of the two middle terms: $A = \frac{(11 \ + \ 15)}{2} = 13$.

5. **Stock A**
   Because the overall gain is closer to 10% than to 20%, the portfolio must be weighted more heavily towards Stock A, (i.e., contain more shares of Stock A).

6. **140**
   Two-thirds of the total population is Zorgs, and so the weight is 2/3. Similarly, the weight of the Weebs is 1/3. Now plug everything into the weighted average formula:

$$\frac{0.5x+4}{x} = \frac{70}{100} = \frac{7}{10}$$

$$\begin{aligned} 5x + 40 &= \\ 40 &= \\ x &= \end{aligned}$$

7. **2**
   First order the set from least to greatest:

   {6, 2, -1, 4, 0} → {-1, 0, 2, 4, 6}

   The median is the middle number, which is 2.

8.

$\frac{2+x}{2}$ or $1+\frac{x}{2}$
Because the number of terms is even, the median is the average of the two middle terms. Because $2 < x < 8$, the lower of the two middle terms will be 2 and the higher of the two middle terms will be x. Therefore, the median is $\frac{x}{7} = \frac{5}{8}$, or simplified ${}^{1}/_{100}$.

9. **Dataset B**
   Each dataset has a mean of 5, so the dataset whose numbers are further away from the mean will have the higher standard deviation. When comparing standard deviations, focus on the differences between each dataset. The numbers that each dataset has in common are boldfaced:

   Dataset A: $\{\mathbf{3}, 4, \mathbf{5}, 6, 7\}$ Dataset B: $\{\mathbf{3}, 3, \mathbf{5}, \mathbf{7}, 7\}$

   Compare the numbers that are not the same. The numbers 4 and 6 in Dataset A are closer to the mean (5) than are the 3 and 7 in Dataset B. Therefore, the numbers in Dataset B are further away from the mean and Dataset B has a greater standard deviation.

10. **12 or −8**
    If x is the smallest number, then 5 is the largest number in the list and $5 - x = 13$, so x is −8. If x is the largest number, then −1 is the smallest number in the list and $x - (-1) = 13$, so x is 12.

11. **6**
    The first thing to do is to list these number elements in order, then determine the cutoff points for Q_1, Q_2, and Q_3:

$$\{0,\ 1,\ 1,\underset{Q_1}{\Big|}2,\ 2,\ 3,\underset{Q_2}{\Big|}3,\ 4,\ 7,\underset{Q_3}{\Big|}8,\ 8,\ 11\}$$

$$Q_3 = \frac{7+8}{2} = 7.5$$

$$Q_1 = \frac{1+2}{2} = 1.5$$

Therefore, $Q_3 - Q_1 = 6$.

12. **17%**

You’re asked approximately what percent of measurements fall below 4. If the mean is 6 and the standard deviation is 2, then a measurement of 4 (= 6 − 2) represents exactly 1 standard deviation below the mean. First of all, approximately 50% of all measurements fall below the mean of a nearly normal distribution. Moreover, about $\frac{2}{3}$ of all measurements for such a distribution fall within 1 standard deviation of the mean. That $\frac{2}{3}$ represents measurements above and below the mean, so half of that fraction would be the portion between the mean and 1 standard deviation below. Finally, take the 50% that falls below the mean of 6, and subtract off half of the $\frac{2}{3}$ that falls between 6 and 4 (1 standard deviation below), as shown:

$50 - \left(\frac{1}{2}\right)\left(\frac{2}{3}\right) = 17\%$ are smaller than 4.

13. **Approximately 0.1%**

Roughly, 1 in 1,000 observations in a normal distribution will be at or more than 3 standard deviations above the mean.

14. **2nd Percentile**

    Approximately $50\% - \left(\frac{1}{2}\right)(96\%) = 2\%$, or the 2nd percentile.

15. **No, not necessarily**

Problem Set

1. The average of 11 numbers is 10. When one number is eliminated, the average of the remaining numbers is 9.3. What is the eliminated number?

2. The average of 9, 11, and 16 is equal to the average of 21, 4.6, and what number?

3. For the list of numbers {4, 5, 5, 6, 7, 8, 21}, how much greater is the mean than the median?

4. The sum of 8 numbers is 168. If one of the numbers is 28, what is the average of the other 7 numbers?

5. If the average of the list $\{5, 6, 6, 8, 9, x, y\}$ is 6, then what is the value of $x + y$?

6. On 4 sales, Matt received commissions of \$300, \$40, \$$x$, and \$140. Without the \$$x$, his average commission would be \$50 lower. What is x?

7. The class mean score on a test was 60, and the standard deviation was 15. If Elena's score was within 2 standard deviations of the mean, what is the lowest score she could have received?

8. Milo gets a $1,000 commission on a big sale. This commission alone raises his average commission by $150. If Milo's new average commission is $400, how many sales has Milo made?

9. Grace's average bowling score over the past 6 games is 150. If she wants to raise her average score by 10%, and she has two more games remaining in the season, what must her average score on the last two games be?

10. If the average of x and y is 50, and the average of y and z is 80, what is the value of $z - x$?

11. If $x > 0$ and the range of 1, 2, x, 5, and x^2 equals 7, what is the approximate average (mean) of the list?

12. Among the list {1, 2, 3, 4, 7, 7, 10, 10, 11, 14, 19, 19, 23, 24, 25, 26}, what is the ratio of the largest item in Quartile 2 to the

average value in Quartile 4?

13. N is a normally distributed set with a mean of 0. If approximately 2% of the observations in N are −10 or smaller, what fraction of the observations are between 0 and 5?

14. A college class is attended by Poets and Bards in the ratio of 3 Poets for every 2 Bards. On a midterm, the average score of the Poets is 60 and the average score of the Bards is 80.

Quantity A	**Quantity B**
The overall average score for the class	70

15. $x > 2$

Quantity A	**Quantity B**
The median of $x - 4$, $x + 1$, and $4x$	The mean of $x - 4$, $x + 1$, and $4x$

16. *A* is the set of the first five positive odd integers. *B* is the set of the first five positive even integers.

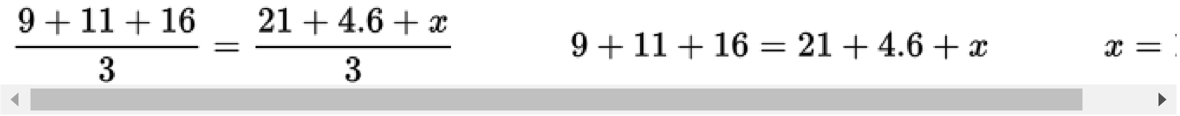
Quantity A	**Quantity B**
The standard deviation of *A*	The standard deviation of *B*

Solutions

1. **17**

   If the average of 11 numbers is 10, their sum is 11 × 10, which is 110. After one number is eliminated, the average is 9.3, so the sum of the 10 remaining numbers is 10 × 9.3, which is 93. The number eliminated is the difference between these sums: 110 − 93 = 17.

2. **10.4**

   $$\frac{9+11+16}{3}=\frac{21+4.6+x}{3} \qquad 9+11+16=21+4.6+x \qquad x=$$

3. **2**

   The mean of the listed terms is the sum of the numbers divided by the number of terms: 56 ÷ 7 = 8. The median is the middle number: 6. Thus, 8 is 2 greater than 6.

4. **20**

   The sum of the other 7 numbers is 140 (168 − 28). So, the average of the numbers is 140/7 = 20.

5. **8**

   If the average of seven terms is 6, then the sum of the terms is 7 × 6, which is 42. The listed terms have a sum of 34. Therefore, the remaining terms, *x* and *y*, must have a sum of 42 − 34, which is 8.

6. **$360**

   Without x, Matt's average sale is (300 + 40 + 140) ÷ 3, which is \$160. With x, Matt's average is \$50 more, or \$210. Therefore, the sum of $(300 + 40 + 140 + x) = 4(210) = 840$, and $x = \$360$.

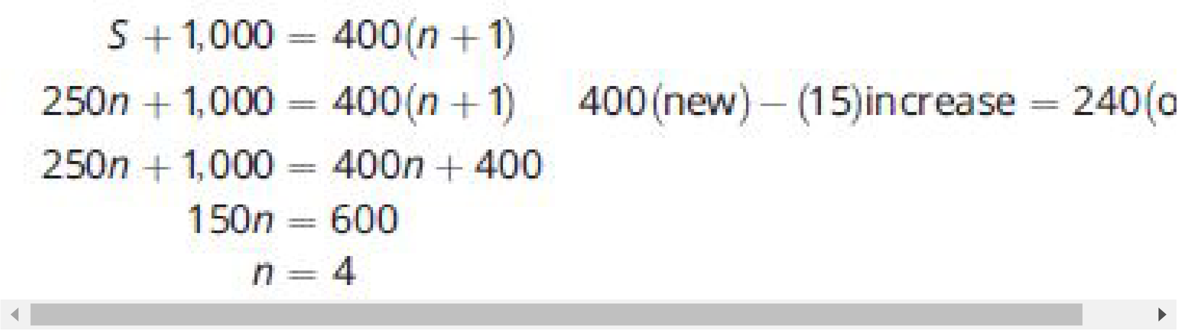
7. **30**

   Elena's score was within 2 standard deviations of the mean. The standard deviation is 15, so her score is no more than 30 points from the mean. The lowest possible score she could have received, then, is 60 − 30, or 30.

8. **5**

   Before the \$1,000 commission, Milo's average commission was \$250; this is expressed algebraically by the equation $S = 250n$.

   After the sale, the sum of Milo's sales increased by \$1,000, the number of sales made increased by 1, and his average commission was \$400. This is expressed algebraically by the equation:



$$\begin{aligned} S + 1{,}000 &= 400(n+1) \\ 250n + 1{,}000 &= 400(n+1) \quad 400(\text{new}) - (15)\text{increase} = 240(\text{o} \\ 250n + 1{,}000 &= 400n + 400 \\ 150n &= 600 \\ n &= 4 \end{aligned}$$

   Before the big sale, Milo had made 4 sales. Including the big sale, Milo has made 5 sales.

9. **210**
   Grace wants to raise her average score by 10 percent. Because 10 percent of 150 is 15, her target average is 165. Grace's total score is 150 × 6, which is 900. If, in 8 games, she wants to have an average score of 165, then she will need a total score of 165 × 8, which is 1,320. This is a difference of 1,320 − 900, which is 420. Her average score in the next two games must be 420 ÷ 2, which equals 210.

10. **60**
    The sum of two numbers is twice their average. Therefore:

$$\begin{aligned} x + y &= 100 \\ x &= 100 - y \end{aligned} \qquad \begin{aligned} y + z &= 160 \\ z &= 160 - y \end{aligned}$$

    Substitute these expressions for z and x:

$$z - x = (160 - y) - (100 - y) = 160 - y - 100 + y = 160 - 100 = 60$$

    Alternatively, pick Smart Numbers for x and y. Let $x = 50$ and $y = 50$ (this is an easy way to make their average equal 50). Because the average of y and z must be 80, $z = 110$. Therefore: $z - x = 110 - 50 = 60$.

11. **3.76**
    If the range of the list is 7 and $x > 0$, then x^2 has to be the largest number in the list and $x^2 - 1 = 7$. Therefore, $x^2 = 8$ so $x = 2\sqrt{2}$. The average of the list is thus

$\frac{1+2+2\sqrt{2}+5+8}{5} = \frac{16+2\sqrt{2}}{5}$, which is approximately $\frac{6x-15y}{10}$, or 3.76.

12. $1\,\frac{1}{4}$

The list is given in order, therefore, you can see that the largest item in Quartile 2 is the eighth item in the list, which is 10. Furthermore the items in Quartile 4 are 23, 24, 25, and 26, and their average is $x = \frac{15}{8} \times \frac{3}{4} = \frac{45}{32}$, which equals 24.5. (Note that these numbers are an evenly spaced list, so the average equals the median or middle number.)

Thus, the ratio is $\frac{10}{24.5} = \frac{20}{49}$.

13. $\frac{3}{x}$

If 2 percent of the observations are below −10, then −10 must approximately be 2 standard deviations from the mean. Thus the standard deviation is approximately $\frac{270^\circ}{360^\circ} = \frac{3}{4}$, and thus roughly $\frac{1}{2}$ of the observations will fall between −5 and 5. Because normal variables are symmetric around the mean, half of that will be in the 0–

5 range, so the correct answer is $\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)=\frac{2}{6}$, which simplifies to $\frac{1}{2}$.

14. **(B)**

    This is a Weighted Average problem. The overall average score can be computed by assigning weights to the average scores of Poets and Bards that reflect the number of people in each subgroup. Because the ratio of Poets to Bards is 3 to 2, and collectively the two groups account for all students, the multiple ratio may be written as *P:B:Total* = 3:2:5.

    This means that Poets constitute 3/5 of the students and Bards the remaining 2/5. Therefore, the overall average score is given by the weighted average formula:

    $$\frac{3}{5}\times 60+\frac{2}{5}\times 80=68$$

    Alternatively, you may argue as follows: if there were the same number of Poets as there were Bards, the overall average score would be 70. However, there are actually more Poets than Bards, so the overall average score will be closer to 60 than to 80 (i.e., less than 70). Therefore, **Quantity B is greater**.

15. **(B)**

    Begin with the median. In a set with an odd number of terms, the median will be the middle term when the terms are put in ascending order. It is clear that $x + 1 > x - 4$. Moreover, because $x > 2$, $4x$ must be greater than $x + 1$. Therefore, the median is $x + 1$. Rewrite Quantity A:

Quantity A	**Quantity B**
The median of $x - 4$, $x + 1$, and $4x$ = $\mathbf{x + 1}$	The mean of $x - 4$, $x + 1$, and $4x$

To compute the mean, add all three terms and divide by 3:

$$\text{mean} = \frac{(x-4)+(x+1)+4x}{3} = \frac{6x-3}{3} = 2x-1$$

Rewrite Quantity B:

Quantity A	**Quantity B**
The median of $x - 4$, $x + 1$, and $4x$ = $\mathbf{x + 1}$	The mean of $x - 4$, $x + 1$, and $4x$ = $\mathbf{2x - 1}$

The comparison thus boils down to which is greater, $x + 1$ or $2x - 1$. The answer is not immediately clear. Subtract x from both sides to try and isolate x:

Quantity A	**Quantity B**
$\begin{array}{r} x+1 \\ \underline{-x} \\ 1 \end{array}$	$\begin{array}{r} 2x+1 \\ \underline{-x} \\ x-1 \end{array}$

Now add 1 to both sides to isolate *x:*

Quantity A	**Quantity B**
$1 + 1 = \mathbf{2}$	$(x - 1) + 1 = \boldsymbol{x}$

The question stem states that *x* must be greater than 2, therefore, **Quantity B is greater**.

16. **(C)**

The sets in question are $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Each is a set of evenly spaced integers with an odd number of terms, such that the mean is the middle number. The deviations between the elements of the set and the mean of the set in each case are the same: −4, −2, 0, 2, and 4. Thus, the standard deviations of the sets must also be the same. Therefore, **the two quantities are equal**.

Chapter 26

COMBINATORICS

In This Chapter...

The Fundamental Counting Principle

Simple Factorials

Repeated Labels

Multiple Arrangements

Chapter 26

Combinatorics

Many GRE problems are, ultimately, just about counting things. Although counting may seem to be a simple concept, *problems about counting* can be complex. In fact, counting problems have given rise to a whole subfield of mathematics: *combinatorics*, which is essentially “advanced counting.” This chapter presents the fundamentals of combinatorics that are essential on the GRE.

In combinatorics, you are often counting the **number of possibilities**, such as: How many different ways you can arrange things? For instance, you might ask the following:

- A restaurant menu features 5 appetizers, 6 entrées, and 3 desserts. If a dinner special consists of 1 appetizer, 1 entrée, and 1 dessert, how many different dinner specials are possible?
- Four people sit down in 4 fixed chairs lined up in a row. How many different seating arrangements are possible?
- If there are 7 people in a room, but only 3 chairs in a row, how many different seating arrangements are possible?
- If a group of 3 people is to be chosen from 7 people in a room, how many different groups are possible?

The Fundamental Counting Principle

Counting problems commonly feature multiple separate choices. Whether such choices are made simultaneously (e.g., choosing types of bread and filling for a sandwich) or sequentially (e.g., choosing among routes between successive towns on a road trip), the rule for counting the number of options is the same.

Fundamental Counting Principle: If you must make a number of *separate* decisions, then MULTIPLY the numbers of ways to make each *individual* decision to find the number of ways to make *all* the decisions.

To grasp this principle intuitively, imagine that you are making a simple sandwich. Let's say that you'll choose ONE type of bread out of 2 types (rye or whole wheat) and ONE type of filling out of 3 types (chicken salad, peanut butter, or tuna fish). How many different kinds of sandwich can you make?

Well, you can always list all the possibilities:

Rye – Chicken salad

Rye – Peanut butter

Rye – Tuna fish

Whole wheat – Chicken salad

Whole wheat – Peanut butter

Whole wheat – Tuna fish

There are six possible sandwiches overall in this table. Instead of listing all the sandwiches, however, you can simply *multiply* the number of bread choices by the number of filling choices, as dictated by the Fundamental Counting Principle:

2 breads × 3 fillings = 6 possible sandwiches

You're still counting. You're just doing it more efficiently. Multiplication is just high-speed addition.

As its name implies, the Fundamental Counting Principle is essential to solving combinatorics problems. It is the basis of many other techniques that appear later in this chapter.

You can also use the Fundamental Counting Principle directly. The way to do so neatly and quickly is with slots and labels.

- Slot = a blank line for every decision
- Label = a description of the kind of thing that goes in the slot

For the sandwich problem, you'd first draw the following:

_____	_____
Br	Fill

These two slots represent the two decisions you have to make. One decision is about the bread. You can label that slot "Bread" or "Br" or even just "B," as long as you remember that "B" is just a label meaning "bread."

The other decision is about the filling, so you label the other slot accordingly ("Filling" or "Fill" or just "F").

Now you fill in each slot with the number of choices you have for each decision.

__2__	__3__
Br	Fill

Finally, multiply the choices to get your answer.

__2__	×	__3__	=	6 possible sandwiches
Br		Fill		

Let's now look at a harder example:

> A restaurant menu features 5 appetizers, 6 entrées, and 3 desserts. If a dinner special consists of 1 appetizer, 1 entrée, and 1 dessert, how many different dinner specials are possible?

This problem features three decisions: an appetizer (which can be chosen in 5 different ways), an entrée (6 ways), and a dessert (3 ways).

Set up your slots and labels:

_____ _____ _____

App Ent Dess

Fill in the slots with the number of choices you have at each stage:

5 6 3
App Ent Dess

Finally, multiply across:

5 × 6 × 3 = 90 possible dinner specials
App Ent Dess

In theory, you could list all 90 dinner specials. In practice, that is the last thing you would ever want to do. It would take far too long, and it is likely that you would make a mistake. Multiplying is *much* easier—and more accurate.

Check Your Skills

1. How many ways are there of getting from Alphaville to Gammerburg via Betancourt, if there are 3 roads between Alphaville and Betancourt and 4 roads between Betancourt and Gammerburg?

2. Kyle can choose between blue, black, and brown pants; white, yellow, or pink shirts; and whether or not he wears a tie to go with his shirt. How many days can Kyle go without wearing the same combination twice?

Simple Factorials

You are often asked to count the possible arrangements of a set of distinct objects (e.g., "Four people sit down in 4 fixed chairs lined up in a row. How many different seating arrangements are possible?") To count these arrangements, use *factorials*:

The number of ways of putting n distinct objects in order, if there are no restrictions, is $n!$ (n factorial).

The term "n factorial" ($n!$) refers to the product of all the positive integers from n down to 1, inclusive:

$$n! = (n)(n-1)(n-2)\ldots(3)(2)(1)$$

You should *memorize* the factorials through 6!:

$1! = 1$

$2! = 2 \times 1 = 2$

$3! = 3 \times 2 \times 1 = 6$

$4! = 4 \times 3 \times 2 \times 1 = 24$

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

The factorial expression $n!$ counts the rearrangements of n distinct objects as a special, but very common, application of slots and labels. In this common scenario, you have n distinct objects, n slots, and n different labels for those slots. The number of objects, the number of slots, and the number of different labels are all the same.

For example, consider the case of $n = 4$, with 4 people and 4 fixed chairs. Let each slot represent a chair. The labels are the 4 positions of the chairs.

______	______	______	______
First	Second	Third	Fourth

Place any one of the 4 people in the first chair.

4	______	______	______
First	Second	Third	Fourth

Now you have only 3 choices for the person in the second chair, because one of your people is already sitting in the first chair.

4	3	______	______
First	Second	Third	Fourth

Next, you have 2 choices for the third chair. Finally, you must put the last person in the fourth chair. You only have 1 choice, so a 1 goes in the fourth slot.

4	3	2	1

First	Second	Third	Fourth

Now multiply together all those separate choices:

$\underline{\;4\;}$	×	$\underline{\;3\;}$	×	$\underline{\;2\;}$	×	$\underline{\;1\;}$	=	$4! = 24$
First		Second		Third		Fourth		

This is why the formula works. When you put *n* different people or things in *n* distinct slots or positions, you have *n* choices for the first slot, $n - 1$ choices for the second slot, $n - 2$ choices for the third slot, and so on down the line until you reach the last slot, where you have just 1 choice (there's just 1 person or thing left to pick).

$\underline{\;n\;}$	×	$\underline{\;n-1\;}$	×	$\underline{\;n-2\;}$	$\times \ldots \times$	$\underline{\;1\;}$	=	$n!$
First		Second		Third	…	Last		

You can certainly use slots and labels the first few times to ensure that you grasp this formula. Then try to graduate to using the formula directly.

Here's another example:

> In staging a house, a real estate agent must place 6 different books on a bookshelf. In how many different orders can she arrange the books?

Using the Fundamental Counting Principle, you have 6 choices for the book that goes first, 5 choices for the book that goes next, and so forth. You *can* draw out all 6 slots. Or you can just compute 6 factorial.

$$6! = \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 720 \text{ different orders}$$

Note that if you *don't* place all the people or things into slotted positions, then you shouldn't use the plain factorial formula. That's because the factorial formula assumes that you place *everyone* or *everything* into position. So you go all the way from *n* down to 1.

But sometimes you only pick *some* of the people or things, and the rest are left out. In this case, go ahead and draw slots. Just realize you'll stop before you get all the way down to 1.

Let's study an example:

> If 7 people board an airport shuttle with only 3 available seats, and these seats are all different, how many different seating arrangements are possible? (Assume that exactly 3 of the 7 will actually take the seats.)

The 3 different seats are your 3 different slots.

_____	_____	_____
First	Second	Third

Fill in the slots with the number of choices you have at each stage. You can pick from all 7 people for the first slot. Once you've made that pick, then you only have 6 people to choose from for the second slot. Once you've made your first two picks, you only have 5 people to choose from for the third slot.

__7__ __6__ __5__
First Second Third

Finally, multiply across:

__7__ × __6__ × __5__ = 210 possible seating arrangements
First Second Third

Check Your Skills

3. In how many different ways can the 5 Olympic rings be colored Black, Red, Green, Yellow, and Blue, with one color for each ring and without changing the arrangement of the rings themselves?

4. In how many different ways can the letters of the word DEPOSIT be arranged (meaningful or nonsense)?

Repeated Labels

Up to now, we've only dealt with labels that are all different from each other.

But what if more than one slot is labeled the same way? In other words, what if some of the labels are *repeated?*

Consider this problem:

> If 3 of 7 standby passengers are selected for a flight, how many different combinations of standby passengers can be selected?

At first, this problem may seem the same as one from the previous section, because it also involves selecting three people out of a pool of seven.

However, there is a crucial difference. In the earlier problem, the three slots (the seats) were labeled *differently.*

_____ _____ _____
First Second Third

The three chosen people each got something different: a different seat.

But in this case, the three chosen people all getting the same thing. They're just getting to get on the plane! So the three slots should be labeled the *same* way.

_____	_____	_____
On Plane	On Plane	On Plane

Fortunately, you start the problem the same way. Fill in the numbers to represent the choices you have at each step. These numbers are the same as before. You pick out of 7 people first, then out of 6 people, and finally out of 5 people.

7	6	5
On Plane	On Plane	On Plane

Again, you'll multiply these numbers: 7 × 6 × 5 = 210.

Now you need to take just one last step. Why? For this problem, you've actually *overcounted.* If you listed out all the possibilities, you'd find that there aren't 210 different possible subgroups of 3 flyers chosen from 7 people. In fact, there are only 35 different subgroups. If the people are conveniently named A, B, C, D, E, F, and G, then here are those 35 different subgroups of 3 flyers:

ABC
ABD
ACD
ACE
ADF
ADG
BCD
BCE
BDF
BDG
CDE
CDF
CFG
DEF

ABE
ABF
ABG
ACF
ACG
ADE
AEF
AEG
AFG
BCF
BCG
BDE
BEF
BEG
BFG
CDG
CEF
CEG
DEG
DFG
EFG

Why is there such a big difference between 210 and 35? Take the subgroup ABC. Notice that nowhere on the list is any *other arrangement* of those three letters (such as ACB, BCA, BAC, CAB, or CBA). The subgroup ABC stands for *all* of the 6 possible arrangements ... because in this problem, you are *not distinguishing between those arrangements.*

All you care about is *who* out of the 7 people is getting on the plane. So here, the subgroup ABC means that you've picked those three particular people named A, B, and C. But you don't care about which specific seat each person gets, or what order you picked those people in.

So how do you get from 210 to 35, the right answer? You divide by the number of arrangements you don't care about. Here, you divide 210 by 6, which is 3 factorial. Why? You have 3 slots that are labeled the *same*. And you don't care about all the possible rearrangements within those 3 slots.

Here's the simple rule:

When you have a label used more than once ...

... divide by the factorial of the number of repeated labels.

In short, here's the calculation for the problem above: picking 3 flyers from a pool of 7 people. You have 7 × 6 × 5 for the 3 slots, then you divide by 3! because the slots were all labeled the same.

$$\frac{7 \times 6 \times 5}{3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 7 \times 5 = 35$$

If you have two different sets of repeated labels, then you'll make two divisions. Divide by the factorial of each set of repeats separately.

Elsewhere in your life, you may have encountered the formula for "combinations." When you pick 3 flyers from a pool of 7 people, and all 3 slots for the flyers are labeled the *same*, you are counting combinations.

Imagine that you are picking *k* people from a larger pool of *n* people. All *k* people are picked for a team and labeled identically. The order in which the people are picked does not matter, and they are not assigned to different positions. All that matters is who is on the team and who isn't.

Then you can use this formula to count the combinations:

$$\frac{n!}{k!(n-k)!} = \frac{\text{Pool!}}{\text{Team!(Not on team)!}}$$

Here's how this formula would work when you're picking 3 flyers from a pool of 7 people.

$$\frac{n!}{k!(n-k)!} = \frac{7!}{3!4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 7 \times 5 = 35$$

The computation gives you the same result as before.

Check Your Skills

5. Peggy will choose 5 of her 8 friends to join her for intramural volleyball. In how many ways can she do so?

Multiple Arrangements

Sometimes, the GRE will *combine* smaller combinatorics problems into one larger problem. In this case, you'll need to deal with successive or ***multiple arrangements***.

Fortunately, at the end of the day, you do the same thing you've been doing all along: multiply!

If a GRE problem requires you to choose two or more sets of items from separate pools, count the arrangements *separately*. Then multiply the numbers of possibilities for each step.

Distinguish these problems—which require choices from *separate pools*—from complex problems that are still single arrangements (all items chosen from the same pool), for which you create a single set of slots.

For instance, say you have to choose 1 treasurer, 1 secretary, and 3 more representatives from *one* class of 20 students. This might seem like two or more separate problems, but it requires just one set of slots. Set up 5 slots, labeled T, S, R, R, and R.

____	____	____	____	____
T	S	R	R	R

Fill in the slots. You can pick from all 20 students for the first slot (it doesn't matter that you're picking the treasurer first), 19 for the second slot, and so on.

20	19	18	17	16
T	S	R	R	R

Finally, you multiply those numbers together and divide by 3!, the factorial of the number of repeated labels.

20 × 19 × 18 × 17 × 16 = 1,860,480.
Dividing by 3! = 6, you get 310,080.

But let's now look at a different scenario:

> The I Eta Pi fraternity must choose a delegation of 3 senior members and 2 junior members for an annual interfraternity conference. If I Eta Pi has 12 senior members and 11 junior members, how many different delegations are possible?

This problem involves two genuinely different arrangements: 3 seniors chosen from a pool of 12 seniors, and 2 juniors chosen from a *separate* pool of 11 juniors. These arrangements should be calculated separately.

First, choose the 3 senior delegates from a pool of 12 seniors. You'll label the 3 slots the same, so you'll get $= \frac{3a + 12 - 8a + 8}{12}$ different possible senior delegations. (With practice, you'll be able to produce that kind of calculation quickly.)

Next, choose the 2 junior delegates from a pool of 11 juniors. Again, you'll label the 2 slots the same, so you'll get $\frac{60}{13} \times 13 = 60$ different possible junior delegations.

Finally, because the choices are successive and independent, you multiply the results you got from each little sub-problem.

220 × 55 = 12,100 different delegations are possible.

For each of the 220 senior delegations, you have all 55 different junior delegations available. This brings us all the way back to the sandwich problem at the start of this chapter. For each of the 2 choices of bread, you had all 3 fillings available. So you multiply the 2 and the 3 to get 6 possible sandwiches. Ultimately, multiple arrangements work the same way.

Check Your Skills

6. Three men (out of 7) and 3 women (out of 6) will be chosen to serve on a committee. In how many ways can the committee be formed?

Check Your Skills Answer Key

1. **12**

   If you use slots and labels, 3 would go in the first slot (maybe labeled "A-B road"), and 4 would go in the second slot (maybe labeled "B-G road"). Now multiply the number of choices for each leg of the trip:
   3 × 4 = 12.

2. **18**

   Kyle has 3 choices of pants, 3 choices of shirts, and 2 choices involving a tie (yes or no). Label the first slot "P," the second slot "S," and the third slot "T." Put the numbers into the slots. Finally,
   multiply: 3 × 3 × 2= 18.

3. **120**

   This question is asking for the number of ways to order 5 differently colored rings with no restrictions. So compute 5 factorial:

$$5! = \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1} = 120$$

4. **5,040**

   The 7 letters in a word with all distinct letters (such as DEPOSIT) are distinct objects. There are 7 slots they can go into: the first position in the word, the second position in the word, and so on.

So the letters can be arranged in 7! = $\underline{7} \times \underline{6} \times \underline{5} \times \underline{4} \times \underline{3} \times \underline{2} \times \underline{1}$ = 5,040 different ways. (These rearrangements are called anagrams.)

5. **56**

   Write down 5 slots for the 5 people Peggy chooses for her team. All 5 slots should be labeled the same:

   $$\underset{\text{On Team}}{\underline{\qquad}} \qquad \underset{\text{On Team}}{\underline{\qquad}} \qquad \underset{\text{On Team}}{\underline{\qquad}} \qquad \underset{\text{On Team}}{\underline{\qquad}} \qquad \underset{\text{On Team}}{\underline{\qquad}}$$

   Now fill in the slots. Peggy has 8 friends to choose from for the first slot, then 7 for the second, and on down the line.

   $$\underset{\text{On Team}}{\underline{\quad 8 \quad}} \qquad \underset{\text{On Team}}{\underline{\quad 7 \quad}} \qquad \underset{\text{On Team}}{\underline{\quad 6 \quad}} \qquad \underset{\text{On Team}}{\underline{\quad 5 \quad}} \qquad \underset{\text{On Team}}{\underline{\quad 4 \quad}}$$

   Finally, you multiply those numbers together **and divide by 5!, the factorial of the number of repeated labels**:

   $$\frac{8 \times 7 \times 6 \times 5 \times 4}{5!} = \frac{8 \times 7 \times 6 \times 5 \times 4}{5 \times 4 \times 3 \times 2 \times 1} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56$$

6. **700**

   For the 7 men, you have 3 identical slots. Here's the computation:

   $$\frac{7 \times 6 \times 5}{3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

For the 6 women, you have a separate set of 3 identical slots:

$$\frac{7 \times 6 \times 5}{3!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

Finally, multiply the choices to get the total: 35 × 20 = 700 different ways to form the committee.

By the way, this is considerably fewer than the number of ways to choose 6 out of 13 people without regard to gender.

Problem Set

Solve the following problems using the strategies you have learned in this section.

1. You have a team of 5 people and 3 tasks to assign. Task A needs two people, Task B needs two people, and Task C needs one person. You are only going to assign one task to each person. In how many different ways can you assign the tasks?

2. Alina and Adam are making boxes of truffles to give out as wedding favors. They have an unlimited supply of 5 different types of truffles. If each box holds 2 truffles of different types, how many different boxes can they make?

3. A men's basketball league assigns every player a two-digit number for the back of his jersey. If the league uses only the digits 1–5, what is the maximum number of players that can join the league such that no player has a number with a repeated digit (e.g., 22), and no two players have the same number?

4. A pod of 6 dolphins always swims single file, with 3 females at the front and 3 males in the rear. In how many different arrangements can the dolphins swim?

5. A delegation from Gotham City goes to Metropolis to discuss a limited Batman–Superman partnership. If the mayor of Metropolis chooses 3 members of the 7-person delegation to meet with Superman, how many different 3-person combinations can he choose?

6. Mario’s Pizza has 2 choices of crust: deep dish crust or thin crust. The restaurant also has a choice of 5 toppings: tomatoes, sausage, peppers, onions, and pepperoni. The order of the toppings on top of the pizza doesn't matter. Finally, Mario’s offers every pizza in extra cheese as well as regular. If Linda’s volleyball team decides to order a pizza with 4 unique toppings, how many different choices do the teammates have at Mario’s Pizza?

7. Country X has a four-digit postal code assigned to each town, such that the first digit is non-zero, and none of the digits is repeated.

Quantity A	**Quantity B**
The number of possible postal	4,500

codes in Country X

8. Eight athletes compete in a race in which a gold, a silver and a bronze medal will be awarded to the top three finishers, in that order.

Quantity A	**Quantity B**
The number of ways in which the medals can be awarded	$8 \times 3!$

9. Lothar has 6 stamps from Utopia and 4 stamps from Cornucopia in his collection. He will give two stamps of each type to his friend Peggy Sue.

Quantity A	**Quantity B**
The number of ways Lothar can give 4 stamps (two of each type) to Peggy Sue	100

Solutions

1. **30**

   Set up 5 slots, because everyone is going to get a task. Label two of the slots A, two other slots B, and the last slot C. Put in the numbers 5 on down to 1.

5	4	3	2	1
A	A	B	B	On Team

   Finally, you multiply those numbers together, making the product 5 × 4 × 3 × 2 × 1, which is 120. However, there are two sets of 2 repeated labels each, so you must divide by 2! twice to account for these two sets of repeats.

   There are $\frac{5 \times 4 \times 3 \times 2 \times 1}{2! \times 2!} = 30$ possible task assignments.

2. **10**

   In every combination, 2 types of truffles will be in the box. You can set up two slots, both labeled "In Box."

In Box	In Box

Put in the choices you have: 5 for the first slot and 4 for the second (because the boxes have to contain *different* types of truffles).

5	4
In Box	In Box

Multiply 5 by 4 *and then divide by 2!*, because you have 2 repeated labels. Because 2! = 2 × 1 = 2, the result is

(5 × 4) ÷ 2 = 10 possible boxes of truffles

This problem can also be solved with the formula for combinations, because it is a combination of 2 items chosen from a set of 5 (in which order does not matter). Therefore, there are $\frac{5!}{2! \times 3!} = 10$ possible boxes.

3. **20**
   In this problem, each of the digits 1–5 can be either the tens digit, the units digit, or not a digit in the jersey number. What you're really counting is the number of unique jersey numbers.

   Make two slots, one for the tens digit and one for the units digit. You have 5 choices for the tens digit and then only 4 choices for the units digit (since you cannot use the same digit again), resulting in 5 × 4 = 20 possibilities. The slot labels are different (Tens and Units), so you're done.

In a pinch, you can also just list out the jersey numbers, since the number of possibilities is relatively small. Even if you stop partway through, this can be a good way to start, so that you get a sense of the problem:

12, 13, 14, 15, 21, 23, 24, 25, 31, 32, 34, 35, 41, 42, 43, 45, 51, 52, 53, 54 = 5 groups of 4 = 20

4. **36**
   There are 3! ways in which the 3 females can swim. There are 3! ways in which the 3 males can swim. Therefore, there are 3! × 3! ways in which the entire pod can swim:

   $$3! \times 3! = 3 \times 2 \times 1 \times 3 \times 2 \times 1 = 6 \times 6 = 36$$

   This is a multiple arrangements problem, in which you have two separate pools (females and males).

5. **35**
   In this problem, 3 people are chosen for the Superman meeting from a larger pool of 7 people in the delegation.

   Set up 3 slots, all labeled "S" for you know who. Fill the slots in: 7 for the first slot, then 6 and 5.

7	6	5
S	S	S

After you multiply the numbers, you need to divide by 3 factorial, because the three slots are labeled identically.

$$\frac{5}{20} + \frac{8}{20} = \frac{13}{20}$$

Alternatively, you can use the textbook combinations formula:

$$\frac{n!}{k!(n-k)!} = \frac{7!}{3!4!} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

Note that you must divide by both 3! and 4! in this formula.

6. **20**

   Consider the toppings first. Set up 4 slots for the toppings, and since order literally doesn't matter, you can label all the slots the same. Fill in the numbers from 5 on down (notice that the 4 toppings are unique).

5	4	3	2
T	T	T	T

Now multiply the numbers, and don't forget to divide by 4! because the labels are all the same. So the number of 4-topping pizzas is actually only

$$g(y) = y^2 - \frac{1}{y+1}$$

Alternatively, use the combinations formula to count the combinations of toppings: $\frac{5!}{4! \times 1!} = 5$. Or use an intuitive approach: choosing 4 toppings out of 5 is equivalent to choosing the 1 topping that will not be on the pizza. There are only 5 ways to do that.

If each of these pizzas can also be offered in 2 choices of crust, there are 5 × 2 = 10 possible pizzas with 4 toppings and either crust.

Finally, the same logic applies for extra cheese and regular. The number of pizzas with 4 toppings, either crust, and either version of cheese is 5 × 2 × 2 = 20.

7. **(A)**
   Use slots to solve this problem. The first slot can be filled by any one of the digits from 1 through 9, because 0 is disallowed. The second digit has no restriction involving 0; however, the digit that was used in the first slot may not be reused. Thus, the second slot also has 9 possibilities. The third and fourth slots may not use previously used digits, so they may be filled with 8 and 7 different digits, respectively. The total number of possible postal codes is therefore:

$$9 \times 9 \times 8 \times 7 = 4{,}536$$

Quantity A	**Quantity B**
The number of possible postal codes in Country X = **4,536**	4,500

Therefore, **Quantity A is greater**.

8. **(A)**

Set up three slots, labeled *G*, *S,* and *B* to indicate the 3 different medals. You have 8 choices for the gold, 7 choices for the silver, and 6 choices for the bronze. So the number of ways to award the medals is 8 × 7 × 6 = 336.

Compare this number to 8 × 3!:

$$8 \times 3! = 8 \times 3 \times 2 \times 1 = 8 \times 6 = 48$$

Rewrite the quantities:

Quantity A	**Quantity B**
The number of ways in which the medals can be awarded = **336**	8 × 3! = **48**

Quantity A is greater.

9. **(B)**

This exercise can be regarded as two successive “pick a group” problems. First, Lothar picks 2 out of 6 Utopian stamps, and then 2 out of 4 Cornucopian stamps. Each selection may be computed by using the combinations formula to compute the number of groups of size 2 of each type of stamp. To get the total number of ways, multiply the Utopian result and the Cornucopian result:

$$\text{Total number of ways} = \left(\frac{6!}{2!4!}\right) \times \left(\frac{4!}{2!2!}\right)$$

$$= \left(\frac{6 \times 5}{2 \times 1}\right) \times \left(\frac{4 \times 3}{2 \times 1}\right) = 15 \times 6 = 90$$

<u>Quantity A</u>	<u>Quantity B</u>
The number of ways Lothar can give four stamps (two of each type) to Peggy Sue = **90**	100

Quantity B is greater.

Chapter 27

PROBABILITY

In This Chapter...

"1" Is the Greatest Probability

More Than One Event: "AND" versus "OR"

The "1 – x" Probability Trick

More Forks in the Road

Chapter 27

Probability

Probability is a quantity that expresses the chance, or likelihood, of an event. In other words, it measures how often an event will occur in a long series of repeated trials.

For events with countable outcomes, probability is defined by the following fraction:

$$\text{Probability} = \frac{\text{Number of } desired \text{ or } successful \text{ outcomes}}{\text{Total number of } possible \text{ outcomes}}$$

This fraction assumes all *outcomes are equally likely.* If not, the math can be more complicated (more on this later).

As a simple illustration, rolling a die (singular for dice) has six possible outcomes: 1, 2, 3, 4, 5, and 6. The probability of rolling a “5” is 1/6, because the “5” corresponds to only one of those outcomes. The probability of rolling a prime number, though, is 3/6, which simplifies to 1/2, because in that case, three of the outcomes—2, 3, and 5—are considered successes.

Again, all the outcomes must be equally likely. For instance, you could say that the lottery has only two “outcomes”—win or lose—but that does not mean the probability of winning the lottery is 1/2. If you want to calculate the correct probability of winning the lottery, you must find *all of the possible* equally likely outcomes. In other words, you have to count up all the specific combinations of differently numbered balls in the lottery to determine the correct probability of winning the lottery.

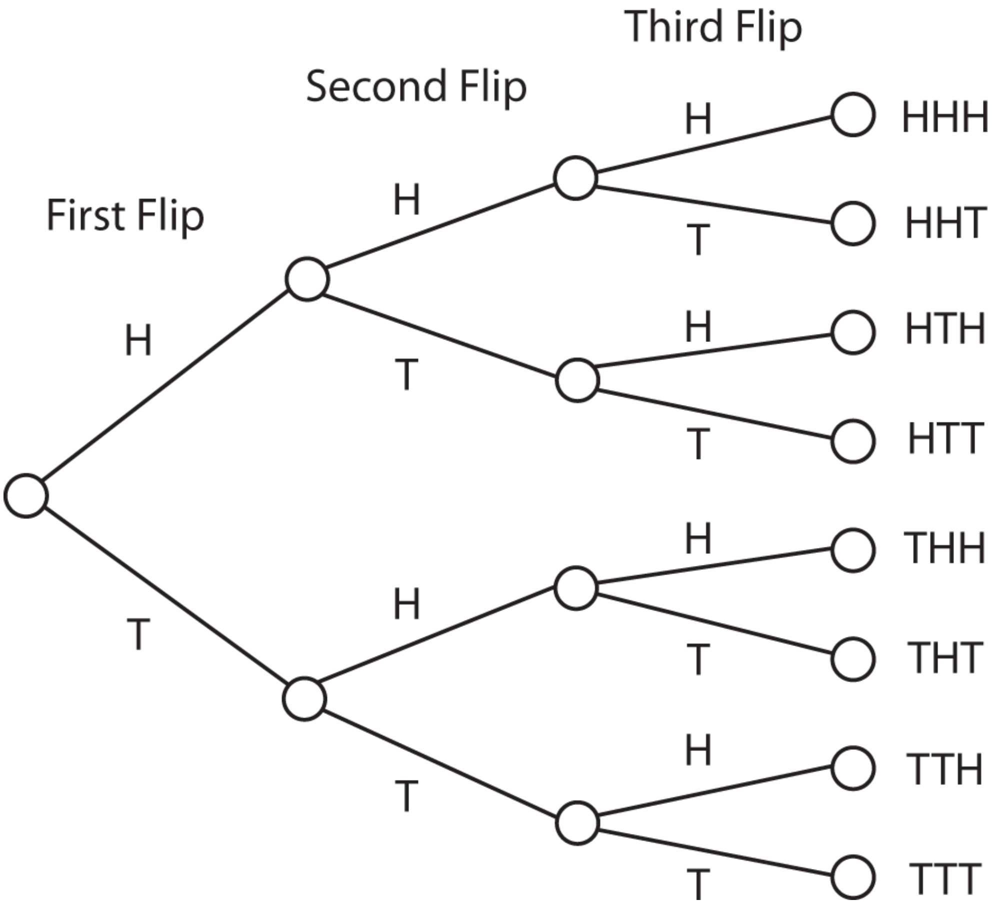
In some problems, you will have to think carefully about how to break a situation down into equally likely outcomes. Consider the following problem:

> If a fair coin is tossed three times, what is the probability that it will turn up heads exactly twice?

You may be tempted to say that there are four possibilities—no heads, 1 head, 2 heads, and 3 heads—and that the probability of 2 heads is thus 1/4. You would be wrong, though, because those four outcomes are not equally likely. You are much more likely to get 1 or 2 heads than to get all heads or all tails. Instead, you have to formulate equally likely outcomes in terms of the outcome of each flip:

HHH HHT HTH THH HTT THT
TTH TTT

If you have trouble formulating this list from scratch, you can use a **counting tree**, which breaks down possible outcomes step by step, with only one decision at each branch of the tree. An example is to the left.



These eight outcomes are equally likely, because the coin is equally likely to come up heads or tails at each flip. Three outcomes on this list (HHT, HTH, THH) have heads exactly twice, so the probability of exactly two heads is 3/8.

This result can also be written:

P(exactly 2 heads) = 3/8.

"1" Is the Greatest Probability

The greatest probability—the *certainty* that an event will occur—is 1. Thus, a probability of 1 means that the event must occur. For example:

The probability that you roll a fair die once, and it lands on a number less than seven, is certain, or 1:

$$\frac{\text{Number of } successful \text{outcomes}}{\text{Total number of possible outcomes}} = \frac{6}{6} = \mathbf{1}$$

As a percent, this certainty is expressed as 100%.

The lowest probability—the *impossibility* that an event will occur—is 0. Thus, a probability of 0 means that an event will NOT occur. For example, the probability that you roll a fair die once and it lands on the number 9 is impossible—a probability of 0:

$$\frac{\text{Number of } successful \text{outcomes}}{\text{Total number of possible outcomes}} = \frac{0}{6} = \mathbf{0}$$

As a percent, this impossibility is expressed as 0%.

Thus, probabilities can also be expressed as percents between 0 percent and 100 percent, inclusive, or fractions between 0 and 1, inclusive.

More Than One Event: “AND” versus “OR”

Probability problems that deal with multiple events usually involve two primary operations: multiplication and addition. The key to understanding probability is to understand *when you must multiply* and *when you must add.*

Assume that *X* and *Y* are *independent* events. Two events are said to be *independent* if the likelihood of one occurring does not depend on the likelihood of the other occurring. **To determine the probability that event *X* AND event *Y* will both occur, MULTIPLY the two probabilities together.** Note that the events must be independent for this to work.

For example:

> What is the probability that a fair coin flipped twice will land on heads both times?

This is an “AND” problem, because it is asking for the probability that the coin will land on heads on the first flip AND on the second flip.

Notice that there's a *sequence* of events here: the first flip happens, *then* the second flip happens. Fortunately, many “AND” problems involve a sequence like this one, or you can pretend that there is one. For instance, if you are asked about a coin flip and a roll of a die, you can pretend that the coin flip comes before the die roll.

Either way, you can imagine the sequence of events as a series of “forks in the road.” In the case of the two coin flips, the first fork is the first flip, and you want to take the “heads” path (not the “tails” one). Then you come to the *second* fork, and again, you want to take the "heads" path.

The probability that the coin will land on heads on the first flip is 1/2. The probability that the coin will land on heads on the second flip is 1/2. These events *are* independent of each other.

Therefore, to determine the probability that the coin will land on heads on both flips, multiply the probabilities: $\frac{2}{9} + \frac{5}{9} = \frac{7}{9}$.

Note that the probability of having BOTH flips come up heads (1/4) is less than the probability of having just one flip come up heads (1/2). This should make intuitive sense. If you define success in a more constrained way (e.g., “to win, BOTH this AND that have to happen”), then the probability of success will be lower. In other words, you have to take the correct path at the first fork AND at the second fork. That's harder to do.

The operation of multiplication should also make sense. Typical probabilities are fractions between 0 and 1. When you multiply together two such fractions, you get a *smaller* result, which means a lower probability.

Now assume that *X* and *Y* are *mutually exclusive* events (meaning that the two events cannot both occur). **To determine the probability that event *X* OR event *Y* will occur, ADD the two probabilities together.**

For example:

> What is the probability that a fair die rolled once will land on *either* 4 or 5?

This is an "OR" problem, because it is asking for the probability that the die will land on either 4 **OR** 5. The probability that the die will land on 4 is 1/6. The probability that the die will land on 5 is 1/6. The two outcomes are mutually exclusive: the die cannot land on BOTH 4 and 5 at the same time.

Therefore, to find the probability that the die will land on either 4 or 5, add the probabilities: $\frac{40\pi}{100\pi} = \frac{4}{10} = \frac{2}{5}$.

In this case, you don't have a *sequence* of events. There is just one roll of the die. The two events you care about are two different possible outcomes from that same roll.

In other words, you don't have two successive "forks in the road" as you did before. Now you have just *one* fork—one roll of the die—with six different paths coming out of it, one for each possible number you can roll (1, 2, 3, 4, 5, or 6). The two paths you care about (rolling a 4 or rolling a 5) are two possible events resulting from the *same* fork in the road. This situation is common for "OR" problems.

Note that the probability of having the die come up either 4 or 5 (1/3) is greater than the probability of a 4 by itself (1/6) or of a 5 by itself (1/6). This should make intuitive sense. If you define success in a less constrained way (e.g., "I can win EITHER this way OR that way"), then the probability of success will be higher. The operation of addition should also make sense. Typical probabilities are fractions between 0 and 1. When you add together two such fractions, you get a *larger* result, which means a higher probability.

All that said, the majority of probability questions on the GRE are of the "AND" variety. When in doubt, *multiply*. Most GRE probability problems just want you to multiply two or three fractions together.

Check Your Skills

1. If a die is rolled twice, what is the probability that it will land on an even number both times?

2. Eight runners in a race are equally likely to win the race. What is the probability that the race will be won by the runner in lane 1 OR the runner in lane 8?

Advanced note: For adding "OR" probabillities, up until now it has been assumed that the events are *mutually exclusive* (meaning that both events cannot occur). What happens if the events are *not* mutually exclusive?

If that is the case, and you simply add the probabilities, you will be double-counting the instances when *both* events occur. Thus, you must *subtract out* the probability that both events occur.

If events *X* and *Y* are not mutually exclusive, then *P*(*X* OR *Y*) = *P*(*X*) + *P*(*Y*) − *P*(*X* AND *Y*). For example:

> Suppose a box contains 20 balls. Ten balls are white and marked with the integers 1–10. The other 10 balls are red and marked with the integers 11–20. If one ball is selected, what is the probability that the ball will be white OR will be marked with an even number?

Because half the balls are white and half are marked with an even number *P*(white) + *P*(even) would give you $\frac{4}{4} + \frac{1}{4}$, which equals 1. **This is incorrect!** You must subtract out the probability that the ball is both white AND marked with an even number. There are 5 such balls out of 20. Thus, the correct answer is: *P*(white or even) = *P*(white) + *P*(even) − *P*(white AND even) = $\mathbf{area} = \frac{1}{2}\,\mathbf{(base)} \times \mathbf{(height)}$.

Check Your Skills

3. A fair die is rolled and a fair coin is flipped. What is the probability that either the die will come up 2 or 3, OR the coin will land heads up?

The “1 – x” Probability Trick

As shown in the previous section, you can solve “OR” problems (explicit or disguised) by combining the probabilities of individual events. If there are many individual events, though, such calculation may be tedious and time-consuming. The good news is that you may not have to perform these calculations. In certain types of “OR” problems, the probability of the desired event not happening may be much easier to calculate.

For example, in the previous section, you could have calculated the probability of getting at least one head on two flips by considering how you would NOT get at least one head. However, it would not be too much work to compute the probability directly, using the slightly more complicated “OR” formula.

Say, however, that a salesperson makes five sales calls, and you want to find the likelihood that he or she makes *at least* one sale. If you try to calculate this probability directly, you will have to confront five separate possibilities that constitute “success”: exactly one sale, exactly two sales, exactly three sales, exactly four sales, or exactly five sales. It would seem that you would have no choice but to calculate each of those probabilities separately and then add them together. This will be far too much work, especially under timed conditions.

However, consider the probability of *failure*—that is, the salesperson *does not* make at least one sale. Now you have only one possibility to consider: 0 sales. You can now calculate the probability in which you are interested, because for *any* event, the following relationship is true:

Probability of SUCCESS + **Probability of FAILURE** = **1**
(the event happens) (it does *not* happen)

If, on a GRE problem, "success" contains ***multiple possibilities****—especially if the wording contains phrases such as* ***"at least"*** *and* ***"at most"****—then consider finding the probability that success* ***does not happen****. If you can find this "failure" probability more easily (call it* x*), then the probability you really want to find will be* ***1*** *−* ***x****.* For example:

> What is the probability that, on three rolls of a single fair die, AT LEAST ONE of the rolls will be a 6?

You could list all the possible outcomes of three rolls of a die (1–1–1, 1–1–2, 1–1–3, etc.), and then determine how many of them have at least one 6, but this would be very time-consuming. Instead, it is easier to think of this problem in reverse before solving:

> Failure: What is the probability that NONE of the rolls will yield a 6?

On each roll, there is a $\frac{1}{2}$ probability that the die will NOT yield a 6. Thus, the probability that on all three rolls the die will *not* yield a 6 is:

$$\frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} = \frac{125}{216}.$$

Now, success was originally defined as rolling at least one 6. Because you have found the probability of failure, you can answer the original question by subtracting this probability from 1:

The probability that at least one 6 will be rolled is $1 - \frac{125}{216} = \frac{91}{216}$.

Check Your Skills

4. If a die is rolled twice, what is the probability that it will land on an even number at least once?

More Forks in the Road

Sometimes the outcome of the first event will affect the probability of a subsequent event. For example:

> In a box with 10 blocks, 3 of which are red, what is the probability of picking out a red block at random on each of your first two tries? Assume that you do NOT replace the first block after you have picked it.

Because this is an “AND” problem, you must find the probability of both events and multiply them together. Consider how easy it is to make the following mistake:

You compute the probability of picking a red block on your first pick as $\frac{23}{7}$.

You compute the probability of picking a red block on your second pick as $\frac{23}{7}$.

So you compute the probability of picking a red block on both picks as $40\% < \frac{8}{18} < 0.8 \ldots$

Oops. This solution is *incorrect*, because it does not take into account how the first event influences the second event in this problem. When you *don't* put that first block back and restore the original state of things, the first pick affects the second pick.

In other words, for this kind of problem, the road you take at the first fork matters at the second fork.

Concretely, you incorrectly computed the second probability in the example above. The two events are NOT independent.

If a red block is chosen on the first pick, then the number of blocks now in the box has decreased from 10 to **9**. Additionally, the number of red blocks now in the box has decreased from 3 to **2**.

So the probability of choosing a red block on the second pick, *given that you chose red on the first pick,* is different from the probability of choosing a red block on the first pick, before anything was ever chosen.

Fortunately, you can still think of this problem as a sequence of forks in the road. You will still multiply probabilities. Just make sure you compute the probability at the *second* fork differently, taking into account the path you took from the first fork.

The CORRECT solution to this problem is as follows:

> The probability of picking a red block on your first pick is $\frac{23}{7}$.

The probability of picking a red block on your second pick, *given that you already picked a red block on your first pick,* is $\frac{1}{2}$. There are two red blocks at this point, and you're choosing at random out of a set of nine.

Therefore, the probability of picking a red block on both picks is $x = \frac{48 \times 600}{2 \times 144} = 100$.

Do not forget to analyze events by considering whether one event affects subsequent events. The first roll of a die or flip of a coin has no effect on any subsequent rolls or flips. However, the first pick of an object out of a box *does* affect subsequent picks if you do not replace that object. This scenario is called “no replacement” or “without replacement.”

In such a scenario, always adjust the subsequent probabilities. For this problem, you adjust the second pick's probability from $\frac{23}{7}$ to $\frac{1}{2}$.

If you *are* supposed to replace the object, the problem should clearly tell you so. In that sort of scenario (called “with replacement”), the first pick does not affect the second pick. That's because you're deliberately restoring the original state of affairs, and you don't have to adjust subsequent probabilities.

Again, you always assume that "original restoration" is the case with coin flips and rolls of a die. With each flip or roll, you're starting fresh. So getting heads on a second or third coin flip is still just $\frac{1}{2}$, with no adjustment.

Check Your Skills

5. A drawer contains 7 white shirts and 3 red shirts. What is the probability of picking a white shirt and then a red shirt, if the first shirt is not put back in?

Check Your Skills Answer Key

CHECK YOUR SKILLS

1. $\frac{1}{2}$

   For each roll of the die, the probability of an even number is 3/6, which simplifies to 1/2. Multiply the individual probabilities because the two outcomes are independent: *P* = 1/2 × 1/2 = 1/4.

2. $\frac{1}{2}$

   P(1) = 1/8, *P*(8) = 1/8, *P*(1 or 8) = 1/8 + 1/8 = 2/8 = 1/4.

3. $\frac{1}{2}$

   Rolling a 2 or 3 on the die and flipping a heads on the coin are not mutually exclusive (that is, it is possible for both events to happen). Thus, *P*(one event OR the other event) = *P*(one event) + *P*(the other event) − *P*(both events). In this scenario, *P*(2 or 3 on the die) = 1/6 + 1/6 = 2/6. *P*(heads on the coin) = 1/2 = 3/6. *P*(both) = 2/6 × 1/2 = 1/6 because the die roll and coin flip are independent events. Therefore, *P*(2 or 3 OR heads) = 2/6 + 3/6 − 1/6 = 4/6 = 2/3.

4.

$$\frac{1}{2}$$

If the die does not land on an even number at least once, then it must have landed on an odd number both times. For each throw, the probability of an odd number is 3/6 = 1/2. Multiply the individual probabilities to get the probability of two odd numbers in a row: x = 1/2 × 1/2 = 1/4. Then the probability of at least one even number is 1 − x = 1 − 1/4, which is 3/4.

5. $\frac{23}{7}$

There are 10 shirts total, with seven white shirts and three red shirts.

Probability of picking a white shirt first: 7/10.

Probability of picking a red shirt next, given that a white shirt was chosen first (all three reds are still there, but there are only nine shirts now to choose from): 3/9 = 1/3.

Probability of picking white first, then red: 7/10 × 1/3 = 7/30.

Problem Set

Solve the following problems. Express probabilities as fractions or percentages unless otherwise instructed.

1. What is the probability that the sum of two dice rolls will yield a 4 OR 6?
2. What is the probability that the sum of two dice rolls will yield anything but an 8?
3. What is the probability that the sum of two dice rolls will yield a 7, and then when both are thrown again, their sum will again yield a 7?
4. What is the probability that the sum of two dice rolls will yield a 5, and then when both are thrown again, their sum will yield a 9?

5. At a certain pizzeria, 1/6 of the pizzas sold in a week were cheese, and 1/5 of the OTHER pizzas sold were pepperoni. If Brandon bought a randomly chosen pizza from the pizzeria that week, what is the probability that he ordered a pepperoni?

6. John invites 12 friends to a dinner party, half of whom are men. Exactly one man and one woman are bringing desserts. If one person from this group is selected at random, what is the probability that it is a man who is not bringing a dessert OR that it is a woman?

7. A fair coin is flipped 5 times.

Quantity A	**Quantity B**
The probability of getting more heads than tails	$\frac{1}{2}$

8. A jar contains 3 red and 2 white marbles. Two marbles are picked without replacement.

Quantity A	**Quantity B**
The probability of picking two red marbles	The probability of picking exactly one

red and one white marble

9. A die is rolled *n* times, where *n* is at least 3.

Quantity A	**Quantity B**
The probability that at least one of the throws yields a 6	$\frac{1}{2}$

Solutions

1. $\frac{1}{2}$

   There are 36 ways in which 2 dice can be thrown (6 × 6 = 36). The combinations that yield sums of 4 and 6 are 1 + 3, 2 + 2, 3 + 1, 1 + 5, 2 + 4, 3 + 3, 4 + 2, and 5 + 1, for a total of 8 different combinations. Therefore, the probability is 8/36, which simplifies to 2/9.

2. $\frac{23}{7}$

   Solve this problem by calculating the probability that the sum WILL yield a sum of 8, and then subtract the result from 1. There are 5 combinations of 2 dice that yield a sum of 8: 2 + 6, 3 + 5, 4 + 4, 5 + 3, and 6 + 2. (Note that 7 + 1 is not a valid combination, as there is no 7 on a standard die.) Therefore, the probability that the sum will be 8 is 5/36, and the probability that the sum will NOT be 8 is 1 − 5/36, which equals 31/36.

3. $\frac{23}{7}$

   There are 36 ways in which 2 dice can be thrown (6 × 6 = 36). The combinations that yield a sum of 7 are 1 + 6, 2 + 5, 3 + 4, 4 + 3, 5 + 2, and 6 + 1, for a total of 6 different combinations. Therefore, the probability of rolling a 7 is 6/36, which simplifies to 1/6. To find the probability that this will happen twice in a row, multiply 1/6 by 1/6 to get 1/36.

4. $\frac{23}{7}$

   First, find the individual probability of each event. The probability of rolling a 5 is 4/36, or 1/9, since there are 4 ways to roll a sum of 5 (1 + 4, 2 + 3, 3 + 2, and 4 + 1). The probability of rolling a 9 is also 4/36, or 1/9, since there are 4 ways to roll a sum of 9 (3 + 6, 4 + 5, 5 + 4, and 6 + 3). To find the probability that both events will happen in succession, multiply: 1/9 × 1/9 = 1/81.

5. 1/6

   If 1/6 of the pizzas were cheese, 5/6 of the pizzas were not. Because 1/5 of these 5/6 were pepperoni, multiply to find the total portion: 1/5 × 5/6 = 5/30 = 1/6. If 1/6 of the pizzas were pepperoni, there is a 1/6 chance that Brandon bought a pepperoni pizza.

6. $\frac{11}{12}$

   Six women are invited and 5 men *who are not bringing a dessert* are invited. Thus, 6 + 5, which is 11, out of 12 would fit the description.

7. **(C)**

   Because heads and tails are equally likely, it follows that the probability of getting more heads than tails should be exactly the same as the probability of getting more tails than heads. The only remaining option is that you might get equally as many heads and tails. However, because the total number of coin flips is an odd number, the latter is impossible. Therefore, the probability of getting more heads than tails must be exactly 1/2. (It is, of course, also possible to compute this probability directly by considering the cases of getting 5, 4, or 3 heads separately. However, this approach would be very time-consuming.)

Another way of thinking about it is that, for every set of flips that has more heads than tails, there is a corresponding set of flips, in which every flip gets the opposite result, that has more tails. For instance, the sequence of throws *HHHHH* is balanced by the sequence *TTTTT*. The sequence *HHHHT* is balanced by the sequence *TTTTH*.

Therefore, **the two quantities are equal**.

8. **(B)**
First, compute the probability of picking two red marbles. This is given by:

$$P(RR) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$$

Next, consider the probability of picking a red marble followed by a white marble:

$$P(RW) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$$

However, this is not the only way to pick one red AND one white marble; you could have picked the white one first, followed by the red one:

$$P(RW) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$$

This event is mutually exclusive from picking a red marble followed by a white marble. Thus, the total probability of picking one red AND one white marble is the sum of the probabilities of *RW* and *WR*, yielding an answer of:

$$P(RW \, \mathrm{OR} \, WR) = \frac{3}{10} + \frac{3}{10} = 2 \times \left(\frac{3}{10}\right) = \frac{6}{10} = \frac{3}{5}$$

Quantity A	**Quantity B**
The probability of picking two red marbles = **3/10**	The probability of picking one red and one white marble = **3/5**

Therefore, **Quantity B is greater**.

9. **(D)**

The easiest way to compute the probability in question is through the "1 − *x*" shortcut. To do so, imagine the opposite of the event of interest, namely, that *none* of the *n* throws yields a 6. The probability of a single throw not yielding a 6 is 5/6, and because each throw is independent, the cumulative probability of none of the *n* throws yielding a 6 is found by multiplication:

$$P(\text{No 6 in } n \text{ throws}) = \left(\frac{5}{6}\right)^n$$

Powers of fractions less than one get smaller as the exponent increases. Thus, this probability will become very small for large values of *n*, such that the probability of getting at least one 6 (which is $1 - \left(\frac{5}{6}\right)^n$) will come closer and closer to 1. Thus, as *n* increases, it becomes more and more certain that a 6 will be thrown. The question now is, What is the smallest that the probability of getting at least one six could be? To answer that question, you should set *n* to its lowest possible value, which is 3. In that case, the probability of never getting a 6 is given by:

$$P(\text{No 6 in three throws}) = \left(\frac{5}{6}\right)^3 = \frac{125}{216}$$

Use the calculator
to
compute the
numerator
and denominator
separately.

However, the probability of getting at least one 6 in three throws is given by:

$$P(\text{At least one 6 in three throws}) = 1 - \frac{125}{216} = \frac{91}{216}$$

This value is less than 1/2. As you saw earlier, however, as *n* grows, it becomes ever more likely that at least one throw will yield a 6, so that the probability eventually surpasses 1/2. Thus, Quantity A can be less than or greater than 1/2. Thus, **the relationship cannot be determined from the information given**.

Chapter 28

MINOR PROBLEM TYPES

In This Chapter...

Optimization

Grouping

Overlapping Sets

Chapter 28

Minor Problem Types

The GRE occasionally contains problems that fall into one of three categories:

> *Optimization*: maximizing or minimizing a quantity by choosing optimal values of related quantities.
> *Grouping*: putting people or items into different groups to fit some criteria.
> *Overlapping sets*: people or items that can belong in one of two groups, neither, or both.

You should approach all three of these problem types with the same general outlook, although it is unlikely that you will see more than one of them on the same administration of the GRE. The general approach is to focus on **extreme scenarios**.

You should mind the following three considerations when considering any grouping or optimization problem:

> Be aware of both ***explicit constraints*** (restrictions actually stated in the text) and ***hidden constraints*** (restrictions implied by the real-world aspects of a problem). For instance, in a problem requiring

the separation of 40 people into 6 groups, hidden constraints require the number of people in each group to be a positive whole number.

In most cases, you can maximize or minimize quantities (or optimize schedules, etc.) by ***choosing the highest or lowest values*** of the variables that you are allowed to select.

For ***overlapping sets***, remember that people/items that fit in both categories lessen the number of people/items in just one category. Thus, all other things being equal, the ***more people/items in "both," the fewer in "just one" and the more in "neither."***

Optimization

In general optimization problems, you are asked to maximize or minimize some quantity, given constraints on other quantities. These quantities are all related through some equation.

Consider the following problem:

> The guests at a football banquet consumed a total of 401 pounds of food. If no individual guest consumed more than 2.5 pounds of food, what is the minimum number of guests that could have attended the banquet?

You can visualize the underlying equation in the following table:

Pounds of food per guest	×	Guests	=	Total pounds of food
At MOST		At LEAST		EXACTLY
2.5	×	???	=	401
maximize		*minimize*		*constant*

Notice that finding the *minimum* value of the number of guests involves using the maximum pounds of food per guest, because the two quantities multiply to a constant. This sort of inversion (i.e., maximizing one thing to minimize another) is typical.

Begin by considering the extreme case in which each guest eats as much food as possible, or 2.5 pounds apiece. The corresponding number of guests at the banquet works out to 401/2.5 = 160.4 people.

However, you obviously cannot have a fractional number of guests at the banquet. Thus, the answer must be rounded. To determine whether to round up or down, consider the explicit constraint: the amount of food per guest is a *maximum* of 2.5 pounds per guest. Therefore, the *minimum* number of guests is 160.4 (if guests could be fractional), and you must *round up* to make the number of guests an integer: 161.

Note the careful reasoning required! Although the phrase "*minimum* number of guests" may tempt you to round down, you will get an incorrect answer if you do so. In general, as you solve this sort of problem, put the extreme case into the underlying equation, and solve. Then round appropriately.

Check Your Skills

1. If no one in a group of friends has more than $75, what is the smallest number of people who could be in the group if the group purchases a flat-screen TV that costs $1,100?

Grouping

In grouping problems, you make complete groups of items, drawing these items out of a larger pool. The goal is usually to maximize or minimize some quantity, such as the number of complete groups or the number of leftover items that do not fit into complete groups. As such, these problems are often really a special case of optimization problems. One approach is to determine the **limiting factor** on the number of complete groups. That is, if you need different types of items for a complete group, figure out how many groups you can make with each item, ignoring the other types (as if you had unlimited quantities of those other items). Then compare your results. For example:

> Orange Computers is breaking up its conference attendees into groups. Each group must have exactly one person from Division A, two people from Division B, and three people from Division C. There are 20 people from Division A, 30 people from Division B, and 40 people from Division C at the conference. What is the smallest number of people who will NOT be able to be assigned to a group?

The first step is to find out how many groups you can make with the people from each division separately, ignoring the other divisions. There are enough Division A people for 20 groups, but only enough Division B people for 15 groups (= 30 people ÷ 2 people per group). As for Division C, there are only enough people for 13 groups, because 40 people ÷ 3 people per group = 13 groups, plus one person left over. So the limiting factor is Division C: only 13 complete groups can be formed. These 13 groups will take up 13 Division A people (leaving 20 − 13 = 7 left over) and 26 Division B people (leaving 30 − 26 = 4 left over). Together with the 1 Division C person left over, there are 1 + 4 + 7 = 12 people will be left over in total.

For some grouping problems, you may want to think about the **most or least evenly distributed** arrangements of the items. That is, assign items to groups as evenly (or unevenly) as possible to create extreme cases.

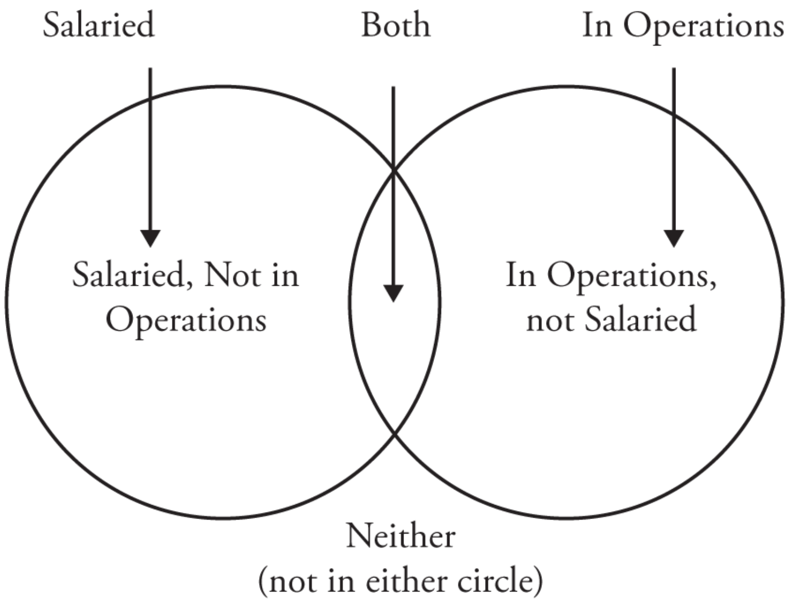
Check Your Skills

2. A salad dressing requires oil, vinegar, and water in the ratio 2:1:3. If Oliver has 1 cup of oil, 1/3 cup of vinegar, and 2 cups of water, what is the maximum number of cups of dressing that he can mix?

Overlapping Sets

In overlapping sets, people or items will be categorized by their "membership" or "non-membership" in either of two groups. For example, workers in a factory could be salaried or non-salaried. They could also work in an Operations role, or *not* work in an Operations role. These problems can be represented by a simple Venn diagram, as shown:

Workers in a Factory



The two key points to note are the following:

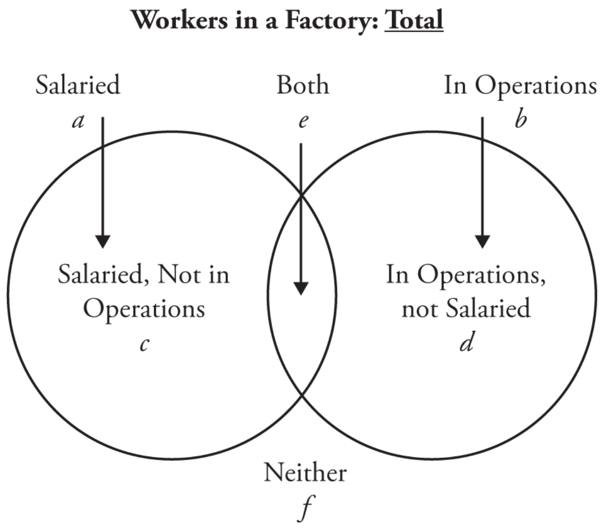
1. The workers will *always* fall into one of four groups:

Salaried and in an Operations role (i.e., “both”)
Salaried and NOT in an Operations role
NOT salaried and in an Operations role
NOT salaried and NOT in an Operations role

Therefore, there are four unknowns in this type of problem, generally (although the question itself may only require that you work with two or three of them).

2. The problem will often give you total amounts for the groups (salaried, and in Operations), and you will have to use logic to figure out whichever unknown the question is asking about.

The various sections can be labeled as follows:



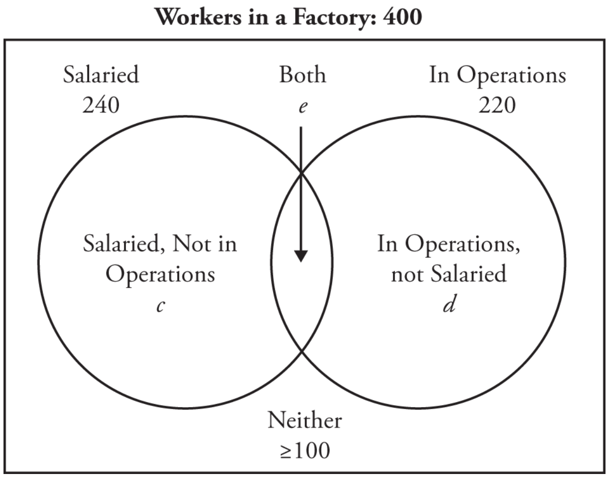
As you can see, $c = a - e$; $d = b - e$, and Total $= a + b - e + f$.

Alternatively, Total $= c + d + e + f$.

Here’s an example:

> At Factory X, there are 400 total workers. Of these workers, 240 are salaried, and 220 work in Operations. If at least 100 of the workers are non-salaried and do not work in Operations, what’s the minimum number of workers who both are salaried and work in Operations?

Graphically, this looks like:



Mathematically, you can use Total = *a* + *b* − *e* + *f*.

$$\begin{aligned}
&400 = 240 + 220 - e + (\geq 100)\\
&e = 240 + 220 - 400 + (\geq 100)\\
&e = 60 + (\geq 100)\\
&e = \ \geq 160
\end{aligned}$$

Thus, at *least* 160 workers are salaried and work in operations.

Check Your Skills

3. Of 320 consumers, 200 eat strawberries and 300 eat oranges. If all 320 eat at least one of the fruits, how many eat both?

Check Your Skills Answer Key

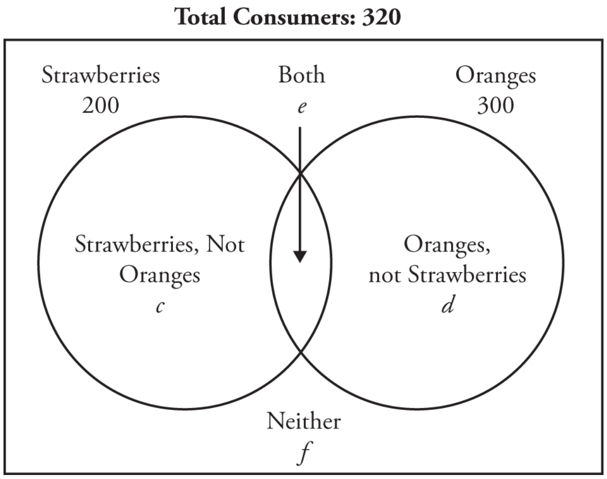
1. **15**
   The group will be as small as possible when everyone contributes as much as they’re able to. The most anyone can contribute is $75, so assume that everyone contributes $75:

   $$x = \frac{15}{8} \times \frac{3}{4} = \frac{45}{32}$$

   However, 14 people contributing $75 would only give $1,050. Therefore, you need to round up. The smallest number of people that could be in the group is 15.

2. **2 cups**
   Try the limits. If Oliver used 1 cup of oil, his recipe would require 1/2 cup of vinegar and 1 1/2 cups of water. He does not have enough vinegar. If he used 1/3 cup of vinegar, he would need 2/3 cups of oil and 1 cup of water, both of which he has. He would then have 2/3 + 1/3 + 1 = 2 cups of dressing. He cannot possibly make more dressing than this, because he does not have any more vinegar.

3. **180**
   Graphically:



Mathematically, you can use Total = $a + b - e + f$.

Because all of the consumers eat at least one of the fruits, $f = 0$. So:

$$\begin{aligned} 320 &= 200 + 300 - e + 0 \\ 320 &= 500 - e \\ e &= 180 \end{aligned}$$

Problem Set

1. Velma has exactly one week to learn all 71 Japanese hiragana characters. If she can learn at most a dozen of them on any one day and will only have time to learn four of them on Friday, what is the least number of hiragana characters that Velma will have to learn on Saturday?

2. Huey’s Hip Pizza sells two sizes of square pizzas: a small pizza that measures 10 inches on a side and costs $10, and a large pizza that measures 15 inches on a side and costs $20. If two friends go to Huey’s with $30 apiece, how many more square inches of pizza can they buy if they pool their money than if they each purchase pizza alone?

3. An eccentric casino owner decides that his casino should only use chips in $5 and $7 denominations. Which of the following amounts cannot be paid out using these chips?

(A) $31
(B) $29
(C) $26
(D) $23
(E) $21

4. A “Collector’s Coin Set” contains a one-dollar coin, a fifty-cent coin, a quarter (= 25 cents), a dime (= 10 cents), a nickel (= 5 cents), and a penny (= 1 cent). The Coin Sets are sold for the combined face price of the currency. If Colin buys as many Coin Sets as he can with the $25 he has, how much change will Colin have left over?

5. A rock band is holding a concert and selling tickets. All of the tickets will either be premium seating OR allow backstage access after the event. They will sell 1,200 premium seating tickets and 500 that will allow backstage access. If 150 of the tickets will both be premium seating and allow backstage access, how many total tickets will they sell?

6. Susan is writing a novel that will be 950-pages long when finished. She can write 10 pages per day on weekdays and 20 pages per day on weekends.

Quantity A	**Quantity B**
The least number of consecutive days it will take Susan to finish her novel	75

7. Jared has four pennies (1 cent), one nickel (5 cents) and one dime (10 cents).

Quantity A	**Quantity B**
The number of different cent values that Jared can achieve using one or more of his coins	20

8. A ribbon 40-inches long is to be cut into three pieces, each of whose lengths is a different integer number of inches.

Quantity A	**Quantity B**
The least possible length, in inches, of the longest piece	15

9. A farmer sells vegetables to 180 different customers. Of these, 90 of them purchase zucchini and 115 of them purchase cauliflower.

Quantity A	**Quantity B**
The number of customers who purchased both zucchini and cauliflower	The number of customers that purchased neither zucchini nor cauliflower

Solutions

1. **7**
   To minimize the number of hiragana that Velma will have to learn on Saturday, consider the extreme case in which she learns *as many* hiragana *as possible* on the other days. She learns 4 on Friday, leaving 71 − 4 = 67 for the other six days of the week. If Velma learns the maximum of 12 hiragana on the other five days (besides Saturday), then she will have 67 − 5(12) = 7 left for Saturday.

2. **25 square inches**
   First, figure the area of each pizza: the small is 100 square inches, and the large is 225 square inches. If the two friends pool their money, they can buy three large pizzas, which have a total area of 675 square inches. If they buy individually, though, then each friend will have to buy one large pizza and one small pizza, so they will only have a total of 2(100 + 225) = 650 square inches of pizza.

3. **(D)**
   This problem is a grouping problem. You have some integer number of 5’s and some integer number of 7’s. Which of the answer choices cannot be the sum? One efficient way to eliminate choices is first to cross off any multiples of 7 and/or 5; this eliminates choice (E). Now, any other possible sums must have at least one 5 and one 7 in them. So you can subtract off 5’s one at a time until you reach a multiple of 7. (It is easier to subtract 5’s than 7’s, because our number system is

base-10.) Choice (A): 31 − 5 = 26; 26 − 5 = 21, a multiple of 7; this eliminates (A). (In other words, 31 = 3 × 7 + 2 × 5.) Choice (B): 29 − 5 = 24; 24 − 5 = 19; 19 − 5 = 14, a multiple of 7; this eliminates (B). Choice (C): 26 − 5 = 21, a multiple of 7; this eliminates (C). So the answer must be choice (D), 23. You can check by successively subtracting 5 and looking for multiples of 7: 23 − 5 = 18, not a multiple of 7; 18 − 5 = 13, also not a multiple of 7; 13 − 5 = 8, not a multiple of 7; and no smaller result will be a multiple of 7 either.

4. **$0.17**
   The first step is to compute the value of a complete "Collector's Coin Set": $1.00 + $0.50 + $0.25 + $0.10 + $0.05 + $0.01 = $1.91. Now, you need to divide $1.91 into $25. A natural first move is to multiply by 10: for $19.10, Colin can buy 10 complete sets. Now add $1.91 successively. Colin can buy 11 sets for $21.01, 12 sets for $22.92, and 13 sets for $24.83. There is $0.17 left over.

5. **1,550**
   You can use the formula Total = $a + b - e + f$. Because all of the tickets will either be premium seating or allow backstage access, f will equal 0. Therefore:

   Total = 1,200 + 500 − 150 = 1,550

6. **(B)**
   In a week consisting of 5 workdays and 2 weekend days, Susan can write:

$5 \times 10 + 2 \times 20 = 90$ pages

Therefore, in 10 consecutive full weeks (i.e., 70 consecutive days), she can write 900 pages of her novel, leaving another 50 pages to be written. The least number of days it would take Susan to write 50 pages is 3: 2 weekend days and 1 weekday. Thus, it is possible for Susan to finish her novel in 73 days. (This assumes that Susan chooses her start day appropriately, so as to take advantage of as many weekends as possible.) Therefore, **Quantity B is greater**.

7. **(B)**
   Jared can achieve any amount from 1 cent to 19 cents: 1 to 4 cents using the pennies, 5 cents with the nickel, 6 to 9 cents using the nickel along with the pennies, 10 cents using the dime, 11 to 14 cents using the dime along with the pennies, 15 cents using the dime and the nickel, and 16 to 19 cents using the dime and nickel along with the pennies. Notice that 19 cents requires every coin Jared possesses, meaning that 19 is the largest possible value. That makes 19 possible values. Therefore, **Quantity B is greater**.

8. **(C)**
   Minimizing the length of the longest piece is equivalent to maximizing the lengths of the remaining pieces, as long as they are shorter than the longest piece. Suppose that the longest piece were 14 inches long (a choice motivated by wanting to be less than the 15 in Quantity B). That would leave $40 - 14 = 26$ inches to be accounted for by the other two pieces.

Because each piece must be a different number of inches long, those pieces cannot each be 13 inches long. This, in turn, implies that one of the two remaining pieces would have to be more than 13 inches long—but then, that piece would be 14 inches long, again violating the constraint that each piece be of a different length. Thus, the longest piece must be at least 15 inches long, and the shorter pieces could then be 12 and 13 inches long, for a total of 40 inches. Thus, **the two quantities are equal**.

9. **(A)**
   Once again using the formula Total = $a + b - e + f$:

$$\begin{aligned} 320 &= 200 + 300 - e + 0 \\ 320 &= 500 - e \\ e &= 180 \end{aligned}$$

Therefore, there will be 25 *more* customers that purchased both zucchini and cauliflower than those who purchased neither. Thus, **Quantity A is greater**.

Unit Six: Quantitative Comparisons

This unit focuses on one of the GRE's most unique quantitative question types. The Quantitative Comparisons section briefs students on how to attack these problems and provides time-saving strategies.

In This Unit...

Chapter 29: QC Basics & Game Plan

Chapter 30: Algebra

Chapter 31: Fractions, Decimals, and Percents

Chapter 32: Geometry

Chapter 33: Number Properties

Chapter 34: Word Problems

Chapter 29

QC BASICS & GAME PLAN

In This Chapter...

Game Plan, Round 1

Game Plan, Round 2

Game Plan, Round 3

Game Plan, Round 4

Chapter 29

QC Basics & Game Plan

Whether you've spent a lot of time with Quantitative Comparison questions, or whether you're brand new to the question format, it's worth reviewing the basics of QC.

1. For QC questions, you need to compare Quantity A with Quantity B and decide whether:

(A) Quantity A is greater.
(B) Quantity B is greater.
(C) The two quantities are equal.
(D) The relationship cannot be determined from the information given.

2. The first three answer choices have an implied *always* before the word "greater" or "equal." The answer choices can really be thought of this way:

(A) Quantity A is *always* greater.
(B) Quantity B is *always* greater.
(C) The two quantities are *always* equal.
(D) The relationship cannot be determined from the information given, or no consistent relationship exists. For instance, it might be that most of the time, Quantity A is greater, but in just one case, Quantity B is greater, or the two quantities are equal.

D IS DIFFERENT FROM ABC

As mentioned above, answer choices (A), (B), and (C) all have an implied *always*. (D) is different. It's the *sometimes* choice: *sometimes* the comparison goes one way, *sometimes* it goes another way. You can even think of (D) as "not always."

This means that you should treat (D) differently from the other answer choices. We'll come back to this idea shortly.

THE NEED FOR A GAME PLAN

Let's pause for a moment. You've now begun to see how QC can be more complicated than it might seem. The answer choices are fixed, but they don't all work the same way. (D) is different, and you should treat it differently somehow...

Hold on. What do you really need?

You need a **QC Game Plan.**

That is, you need a solid, simple, universal approach to Quantitative Comparison questions.

That Game Plan what we'll develop in this chapter. We'll start with the simplest version in Round 1. In Rounds 2, 3, and 4, we'll add more nuances and techniques. Those nuances and techniques, however, will all fit under the simple core Game Plan.

For example:

x is an integer greater than 0.

Quantity A	**Quantity B**
$x(10 - x)$	25

First, notice that you're given some information up front. You know that *x* is an integer greater than 0. So you should restrict your analysis to cases where $x = 1, 2, 3$, etc. That's still an infinite number of possibilities. But whenever you are given information up front, unpack that information. Make sure you understand it and apply it correctly.

Now try some numbers that fit the given constraint. For instance, if x is 4, then Quantity A is $4(6) = 24$. If x is 7, Quantity A is $7(3) = 21$.

At this stage, that means that answer choices (A) and (C) are no longer possible. You know that Quantity A is not *always* bigger, and you know the values in the two quantities are not *always* equal.

~~A~~ B ~~C~~ D

But that doesn’t mean the answer is (B)!

Almost all the time, Quantity B is bigger. But when x is 5, Quantity A is $5(5) = 25$. In that case, and *only* in that case, the values in the two quantities are *equal*. Whoops!

Although *x* makes Quantity B bigger for literally an infinite number of values, just one counterexample is enough to make (B) the wrong answer. Quantity B is ***not always*** larger.

So the answer is **(D)**.

Game Plan, Round 1

Take a look at this example again:

x is an integer greater than 0.

Quantity A	**Quantity B**
$x(10 - x)$	25

How should you approach this, *or any other,* QC problem?

At its core, the QC Game Plan has just two steps.

1. **Simplify** → 2. **Pick Numbers**

Every tactic and technique for QC will fit under one of these two steps.

1. SIMPLIFY

First, you simplify what you're given, as sensibly as you can.

Simplify means "make simpler," of course: reduce the complexity, combine pieces or break them apart, and so on.

It also means "make sense of." Simplifying a piece of information means making sense of it: making it more concrete and understandable.

In the case of the previous problem, everything's already pretty simple. But as you consider how to simplify, you should take extra care with the upfront information ("x is an integer greater than 0").

Namely, you should always **unpack upfront** information.

We did that earlier by translating from math-y language ("x is an integer greater than 0") to concrete examples ("x is 1, 2, 3, etc."). This step ensured that we didn't accidentally consider the wrong kind of numbers.

In the preceding problem, this upfront information wound up not mattering too much. But if that information were "x is *not* an integer" or "x is less than 0," then the answer would change. The value of 5 would be ruled out for x, and the answer would be (B), not (D).

So unpack what comes upfront.

You should also try to simplify what you see in the columns. Quantity B is already very simple: 25. But Quantity A is a more complex expression. And you can do something to it. Should you distribute and rewrite $x(10 - x)$ as $10x - x^2$?

There's no hard and fast rule. The factored form $x(10 - x)$ is at least as simple as the distributed form $10x - x^2$. In fact, it's probably easier to plug numbers into the original factored form. So you can leave the expression alone and move on.

2. PICK NUMBERS

There are QC problems for which you stop at step 1. Why? At step 1, you **might actually finish the problem**.

As you simplify, you pull on a thread, you keep pulling… and the whole problem unravels. You have the answer, and you're done. Be ready to stop there, enter your answer, and go on to the next problem.

But for other QC problems, you simplify as far as you can, but then you can't simplify any further. What do you do?

Well, you should **pick numbers**.

We did this earlier. We tried $x = 4$ and found that Quantity A is 4(6) = 24.

Because Quantity B is 25, we ruled out answer choices (A) and (C). Nice! With just one test number, we eliminated half the possibilities. That's always the case, in fact. You can always eliminate two of the four answer choices by picking one test number per variable and working out what Quantities A and B are. (In this problem, there's just one variable, so we only had to pick one test number.)

The two answer choices you can eliminate on your first try are either (A) and (B), (A) and (C), or (B) and (C). Notice that you can never eliminate (D) just by picking a number. That's because (D) is different: it's the *sometimes* answer.

We tested $x = 4$. But we didn't know at that point whether Quantity B was *always* going to be bigger or just *sometimes.* Maybe a different test case would have a different result. Or maybe it wouldn't.

At this point, you shouldn't look for another case that keeps making Quantity B bigger. You should look for a case that has a *different result.*

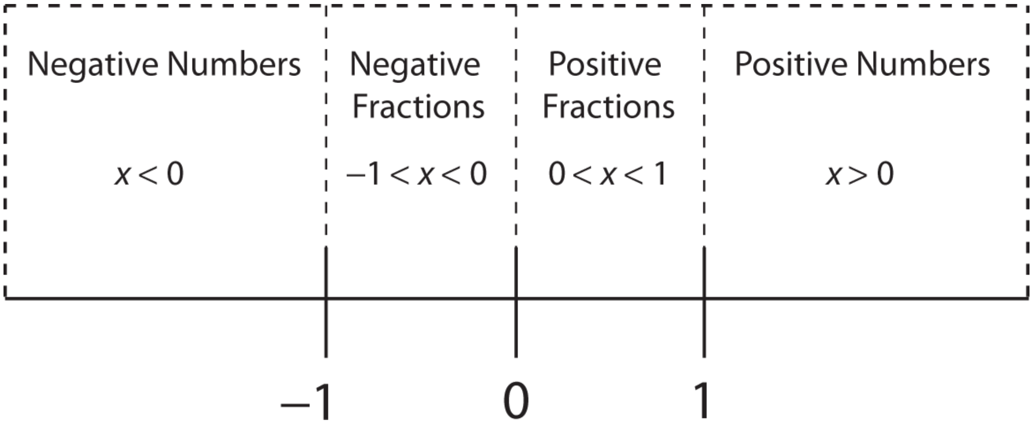
That is, you should **try to prove (D)**. Be a skeptic about (A), (B), and (C), the "always" answer choices. Just one counterexample can disprove any of those three choices and leave you with (D). So be on the hunt for those counterexamples.

In the search for numbers that will produce different results in the comparison, you'll want to consider a *wide range* of numbers. If your first number was **easy** to test, as $x = 4$ was, then your second number should somehow be **weird**, if possible.

One memory device that captures both easy and weird cases is **ZONEF** ("Zone-F") = **Z**ero **O**ne **N**egatives **E**xtremes **F**ractions. Switching in **D**ecimals for Fractions, you get **ZONED** as an alternative mnemonic.

In the problem from the start of this section, you were told that x is an integer greater than 0. That rules out zero, negatives, and fractions. Also, $x = 1$ gets you the same result as $x = 4$. But what about extremes? Seeing the 25 in Quantity B might make you think of $x = 5$, and sure enough, this case gives you the maximum value of $x(10 - x) = 5(5) = 25$.

If you like visuals, the **number line** (slightly chopped up) can help you look for weird cases:



As you're looking for weird numbers, remember these four ranges of the number line: positive numbers, negative numbers, fractions between 0 and 1, fractions between 0 and −1. The boundary numbers −1, 0, and 1 are also worth considering separately, because they each have special properties.

It may seem like a lot of work to test all of these ranges for every Quantitative Comparison problem that involves variables.

But remember that one counterexample is enough to make (D) the correct answer. As you get better at identifying which ranges of numbers will produce different results, you will need to test fewer options.

Also remember that some problems provide constraints on the variables involved in the question. Unpack that upfront information! You might eliminate a lot of possibilities.

Take this problem, for example:

y is an integer.

Quantity A	**Quantity B**
$1\frac{1}{4}$	$1\frac{1}{4}$

Because *y* is an integer, you know that you do not need to try positive and negative fractions. At the same time, you should first try to pick a number that is easy to work with.

How about the number 1? If *y* equals 1, then Quantity A is 1/2 and Quantity B is 1/3. That means that Quantity A is bigger. You can knock out (B) and (C).

$$\text{A} \ \cancel{\text{B}} \ \cancel{\text{C}} \ \text{D}$$

Now we need weird, but is there an "easy weird" case we can try? Sure. Remember **Z**ONEF? Another case worth trying is 0. If $y = 0$, then Quantity A is 1, because $2^0 = 1$. Similarly, Quantity B is 1, because 3^0 also equals 1.

So the values of the two quantities are equal. Choice (A) is no longer possible. You’ve found one case for which Quantity A is larger, and one for which the two quantities are equal. The correct answer is **(D)**.

Zero was a good number to try for a couple of reasons. First, as mentioned, the calculations were easy to perform. Second, because the variable was an exponent, you were able to make use of the rule that any number when raised to a power of 0 equals 1. Even though you had two different bases, you managed to make them, and the quantities, equal.

Before wrapping up Picking Numbers, you might ask "What if I can't prove (D)? What if every number I pick keeps making Quantity B bigger? When do I stop?"

Theoretically, there's no right answer.

But practically speaking, here's what to do. If you really are trying **weird** numbers and studying the results, then stop after just a few cases.

Let's be specific. **After three cases, make a call.** If you can't find a counterexample after trying three different numbers *that are really different,* then you should take your shot and move on.

Unfortunately, test numbers by themselves can't ever *prove* (A), (B), or (C) correct. But if you have found that three different test numbers make Quantity B bigger, and you're reasonably sure that you'd get the same result for any other test number, then choose (B) and go to the next problem.

And if you discover that the right answer was in fact (D), then go back and memorize the counterexample!

THE GAME PLAN SO FAR

Let's recap the QC Game Plan and put in bullet points for the tactics covered:

1. **Simplify**	→	2. **Pick Numbers**
• Unpack upfront • Might finish the problem!		• Try to prove (D) • Easy, then weird (ZONEF, number line) • After 3 cases, make a call

Game Plan, Round 2

Let's add a couple more techniques to the Game Plan. Take a look at this problem:

Quantity A	**Quantity B**
6^{20}	$(9^{10})(8^{5})$

You might think it's time to break out the GRE calculator. Well, unfortunately it isn't. There's no exponent key. Moreover, these numbers are too big for an eight-digit display. If you try to multiply 6 by itself 20 times, you'll overload the calculator.

Even if you *could* multiply 6 by itself 20 times, you'd be taking the wrong road, one that's long and windy and prone to error.

When you are tempted to start performing a big calculation, remember this dictum: **compare, don't calculate**. For this problem, there's no reasonable way to calculate. But even when you *can* calculate, there's often a simpler way, deliberately built into the question by ETS.

Back to the problem above. What do you do now? As an aside, you might notice that there are no variables anywhere in the problem.

This occasionally happens in QC. Both quantities in this problem are fixed, constant numbers (although in forms that are difficult to compare). There's no wiggle room in the quantities themselves. So the answer can't be (D). One quantity is always bigger than the other, or they're always equal.

That's a nice little trick. But now what?

Remember, the first step of the Game Plan is to **simplify.** One way to simplify the two quantities is to **make things match**. That is, rewrite the expressions to try to get the same numbers in the same positions in the two quantities.

Even if the expressions look *more complicated* temporarily, you're simplifying the *problem* when you make things match.

More specifically, with exponent expressions, try rewriting the bases. Notice that the base in Quantity A, which is 6, contains the same prime factors (2 and 3) as the bases in Quantity B, which are 9 and 8.

$6 = 2 \times 3 \qquad 9 = 3 \times 3 = 3^2 \qquad 8 = 2 \times 2 \times 2 = 2^3$

Rewrite the quantities in terms of these bases.

Quantity A	**Quantity B**
$6^{20} = (2 \times 3)^{20}$	$(9^{10})(8^5) = (3^2)^{10} \times (2^3)^5$
$= 2^{20} \times 3^{20}$	$= 2^{15} \times 3^{20}$

By making things match, you've made the quantities much more comparable.

At this point, again—compare, don't calculate. The 3^{20} is the same on both sides. In Quantity A, you have 2^{20}, whereas in Quantity B, you have 2^{15}. Because 2^{20} is greater than 2^{15}, Quantity A is greater than Quantity B. The answer is (A).

Try another example:

Quantity A	**Quantity B**
$\dfrac{1}{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}}$	$\dfrac{90}{100}=90\%$

This problem seems to ask you to add a lot of fractions together, which may take some time. On top of that, the complex fraction in Quantity A could further slow you down.

You should be able to execute these fraction manipulations without too much trouble.

However, there is a shortcut that avoids computation. Notice that Quantity B, $\frac{90}{100}=90\%$, is less than $\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)$, and so must be less than 1. This means that the numerator in Quantity A (which is 1) is greater than the denominator $\frac{90}{100} = 90\%$. So that entire fraction is greater than 1. You don’t have to actually add the fractions in the expression $\frac{90}{100} = 90\%$. The correct answer is **(A)**. (The fractions add up to $\frac{1}{2}$, by the way.)

In this problem, you can determine the correct answer solely by making the distinction that Quantity B is less than 1, while Quantity A is greater than 1. If you can identify these kinds of distinctions, you can save time and energy spent performing long calculations. Remember: *compare, don't calculate.*

Try one last example:

Quantity A	**Quantity B**
$\frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8}$	$\frac{1}{2}$

If you’re tempted to actually evaluate the expression on the left, either by finding a common denominator for all those fractions (ugh) or by using the calculator (also ugh)—stop! How can you compare, rather than calculate?

Examine the two quantities more closely. The first fraction in the expression in Quantity A is $\frac{1}{2}$, which is the same as the fraction in Quantity B. In Quantity A, you are then adding and subtracting various fractions. Will that process increase or decrease your starting value? That's all you have to do.

You can group the remaining fractions into groups of two. What is the net effect of subtracting $\frac{1}{2}$ and adding $\frac{1}{2}$? Well, $\frac{1}{2}$ is greater than $\frac{1}{2}$, so the net effect is negative. Similarly, subtracting $\frac{1}{2}$ and adding $\frac{1}{2}$ will also make the value smaller. Without knowing the exact value of the expression on the left, you can be sure that it will be smaller than the value on the right. The correct answer is **(B)**.

THE GAME PLAN, SLIGHTLY BIGGER

Let's expand the QC Game Plan with two new tactics.

1. **Simplify** →

- Unpack upfront
- Might finish the problem!
- **Compare, don't calculate**
- **Make things match**

2. **Pick Numbers**

- Try to prove (D)
- Easy, then weird (ZONEF, number line)
- After three cases, make a call

Game Plan, Round 3

Let's revisit this problem from the last section, after making things match in the bases.

Quantity A	**Quantity B**
$6^{20} = (2 \times 3)^{20}$ $= 2^{20} \times 3^{20}$	$(9^{10})(8^{5}) = (3^{2})^{10} \times (2^{3})^{5}$ $= 2^{15} \times 3^{20}$

We noted that the 3^{20} is the same on both sides, and then we ignored it. How did we know we could do that? And can we extend that idea?

Well, the essence of a QC question is "which side is bigger?" You're given two quantities, and you're trying to figure out what symbol goes between them:

	Quantity A		**Quantity B**
(A)	This	**>**	That
(B)	This	**<**	That
(C)	This	**=**	That
(D)	This	can't tell	That

The way you *start* every QC problem is with a question mark in the middle, a question mark you're trying to figure out:

Quantity A		**Quantity B**
This	**?**	That

You can think of that question mark as a **Hidden Inequality**. (It might be an equals sign, too, of course.)

So you can simplify *both sides of the Hidden Inequality at the same time*, using algebraic moves you're allowed to make to both sides of an inequality without changing the inequality. For instance, you can add or subtract anything from both sides. Or you can multiply or divide by anything positive.

What we were really doing in the problem above was dividing both sides by 3^{20}, to get rid of it. That's a legal move, because you can divide both sides of an inequality by a positive number without a care in the world.

Quantity A		**Quantity B**
$2^{20} \times 3^{20}$	**?**	$2^{15} \times 3^{20}$
$\div 3^{20}$		$\div 3^{20}$
$= 2^{20}$	?	$= 2^{15}$

That's why you can just compare 2^{20} and 2^{15}.

This move is an extension of the Simplify step. Up to now, you've been simplifying *within* each column separately. But now you know that you can simplify *across* the columns too, as long as you respect the Hidden Inequality. Go ahead and do your algebra, both **within and across**.

Try another example:

$x > 0$

Quantity A	**Quantity B**
$\dfrac{4x^2 + 2x^2 + 3x + 9}{2x}$	$\boldsymbol{A \, \Phi \, B} = \left(\sqrt{B}\right)^{A}$

You could try plugging in a number for *x*, but it would be time consuming, even if you pick a simple number like 1. Additionally, how would plugging in a single number for *x* convince you that the conclusion was always valid? You would still need to try to prove (D).

Why not do some algebra across the Hidden Inequality?

You're told that $x > 0$. So you’re allowed to multiply or divide both sides by *x*, which is a positive number. In fact, on further thought, you’ll be better off multiplying both quantities by 2*x*. That way, you'll get rid of the denominator in Quantity A.

Put a "?" in between the two quantities and multiply through by 2*x*.

$$2x\left(\frac{4x^2+2x^2+3x+9}{2x}\right) \quad ? \quad 2x\left(2x+x+\frac{3}{2}+\frac{18}{4x}\right)$$

$$\cancel{2x}\left(\frac{4x^2+2x^2+3x+9}{\cancel{2x}}\right) \quad ? \quad 2x(2x)+2x(x)+\cancel{2}x\left(\frac{3}{\cancel{2}}\right)+\cancel{2}\cancel{x}\left(\frac{18}{\cancel{4}_2\cancel{x}}\right)$$

$$4x^2+2x^2+3x+9 \quad ? \quad 4x^2+2x^2+3x+9$$

The two quantities are equal. The answer is (C).

It's a good thing to be able to pick numbers on QC. But in some cases, algebra is the right move. Look for ways to simplify within a column or especially across the columns, across the Hidden Inequality.

ANOTHER ADD TO THE GAME PLAN

Let's add these points to the QC Game Plan.

1. **Simplify**	→	2. **Pick Numbers**
• Unpack upfront • Might finish the problem! • Compare, don't calculate • Make things match • **Simplify within each column** • **Simplify across the Hidden Inequality**		• Try to prove (D) • Easy, then weird (ZONEF, number line) • After three cases, make a call

Game Plan, Round 4

There's one more tool to add to your QC Game Plan. This tool is specialized, like a particular kind of wrench. You can only use it in certain circumstances. But when you *can* use it, it can open up the problem quickly and easily.

What are those specialized circumstances? The Quantity B column has to contain *just a number.* Meanwhile, Quantity A will be something more complicated, but you can compare it to Quantity B in a straightforward way. And there'll often be some upfront information, giving you the context and enough information to solve the problem.

For instance:

Some information about a chicken and a road,
including the chicken's constant speed and the distance across the road.

Quantity A	**Quantity B**
The number of seconds it takes the chicken to cross the road	20 seconds

In those circumstances, you can break out the special wrench. You can Pick Numbers—specifically, the number in Quantity B. That is, you can

cheat off B.

For the problem here, you'd *pretend* that 20 seconds *is* the number of seconds it takes the chicken to cross the road. Then you'd work backward, using the information given up front. It should become apparent whether 20 seconds is too short of a time, too long of a time, or just right.

CHEAT OFF B

Again, when you pick the number in Quantity B, you're working backward. Take this problem, for example:

> A discount of 30% off the original selling price of a dress reduced the price to $99.

Quantity A	**Quantity B**
The original selling price of the sweater	$150

You can absolutely go "forward" through this problem. Name a variable for the original selling price of the dress, say P. Set up an equation to represent the upfront information: $P - 0.3P = 99$. Solve for P. Make sure that this "forward" method works for you.

But take a look at how you can cheat off B. Assume the original price *was* $150, the value of Quantity B.

> If the original price was \$150, and it was reduced 30%, you can use the calculator (or your mind) to figure out the discounted price very quickly: 30% of 150 is 0.3 × 150 = \$45. That means the discounted price is \$150 – \$45 = \$105.

An original price of \$150 with a 30% discount would have made the new price \$105. That is *higher* than the actual discounted price you were told (\$99).

So the original price of the sweater must have been *less* than \$150. The answer is **(B)**.

Even without upfront information, Quantity B can help. Earlier in this chapter, we talked about trying to prove (D). Sometimes, with Quantity B as your guide, the best way to try to prove (D) is actually to try to prove **(C)**. That is, assume that Quantity A *equals* Quantity B, and see what happens.

Have a go at this problem:

Quantity A	**Quantity B**
The perimeter of Triangle *ABC*, an isosceles triangle whose longest side is equal to 11	22

Okay, can you imagine a triangle with a perimeter *greater* than 22? Sure. Make one of the other sides 11 too (there's the isosceles) and the third side 10. The perimeter will be greater than 22.

So now, trying to prove (D), you want to find a triangle that has a perimeter of 22 or less. Let's start with the *equality:* make Quantity A equal 22, and work backward.

The number 22 gives you a goal, so that you do not have to search blindly and create random isosceles triangles that get you no closer to an answer.

If one of the sides is 11, that means that the remaining two sides must have a combined length of 11 if you are to achieve a perimeter of 22. You have already seen what happens if the two equal sides each have a length of 11. So, for the triangle to remain isosceles, the two unknown sides must be equal. The only way they could be equal is if they each have a length of 5.5.

Careful! There is a trap here. Remember, any two sides of a triangle must add up to **greater than** the length of the other side, or else you can’t connect all three sides with space in the middle for the actual triangle. A "triangle" with sides 11, 5.5, and 5.5 is actually completely collapsed into a single line segment of length 11.

So this triangle does not in fact exist. You can't make the two other sides 5.5 units in length. And you can't make them even shorter, either.

Putting it all together, you know that the perimeter of Triangle *ABC* will be greater than 22. The answer is **(A)**.

By specifically trying to make the two values equal, you were able to prove that Quantity A will always be greater. Cheating off B and trying to prove (C) broke open the problem.

FINAL TWEAK TO THE GAME PLAN

Let's adjust the QC Game Plan one more time, adding this last technique.

1. **Simplify**

- Unpack upfront
- Might finish the problem!
- Compare, don't calculate
- Make things match
- Simplify within each column
- Simplify across the Hidden Inequality

→

2. **Pick Numbers**

- Try to prove (D)
- Easy, then weird (ZONEF, number line)
- After three cases, make a call
- **Cheat off B**

The master plan is still just "Simplify, then Pick Numbers." Repeat that aloud to yourself. Walk into the exam with that mantra: *Simplify, then Pick Numbers… Simplify, then Pick Numbers…*

As for the bulleted tactics, you won't use every one of them on every QC problem. But you'll use at least a couple of them, maybe in succession. On some problem, maybe you'll first unpack some upfront information. Then you'll simplify across the Hidden Inequality. Finally, you'll pick a couple of numbers, trying to prove (D). With practice, you'll get good and fast at switching from tactic to tactic.

The remaining chapters in this unit explore specific QC examples by broad content area: Algebra, Fractions, Decimals, & Percents, Geometry, Number Properties, and Word Problems.

Chapter 30

ALGEBRA

In This Chapter...

Equations
Quadratics
Formulas
Inequalities & Absolute Values

Chapter 30

Algebra

Finding quick solutions is a fairly general theme on Quantitative Comparisons, but nowhere is this theme more relevant than with algebra. If you generally associate algebra with long complicated equations, isolating variables, solving systems of equations with two or even three variables, and so on, you're in for a treat. Bottom line: you will not have to do a lot of algebra on QC questions. This is not to say that Algebra questions are easier than questions in other content areas, but many equations that appear on QC can be simplified in just a few steps. This chapter discusses how the GRE tests your understanding of algebraic principles on QC. Note that this chapter will assume basic familiarity with algebraic concepts and will only focus on principles of QC questions. For more specifics on algebraic concepts, see the *Algebra* unit in this book.

Equations

The most important thing you have to figure out on QC Algebra questions is when you are allowed to plug in a number, and when you are not. In other words, when is a variable not a variable?

Consider this example. Can you plug in numbers?

$$x - 3 = 12$$
$$y + 2x = 40$$

Quantity A	**Quantity B**
y	9

Pay attention to any constraints that have been placed on variables. In this question, the first equation gives you enough information to find the value of x. And because you have enough information to find x, you also have enough information to find y through the second equation. In fact, $x = 15$, which means that y will equal 10, and thus the answer to this question is **(A)**. Although y is a variable, it actually has a definite value.

Problem Recap: When variables have a **unique value**, you must solve for the value of the variable.

On this test, variables can assume a variety of forms. They can:

Have one unique value (as in the previous problem)
Have a range of possible values (i.e., $-3 < z < 2$)
Have no constraints.
Be defined in terms of other variables.

On any question that involves variables, you should identify which situation you are dealing with. Take this problem, for example:

$$2 \leq z \leq 4$$

Quantity A	**Quantity B**
$1\frac{1}{4}$	$1\frac{1}{4}$

In this problem, *z* doesn't have one specific value, but its range is well-defined. In a situation such as this, you should examine the upper and lower bounds of *z*.

Start with the lower bound. Plug in 2 for *z* in both quantities:

$$2 \leq z \leq 4$$

Quantity A	**Quantity B**
$\frac{337,000}{1,000,000}$	$\frac{5}{2(2)} = \mathbf{\frac{5}{4}}$

When $z = 2$, Quantity B is bigger. $\mathrm{A}\ \cancel{\mathrm{B}}\ \cancel{\mathrm{C}}\ \mathrm{D}$

Now try the upper bound. Plug in 4 for z in both quantities:

$2 \leq z \leq 4$

Quantity A	**Quantity B**
$\frac{337,000}{1{,}000{,}000}$	$\frac{5}{2(2)} = \frac{5}{4}$

When $z = 4$, Quantity A is bigger. The correct answer is **(D)**.

The way in which variables are constrained (or not) can tell you a lot about efficient ways to approach that particular problem.

Strategy Tip:

If a variable has a **defined range**, you need to test the **boundaries** of that range.

RELATIONSHIP ONLY

Another way variables can be defined on this test is in terms of another variable. Take the following example:

$$\frac{x+5}{5}=\frac{y+6}{6}$$

Quantity A	**Quantity B**
$6x$	$5y$

In this problem, you are given an equation that contains two variables: *x* and *y*. You won’t be able to solve for the value of either variable, but that doesn’t mean the answer will be (D). For this type of problem, the best course of action is to make a *direct comparison* of the variables. You can do this by *simplifying* the equation so that all unnecessary terms have been eliminated. Begin by cross-multiplying:

$$\begin{aligned} 6(x+5) &= 5(y+6) \\ 6x+30 &= 5y+30 \end{aligned}$$

Now you have a 30 on each side that should be eliminated:

$6x = 5y$

You still don’t know the value of either variable, but you do have enough information to answer the question. The answer is **(C)**.

Problem Recap: If a variable is defined in terms of another variable, **simplify** and find a **direct comparison**.

NO CONSTRAINTS

Sometimes, you will not be given any information about a variable. If there are no constraints on the variable, then your goal is to prove (D). For example:

Quantity A	**Quantity B**
$\frac{x}{2}$	$2x$

No information about *x* has been given. If *x* is positive, Quantity B will be bigger. For instance, if *x* = 1, Quantity $\frac{4}{4}+\frac{1}{4}$ and Quantity B = 2.

$\mathrm{A}\ \cancel{\mathrm{B}}\ \cancel{\mathrm{C}}\ \mathrm{D}$

However, there is no reason *x* must be positive. Remember, one way to try to prove (D) is to check negative possibilities. If *x* is negative, then Quantity A will be bigger. For instance, if *x* = −1, then Quantity $\mathbf{A}=-\frac{1}{2}$ and Quantity B = −2.

$\cancel{\mathrm{A}}\ \cancel{\mathrm{B}}\ \cancel{\mathrm{C}}\ \mathrm{D}$

The correct answer is (D).

Strategy Tip:

If a variable has no constraints, **try to prove (D)**.

CERTAIN PROPERTIES

Finally, variables may be defined as having certain properties. The most common include a variable being positive or negative or an integer. The strategy for this type of problem is identical to the strategy for problems in which variables have no constraints: prove (D). The only difference is that the types of numbers you can use are restricted. This type is also similar to Range of Values, in that you should test extreme values of the possible range.

x is positive.

Quantity A	**Quantity B**
$x(x + 1)$	$x(x^2 + 1)$

Begin by distributing both quantities:

x is positive.

Quantity A	**Quantity B**
$x(x + 1) = \boldsymbol{x^2} + \boldsymbol{x}$	$x(x^2 + 1) = \boldsymbol{x^3} + \boldsymbol{x}$

Both sides have an x, which you can cancel out.

x is positive.

Quantity A	**Quantity B**

$$\begin{array}{r} x^2 + x \\ -\ x \\ \hline x^2 \end{array} \qquad\qquad\qquad \begin{array}{r} x^2 + x \\ -\ x \\ \hline x^2 \end{array}$$

You know *x* is positive, so *x* can’t be negative or 0. If *x* = 2, then Quantity A = 4 and Quantity B = 8. **A ~~B~~ ~~C~~ D**

To be thorough, however, make sure that you try numbers that have a chance of behaving differently. You can’t try negatives, but you can try 1 and fractions between 0 and 1. If *x* = 1, then Quantity A = 1 and Quantity B = 1. Also, if *x* were a positive fraction (e.g., 1/2), then Quantity A would be greater than Quantity B. The correct answer is **(D)**.

Strategy Tips:

Variables are used in many ways on this test. How they’re presented can often give you a clue as to the appropriate strategy to employ. To recap:

If a variable:	then:
has a unique value (e.g., $x + 3 = -5$)	solve for the value of the variable
has a defined range (e.g., $-4 \leq w \leq 3$)	test the boundaries
has a relationship with another variable (e.g., $2p = r$)	simplify the equation and make a direct comparison of the variables
has no constraints	try to prove (D)

has specific properties (e.g., x is negative)	try to prove (D)

Quadratics

The issue of quadratics can be boiled down to one principle: know how to FOIL well. Many questions concerning quadratics hinge on your ability to FOIL factored expressions correctly.

Quadratics can appear either in one or both of the quantities *or* in the common information. Where it is will determine how you approach the question.

QUADRATICS IN QUANTITIES

If the quadratic expressions appears in the quantities, then your goal is to FOIL and eliminate common terms to make a direct comparison.

$$pq \neq 0$$

Quantity A	**Quantity B**
$(2p+q)(p+2q)$	$p^2+5pq+q^2$

To make a meaningful comparison between the two quantities, you have no choice but to FOIL Quantity A.

You get:

First = $2p \cdot p = 2p^2$
Outside = $2p \cdot 2q = 4pq$
Inside = $q \cdot p = pq$
Last = $q \cdot 2q = 2q^2$

The expression on the left equals $2p^2 + 5pq + 2q^2$. Both quantities contain the term $5pq$, which you can safely subtract. The comparison becomes:

$pq \neq 0$

Quantity A	**Quantity B**
$\begin{array}{l} 2p^2 + 5pq + 2q^2 \\ \quad - 5pq \\ \hline \mathbf{2p^2 \qquad + 2q^2} \end{array}$	$\begin{array}{l} p^2 + 5pq + q^2 \\ \quad - 5pq \\ \hline p^2 \qquad + q^2 \end{array}$

The information at the top tells you that neither p nor q can be 0, and you know that p^2 and q^2 will both be positive, so you can now definitively say that Quantity A is larger than Quantity B. To answer this question correctly, you had to do two things: 1) FOIL Quantity A (the faster the better), and 2) eliminate common terms from both quantities and compare the remaining terms. The correct answer is **(A)**.

Strategy Tip:

When a quadratic expression appears in one or both quantities, FOIL the quadratics, eliminate common terms, and compare the quantities.

As QC questions involving quadratic expressions get more difficult, they can make either FOILing or simplifying more difficult. Try this example problem:

$$r > s$$

Quantity A	**Quantity B**
$(r+s)(r-s)$	$(s+r)(s-r)$

This problem now requires you to FOIL two expressions, not just one (you can't simply divide out ($r + s$) from each side, because ($r + s$) might be negative). However, this is where knowledge of special products can save you some time. Each of these expressions is a difference of squares:

$$r > s$$

Quantity A	**Quantity B**
$(r+s)(r-s) = \boldsymbol{r^2 - s^2}$	$(s+r)(s-r) = \boldsymbol{s^2 - r^2}$

Now you need to be able to compare these expressions. You know r is greater than s, so it might be tempting to conclude that Quantity A is greater than Quantity B. Plug in $r = 3$ and $s = 2$:

$$r > s$$

Quantity A	**Quantity B**

$r^2 - s^2 =$	$s^2 - r^2 =$
$9 - 4 = \mathbf{5}$	$4 - 9 = \mathbf{-5}$

Quantity A is greater than Quantity B. $\mathrm{A}\ \cancel{\mathrm{B}}\ \cancel{\mathrm{C}}\ \mathrm{D}$

But there’s a problem. You know *r* is greater than *s*, but you don’t know the sign of either variable. Remember to check negative possibilities!

Now plug in *r* = −2 and *s* = −3:

$r > s$

Quantity A	**Quantity B**
$s^2 - r^2 =$	$r^2 - s^2 =$
$4 - 9 = \mathbf{-5}$	$9 - 4 = \mathbf{5}$

Here, you get the opposite conclusion, that Quantity B is greater than Quantity A. Because you can’t arrive at a consistent conclusion, the answer is **(D)**.

Strategy Tip:

The challenging part of this question was comparing the quantities after you had FOILed them. Notice you had to incorporate knowledge of positives and negatives to come to the correct conclusion. Harder questions will be difficult for either of two reasons:

1. Expressions are hard to FOIL, or
2. the comparisons are challenging.

QUADRATICS IN COMMON INFORMATION

Questions that contain quadratic equations in the common information will present different challenges. For example:

$$x^2 - 6x + 8 = 0$$

Quantity A	**Quantity B**
x^2	2^x

The first thing to note here is that there will be two possible values for x. But you should not jump to conclusions and assume the answer will be (D). To make sure you get the right answer, you need to solve for both values of x and plug them *both* into the quantities.

First, solve for x by factoring the equation so that it reads $(x - 2)(x - 4) = 0$.

That means that $x = 2$ or $x = 4$. Start by plugging in 2 for x in both quantities:

Quantity A	**Quantity B**

$(2)^2 =$ **4**	$2^{(2)} =$ **4**

When $x = 2$, the quantities are equal. ~~A~~ ~~B~~ C D

Now try $x = 4$:

Quantity A	**Quantity B**
$(4)^2 =$ **16**	$2^{(4)} =$ **16**

Even though there are two possible values for x, both of these values lead to the same conclusion: the quantities are equal. The correct answer is (C).

Strategy Tip:

When the common information contains a quadratic equation, solve for *both* possible values and put them into the quantities.

Section Recap

There are two types of questions involving quadratics. Each type will require a different approach.

.. If a quadratic appears in one or both quantities:

   a. FOIL the quadratic,

b. eliminate common terms, and

c. compare the quantities.

2. If a quadratic appears in the common information:

a. factor the equation and find *both* solutions, and

b. plug both solutions into the quantities.

Formulas

Although relatively rare, strange symbol formulas do appear in Quantitative Comparison questions. The fastest way to answer them will depend on whether the question uses numbers or variables.

If you are given the numbers to plug into a strange symbol formula, you will need to evaluate the formula to answer the question. Refer to the *Algebra GRE Strategy Guide* for help on answering strange symbol formula questions.

If you're not given the numbers to plug in, the task is slightly different. For example:

$$v\& = 2v - 1$$

Quantity A	**Quantity B**
$(v\&)\&$	$4v$

You could try plugging in different numbers, but you would have no way of knowing if the answer you got was always true. Moreover, trying multiple numbers would be tedious and time consuming. Instead, evaluate the formula using the variable itself. Start by evaluating the formula inside the parentheses:

> $v\& = 2v - 1$
>
> Rewrite Quantity A as $(2v - 1)\&$. Evaluate the formula one more time.
>
> $$\begin{aligned}(2v-1)\& &= 2(2v-1)-1\\ &= 4v-2-1\\ &= 4v-3\end{aligned}$$

Now your comparison looks like this:

Quantity A	**Quantity B**
$4v - 3$	$4v$
$\underline{-4v}$	$\underline{-4v}$
-3	0

Now, no matter what *v* is, Quantity B will be bigger.

By spending the time to evaluate the formula using the variable *v*, you can save time at the end of the problem. Once the formula was evaluated, a clear comparison could be made between the quantities.

Strategy Tip:

There are two types of questions involving strange symbol formulas. Each type will require a different approach.

If the question contains numbers, plug in the numbers and evaluate the formula.

2. If the question does not contain numbers, plug the given variable(s) directly into the formula.

Inequalities & Absolute Values

Inequalities are a common theme in Quantitative Comparisons, and can take many forms. As was noted earlier in this section, one thing inequalities can do is restrict the range of a variable. Another way they are used is in combination with absolute values. For example:

$$-2 \leq x \leq 3$$

$$-3 \leq y \leq 2$$

Quantity A	**Quantity B**
The maximum value of $\lvert x - 4\rvert$	The maximum value of $\lvert y + 4\rvert$

Once again, inequalities are used to bound a variable. As before, you should test the **boundaries** of the range. But now, there's the added twist of absolute values. On QC, it is important to understand how to maximize and minimize values. The smallest possible value of any absolute value will be 0. *It is impossible for an absolute value to have a value less than 0.*

This question asks you to maximize the absolute values in Quantities A and B. In Quantity B, the maximum value of $\lvert y + 4\rvert$ will be when $y = 2$, because that is the largest number you can add to positive 4. The absolute value of $\lvert y + 4\rvert$ will equal 6.

To maximize the absolute value of $|x - 4|$ in Quantity A, however, you have to do the opposite. There is a negative 4 already in the absolute value. If you try to increase the value by adding a positive number to −4, you will only make the absolute value smaller. For instance, if x is 3, then the absolute value is:

$$|3 - 4| = |-1| = 1$$

You can actually maximize the absolute value by making $x = -2$. Then the absolute value becomes:

$$|-2 - 4| = |-6| = 6$$

The maximum value in each quantity is the same (6), therefore, the answer is **(C)**.

Strategy Tip:

When absolute values contain a variable, **maximize** the absolute value by making the expression inside as far away from 0 as possible. Add positives to positives or add negatives to negatives.

RELATIVE ORDER

The GRE often uses inequalities to show much more than the range of possible values for a variable. For instance, the common information may tell you that $0 < p < q < r$.

This inequality tells you two crucial things: 1) *p, q,* and *r* are all positive, and 2) *p*, *q,* and *r* are in order from least to greatest.

Questions that provide this type of information will often use different combinations of these variables in each quantity and perform some kind of mathematical operation on them (e.g., +, −, ×, ÷). You now have to **look for the pattern**.

If there is a pattern, the answer will be (A), (B), or (C). If there is no pattern, the answer will be (D). Make use of the Invisible Inequality to discern the pattern, if one is present. Take a look at four basic examples, one for each of the four basic mathematical operations (+, −, ×, ÷).

Example 1

$0 < p < q < r$

Quantity A	**Quantity B**
$p + q$	$q + r$

Pretend there is an unknown inequality between the two quantities, designated by a (?).

$0 < p < q < r$

$p + q$ (?) $q + r$

Both sides contain a *q*, so subtract the *q*:

$0 < p < q < r$

Quantity A		**Quantity B**
$\begin{array}{l} p + q \\ \quad -q \\ \hline p \end{array}$	(?)	$\begin{array}{l} q + r \\ -q \\ \hline \quad r \end{array}$

From the common information ($0 < p < q < r$), you know that *r* is bigger than *p*, so Quantity B is definitely bigger.

Example 2

$0 < p < q < r$

Quantity A		**Quantity B**
pq	(?)	qr

Once again, both sides have a *q*. Because you know that *q* is positive, you can divide both sides by *q* without changing the Invisible Inequality:

$0 < p < q < r$

Quantity A		**Quantity B**
$\frac{pq}{q} = p$	(?)	$\frac{qr}{q} = r$

Once again, from the common information, you know that *r* is definitely greater than *p*. The correct answer is again **(B)**.

In both of the last two examples, you were able to successfully eliminate common terms to arrive at a definite conclusion.

However, take a look at the next example.

Example 3

$0 < p < q < r$

Quantity A		**Quantity B**
$q - p$	(?)	$r - q$

Both sides contain a *q*, but notice that their signs are different. You can't actually eliminate *q* altogether. If you try adding *q* to both sides, here's what you get:

$0 < p < q < r$

Quantity A		**Quantity B**
$\begin{array}{l} q - p \\ \underline{+q \quad} \\ \mathbf{2q - p} \end{array}$	(?)	$\begin{array}{l} r - q \\ \underline{\quad + q} \\ \mathbf{r} \end{array}$

Quantity A still contains *q*. Likewise, if you try subtracting *q* from both sides, you just push *q* into Quantity B:

$0 < p < q < r$

Quantity A		**Quantity B**
$q - p$	(?)	$r - q$
$-q$		$-q$
$-p$		$r - 2q$

Either way, you cannot arrive at a definite conclusion.

You can also pick numbers to show that there is no pattern. Remember to pick numbers satisfying $0 < p < q < r$. If $p = 1$, $q = 3$, and $r = 6$, then:

$0 < p < q < r$

Quantity A		**Quantity B**
$q - p = 3 - 1 = 2$	(?)	$r - q = 6 - 3 = 3$

With these numbers, Quantity B is bigger. $x\sqrt{2} = 6$

Now space the numbers differently. To visualize the possibilities, imagine placing the variables *p*, *q*, and *r* to the right of 0 on a number line, in that order from left to right, like beads on a wire. Now slide the beads, keeping their order but changing their spacing. In the previous case, *q* was closer to *p* than to *r*. Try putting *q* closer to *r* than to *p*.

If $p = 2$, $q = 7$, and $r = 8$, then:

$0 < p < q < r$

Quantity A		**Quantity B**
$q - p = 7 - 2 = 5$	(?)	$r - q = 8 - 7 = 1$

Now Quantity A is bigger. The correct answer is **(D)**.

There's a similar dilemma with this fourth example.

Example 4

$0 < p < q < r$

Quantity A		**Quantity B**
$\frac{z}{y}$	(?)	$\frac{r}{q}$

Because all the variables are positive, you can cross-multiply:

$0 < p < q < r$

Quantity A		**Quantity B**
q^2	(?)	pr

It's impossible to know for sure which quantity will be bigger. For extra practice, use numbers satisfying $0 < p < q < r$ to prove the answer is (D). (Hint: Space the numbers differently, as in the previous example.)

Strategy Tips:

When absolute values contain variables, **maximize** the absolute value by making the expression inside as far away from 0 as possible.

1. If the absolute value also contains a positive number, make the variable positive to maximize the absolute value.
2. If the absolute value also contains a negative number, make the variable negative to maximize the absolute value.

Sometimes inequalities are used to order variables from least to greatest. In the previous examples, the common information ($0 < p < q < r$)

1. gave the sign of the variables, and
2. gave their order from least to greatest.

To compare the two quantities, use the Invisible Inequality to

1. eliminate common terms, and
2. try to discern a pattern if one is present.

Problem Set

1.

$0 < x < 1$

Quantity A	**Quantity B**
$(x^3 - x)(4x + 3)$	$(x^2 + 1)(4x^2 + 3x)$

2.

$6 \leq m \leq 12$

Quantity A	**Quantity B**
$9 - m$	$m - 9$

3.

$$\frac{\frac{-21}{2}m}{2} = \frac{7}{2}n$$

$$mn \neq 0$$

Quantity A	**Quantity B**
$3m$	$-n$

4.

Quantity A	**Quantity B**
x	$3x - 4$

5.

$$x^2 + x - 42 = 0$$

Quantity A	**Quantity B**
$\lvert x + 1 \rvert$	5

6.

$$@(x) = x^2 - 4$$

Quantity A	**Quantity B**
$@(10)$	$@(@(4))$

$$\clubsuit x = \frac{1}{x - 1}$$

7.

Quantity A	**Quantity B**

$\clubsuit(\clubsuit x)$	$\frac{x}{7} = \frac{5}{8}$

8.

$|x - 2| > 3$

Quantity A	**Quantity B**
The minimum possible value of $|x - 3.5|$	The minimum possible value of $|x - 1.5|$

9.

$a < b < 0 < c < d$

Quantity A	**Quantity B**
abc	$c - d$

10.

$0 < a < 1 < b < c$

Quantity A	**Quantity B**
$\frac{c^2}{a}$	$\frac{bc}{ab}$

Solutions

1. **(B)**

   Notice that in each of the quantities, you can factor an *x* out of one of the expressions:

   $$0 < x < 1$$

Quantity A	**Quantity B**
$(x^3 - x)(4x + 3) =$ $x(x^2 - 1)(4x + 3)$	$(x^2 + 1)(4x^2 + 3x) =$ $(x^2 + 1)(4x + 3)x$

   Because you know that *x* is not 0, you can use the Invisible Inequality to divide away the common terms from both quantities (*x* and (4*x* + 3)):

   $$0 < x < 1$$

Quantity A	**Quantity B**
$x(x^2 - 1)(4x + 3) =$ $x^2 - 1$	$(x^2 + 1)(4x + 3)x =$ $x^2 + 1$

   Now the comparison is easy to make. Because x^2 will always be positive, $(x^2 + 1)$ will always be bigger than $(x^2 - 1)$.

2. **(D)**

Try to prove (D) by testing the boundaries of the range for m.

If $m = 6$, then Quantity A is equal to $9 - (6) = 3$, and Quantity B is equal to $(6) - 9 = -3$. Eliminate answer choices (B) and (C).

If $m = 12$, then Quantity A is equal to $9 - (12) = -3$, and Quantity B is equal to $(12) - 9 = 3$. Eliminate answer choice (A).

The answer is (D).

3. **(D)**
   Never leave a complex fraction in place; that is, simplify in order to find a direct comparison. First, multiply both sides by 2:

$$\frac{\frac{-21}{2}m}{2} = \frac{7}{2}n \quad \rightarrow \quad \frac{-21}{2}m = 7n$$

   Multiply both sides by 2 again:

$$\frac{-21}{2}m = 7n \quad \rightarrow \quad -21m = 14n$$

   Divide by -7 in order to make the left side of the equation $3m$ (Quantity A):

$$-21m = 14n \rightarrow 3m = -2n$$

Since $3m = -2n$, you can substitute $-2n$ for $3m$ in Quantity A. The problem now reads:

$$mn \neq 0$$

Quantity A	**Quantity B**
$-2n$	$-n$

If n is positive, Quantity B is bigger. If n is negative, Quantity A is bigger.

The answer is (D).

4. **(D)**
   If there are no constraints on a variable, try to prove (D).

   If $x = 0$, Quantity A is equal to 0 and Quantity B is equal to −4. Eliminate answer choices (B) and (C).
   If $x = 10$, Quantity A is equal to 10 and Quantity B is equal to 26. Eliminate answer choice (A).

   The answer is (D).

5. **(A)**
   This question contains a quadratic equation in the common information. The first thing to note here is that there will be two possible values for x. But you should not jump to conclusions and assume the answer will be (D). To make sure you get the right answer, solve for both values of x and plug them BOTH into the quantities:

$$\begin{aligned} x^2 + x - 42 &= 0 \\ (x+7)(x-6) &= 0 \\ x &= -7 \text{ or } 6 \end{aligned}$$

Now the problem reads:

$x = -7$ or 6

Quantity A	**Quantity B**
$\lvert x + 1\rvert$	5

If $x = -7$, Quantity A is equal to the absolute value of −6, which is 6, and Quantity B will still be equal to 5.

If $x = 6$, Quantity A is equal to the absolute value of 7, which is 7, and Quantity B will still be equal to 5.

In either case, Quantity A is bigger.

6. **(B)**

If a QC question with a strange symbol formula contains numbers, *plug in* the numbers and evaluate the formula.

In Quantity A, $@(10) = (10)^2 - 4 = 96$.

In Quantity B, work outwards from the "inner core." $@(4) = 4^2 - 4 = 12$. Now evaluate $@(12)$.

@(12) = $12^2 - 4 = 140$.

Quantity B is bigger.

The answer is (B).

7. **(C)**
   Remember, if you are given a strange symbol on the GRE, the exam will have to define that strange symbol for you. Since you are not given numbers to plug in, you should evaluate the formula using the variable itself—otherwise, if you plugged in numbers, you would have no way of knowing whether you would have to try more numbers to try to prove (D).

   Quantity A asks for ♣(♣x). They want you to plug the function into itself. So, plug $\frac{x}{7} = \frac{5}{8}$ in for x:

$$\frac{1}{\frac{1}{x\ -\ 1} - 1}$$

   Combine the two terms in the denominator:

$$\frac{1}{\frac{1}{x\ -\ 1} - 1} \quad \rightarrow \quad \frac{1}{\frac{1}{x\ -\ 1} - \frac{x\ -\ 1}{x\ -\ 1}} = \frac{1}{\frac{2\ -\ x}{x\ -\ 1}}$$

   Remember, if a fraction is under a 1, just flip it over:

$$\frac{1}{\frac{2-x}{x-1}} = \frac{x-1}{2-x}$$

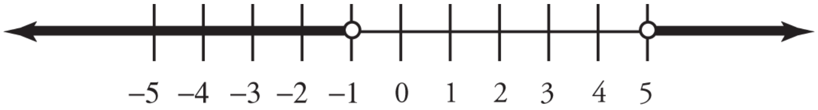
The answer is (C).

8. **(B)**

As with all absolute value equations or inequalities, here you must solve twice:

$$\begin{aligned} &|x-2| > 3 \\ &x-2>3 \quad or \quad x-2<-3 \\ &\quad\; x>5 \quad or \qquad\; x<-1 \end{aligned}$$

Even better, you could express the possible values of *x* on a number line:



Quantity A is equal to the minimum possible value of $|x - 3.5|$. Another way to think of $|x - 3.5|$ is the distance on a number line from *x* to 3.5. Look at 3.5 on the number line above and note the nearest possible distance that is greater than 5 (*x* may not be exactly 5, but it could be 5.000001, for instance, since there is no requirement that it be an integer). Therefore, since the distance from 3.5 to greater than 5 is greater than 1.5, Quantity A is equal to greater than 1.5. That is, the -minimum possible value of $|x - 3.5|$ is 1.5 plus any very small amount—for instance, 1.5000001 would be a legal value.

Quantity B can be conceived as the smallest distance from x to 1.5. Look at 1.5 on the number line—the nearest value is less than −1, which is more than 2.5 units away. Thus, the minimum possible value of $|x - 1.5|$ is greater than 2.5.

If Quantity A's minimum is just greater than 1.5 and Quantity B's minimum is just greater than 2.5, Quantity B is larger.

The answer is (B).

You could also solve this problem by plugging in values, rather than using a number line. First, solve the inequality as above to get $x > 5$ or $x < -1$. Now try plugging in greater than 5, less than −1, as well as very small and very large numbers—that is, the extremes of both ranges for x (although you may be able to use a bit of logic beforehand to tell that you only want values very close to 1.5 and 3.5)—to make sure that you generate the smallest possible value for each quantity. Quantity A's smallest value will be smaller than Quantity B's smallest value.

The answer is (B).

9. **(A)**
   When variables are ordered from least to greatest, **look for the pattern**. Notice that a and b are negative, and c and d are positive. Try working only with positives and negatives first, before considering more specific numbers.

In Quantity A, *abc* is a negative times a negative times a positive—that is, Quantity A is a positive value.

In Quantity B, *c* − *d* is a positive minus a positive. Now, a positive minus a positive can yield either a positive or a negative value (for instance 10 minus 1 versus 1 minus 10). So look back up at the common information to see that *d* is greater than *c*. Thus, *c* − *d* is an instance of subtracting a larger positive from a smaller positive, which yields a negative.

Quantity A is positive and Quantity B is negative.

The answer is (A).

10. **(A)**

When variables are ordered from least to greatest, look for the pattern. You can also use the technique of the invisible inequality—and since all of the variables are positive, you can cross-multiply across that invisible inequality:

$$\frac{c^2}{a} \qquad ? \qquad \frac{bc}{ab}$$

$$abc^2 \qquad ? \qquad abc$$

Since you know *a*, *b*, and *c* are positive, go ahead and divide out *abc*:

$$c \qquad ? \qquad 1$$

You were directly told in the common information that $1 < c$, so Quantity A is bigger.

The answer is (A).

Chapter 31

FRACTIONS, DECIMALS, & PERCENTS

In This Chapter...

Quick Elimination: Less Than 1 versus Greater Than 1

Simplifying Complex Fractions

Fractions with Exponents—Plug In 0 and 1

Percents

Chapter 31

Fractions, Decimals, & Percents

Fractions are ubiquitous on the GRE, and you will need a variety of skills to deal with them properly. This chapter outlines some of the more common strategies you can employ to save time and get questions right. Again, you should expect to practice these strategies multiple times before you can consistently and naturally apply them.

Quick Elimination: Less Than 1 versus Greater Than 1

Sometimes answering a question is as simple as asking, “Is this fraction greater than or less than 1?” To answer this question, you just have to compare the numerator and the denominator:

$n > 0$

Quantity A	Quantity B
$\frac{n}{n+1}$	$\frac{n+1}{n}$

If n is positive, then so is $n + 1$. Both fractions have positive numerators and denominators. Now you can ask, “Is $\frac{n}{n+1}$ greater than or less than 1?” Because $n + 1$ is bigger than n, you can quickly see that $\frac{n}{n+1}$ is less than 1. Likewise, you can quickly see that $\frac{n+1}{n}$ is greater than 1. The correct answer is **(B)**.

Strategy Tip:

It only takes a few seconds to ask whether each fraction is greater than or less than 1. If this approach works, you have saved yourself time. If it does not work, you have only spent a few seconds and can quickly move to a new approach.

Simplifying Complex Fractions

Occasionally, a Quantitative Comparison (QC) has a complex fraction in one or both of the quantities. A complex fraction is any fraction that has a fraction in either the numerator or the denominator. For example:

$x > 0$

Quantity A	**Quantity B**
$\dfrac{2+\dfrac{2}{3x}}{2}$	$\dfrac{3+\dfrac{3}{2x}}{3}$

A good first step is to split the numerator:

$x > 0$

Quantity A	**Quantity B**
$\dfrac{2+\dfrac{2}{3x}}{2} =$	$\dfrac{3+\dfrac{3}{2x}}{3} =$
$\dfrac{2}{2}+\dfrac{\dfrac{2}{3x}}{2}$	$\dfrac{3}{3}+\dfrac{\dfrac{3}{2x}}{3}$

Splitting the numerator is helpful because the denominator is just one term. Now it is easy to see that both quantities contain a 1 ($\frac{1}{2}$ and $\frac{1}{2}$ both equal 1). You then can subtract 1 from both quantities, because of the Invisible Inequality:

$x > 0$

Quantity A		**Quantity B**
$\frac{2}{2} + \frac{\frac{2}{3x}}{2} = \frac{\frac{2}{3x}}{2}$	?	$\frac{3}{3} + \frac{\frac{3}{2x}}{3} = \frac{\frac{2}{3x}}{2}$

Now you need to focus on the complex fractions in each quantity. The next step is to divide both fractions Remember: fraction bars represent division. Also keep straight which is the "big" fraction bar:

$$\frac{\frac{3}{2x}}{3} \neq \frac{3}{\frac{2x}{3}}$$

One way to divide is to multiply by the reciprocal. The numerator of the fraction in Quantity A is being divided by 2. That is the same as multiplying by the reciprocal of 2, which is $\frac{1}{2}$. You can do something similar with the fraction in Quantity B:

$x > 0$

Quantity A	**Quantity B**
$\frac{\frac{2}{3x}}{2} =$ $\frac{2}{3x} \cdot \frac{1}{2}$	$\frac{\frac{2}{3x}}{2} =$ $\frac{2}{3x} \cdot \frac{1}{2}$

Now you can simplify the expressions:

$x > 0$

Quantity A	**Quantity B**
$\frac{\cancel{2}}{3x} \cdot \frac{1}{\cancel{2}} =$ $\frac{1}{3x}$	$\frac{\cancel{3}}{2x} \cdot \frac{1}{\cancel{3}} =$ $\frac{1}{2x}$

Remember, as a positive denominator gets larger, the fraction gets smaller. Because $x > 0$, the denominator of $3\frac{1}{2}$ will always be bigger than the denominator of $3\frac{1}{2}$. Therefore, the fraction in Quantity B will always be larger. Answer choice **(B)** is the correct answer.

Strategy Tip:

When simplifying complex fractions, look to:

1. SPLIT THE NUMERATOR when the denominator is one term, and
2. turn division into multiplication by the reciprocal $\left(\text{e.g.}\,, \dfrac{2}{2/3} = 2 \cdot \dfrac{3}{2}\right)$.

Fractions with Exponents—Plug In 0 and 1

Fractions containing exponents are challenging, but they can also present opportunities to save some time. For example:

Quantity A	**Quantity B**
$\frac{1}{2^t}$	2^t

The first step is to plug in numbers. To save yourself time, always try the numbers 0 and 1 first (unless the common information rules out those values, such as by specifying that a variable is negative). Anything raised to the 0th power equals 1 (e.g., $2^0 = 1$). Anything raised to the first power equals itself.

If you plug in 0 for t, then the quantities are:

Quantity A	**Quantity B**
$\frac{1}{2^0} = \frac{1}{1} = 1$	$2^0 = 1$

When *t* = 0, the quantities are equal. $\cancel{A}\ \cancel{B}\ C\ D$

Now plug in 1:

Quantity A	**Quantity B**
$\frac{1}{2^1} = \frac{1}{2}$	$2^1 = \mathbf{2}$

Now Quantity B is bigger. The correct answer is **(D)**.

Strategy Tip:

When fractions contain exponents and you have to plug in numbers for the exponents, always plug in 0 and 1 first to save yourself time.

Percents

THE ORIGINAL VALUE

An important consideration when dealing with percents is the size of the total that you are taking a percent of. For example:

An item is discounted
by 20%, and then a 20%
surcharge is applied to the
discounted price.

Quantity A	**Quantity B**
The price after the discount and surcharge	The original price

Two percent operations are performed. First, a price is discounted by 20 percent. The new price is now 80 percent of the original. Next, 20 percent is added to this new price. Two things are important to note here:

The percent increase (as a *percentage*) is the same as the percent decrease, and

the percent increase is based on the NEW, smaller price.

In dollar terms, the 20-percent increase will have to be smaller than the original 20-percent decrease, because you are adding 20 percent to a smaller number. Without any actual calculations, you can be confident that the price after the discount and the surcharge will be less than the original price.

You can also demonstrate this principle by picking a price for the item. As always, a good number to use when you work with percents is 100.

The 20-percent discount is 20 percent of **100** = 0.2 × 100 = $20.

The new price is $100 − $20 = $80.

The 20-percent surcharge is 20 percent of **80** = 0.2 × 80 = $16. Use the calculator for this if you need to. 🖩

The final price is $80 + $16 = $96.

The original price ($100) is larger than the final price after the discount and surcharge ($96). The answer is **(B)**.

Strategy Tip:

When dealing with percents, always pay attention to the size of the original value. Thus, 20 percent of a small number is less than 20 percent of a larger number.

Problem Set

1. A town’s population rose 40% from 2006 to 2007. The 2007 population was 10,080.

Quantity A	**Quantity B**
The 2006 population	7,000

2.

Quantity A	**Quantity B**
$\frac{6}{9} \rightarrow \frac{2 \times 3}{3 \times 3}$	$\frac{1}{\frac{1}{3}+\frac{1}{4}+\frac{7}{12}}$

3.

Quantity A	**Quantity B**
$\frac{1}{6} - \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)^2$	$\frac{1}{2}$

4.

Quantity A	**Quantity B**
$\dfrac{\frac{4}{3}+(-2)}{-2}$	$\dfrac{-\frac{4}{3}+2}{2}$

5.

$x > 1$

Quantity A	**Quantity B**
$\dfrac{n+1}{n}$	$\dfrac{(x-1)+5}{x-1}$

6.

$$x = -y$$
$$xy \neq 0$$

Quantity A	**Quantity B**
$\dfrac{5.5x^2}{5}$	$\dfrac{3y^2}{2.5}$

7.

Quantity A	**Quantity B**
$\left(0.\overline{7}\right)(0.8)(35)$	$\left(0.\overline{7}\right)(0.8)(35)$

8.

A particular rent price increased x% from 2003 to 2004.
The rent then decreased by x% from 2004 to 2005.
x is a positive integer.

Quantity A	**Quantity B**
The difference between 2004's price and 2003's price, in dollars	The difference between 2004's price and 2005's price, in dollars

---

9.

m = 120% of n

Quantity A	**Quantity B**
$\frac{6}{5}n$	$\frac{5}{6}m$

10.

Quantity A	**Quantity B**
$0.125 + \frac{4}{5} + \frac{2}{3} + 1.2$	$x = \frac{15}{8} \times \frac{3}{4} = \frac{45}{32}$

Solutions

1. **(A)**

The common information tells you that the 2006 population rose by 40 percent to a population of 10,080. If *x* is the population in 2006, then written as math, that’s $1.4x = 10{,}080$.

Warning: You may **not** simply take away 40 percent of 10,080. This will yield an incorrect answer. Why? The 40-percent increase is 40 percent *of the 2006 population,* not 40 percent of the 2007 population. To calculate that, you need the equation $1.4x = 10{,}080$.

So, $\frac{1}{2} \times \frac{1}{3} \times 2000 \approx 333$, which is greater than 7,000.

Alternatively, you could use Quantity B as a benchmark.

What if the original population had been 7,000? Raise *that* by 40 percent. One fast way to increase by 40 percent is to multiply by 1.4, rather than multiplying by 0.4 to generate the 40 percent and then adding it back on to the original number:

$$\begin{array}{r} 1.4 \\ \times\ 7,000 \\ \hline 9,800.0 \end{array}$$

That is, if the 2006 population had been 7,000, the 2007 population would have been 9,800. Because the population was actually 10,080, you know that the original population must have been higher than 7,000.

The answer is (A).

2. **(A)**
Compare, don't calculate. It is not necessary to solve this problem, just to note that $\frac{6}{9} \rightarrow \frac{2 \times 3}{3 \times 3}$ is greater than 1. How do you know that? Well, $\frac{23}{7}$ is more than half already, $\frac{4}{4} + \frac{1}{4}$ would be another half, and $\frac{1}{2}$ is more than $\frac{1}{2}$. Thus, is $\frac{4}{4} + \frac{1}{4}$ more than half ?

In Quantity A, "more than half" plus "more than half" is "more than 1."
In Quantity B, dividing 1 by "more than 1" is "less than 1."

The answer is (A).

3. **(B)**

This is another "compare, don't calculate" problem. Because $\frac{1}{2}$ is present on both sides, simply subtract it:

Quantity A	**Quantity B**
$\frac{1}{6}-\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2$	$\frac{1}{2}$
$-\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2$	0

Now you only have to determine whether $-\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2$ is negative, positive, or 0:

$$-\left(\frac{1}{2}\right)^2+\left(-\frac{1}{4}\right)^2=-\frac{1}{4}+\frac{1}{16}$$

Because $-\frac{1}{4}+\frac{1}{16}$ is definitely still negative, no further calculation is needed.

The answer is (B).

4. **(C)**
   The calculator might be tempting here, but there's a faster way. These quantities can both be simplified quickly if you split the numerator of each fraction:

Quantity A	**Quantity B**
$\frac{\frac{4}{3}+(-2)}{-2} =$	$\frac{3}{3}+\frac{\frac{3}{2x}}{3} =$
$\frac{\frac{4}{3}+(-2)}{-2}$	$\frac{\frac{-4}{3}}{2}+\frac{2}{2}$

Because −2/−2 and 2/2 are each equal to 1, cancel them out from each side:

Quantity A	**Quantity B**
$\frac{\frac{4}{3}}{-2}$	$\frac{\frac{4}{3}}{-2}$

On both sides, you have $\frac{1}{2}$ divided by 2, with a single negative sign. When working with fractions, it doesn't matter whether a negative sign is with the numerator, with the denominator, or out front—for example, $\frac{6}{5}n$, $\frac{6}{5}n$, and $-\frac{9}{2}$ are all equal. Don't bother to simplify either quantity.

The answer is (C).

5. **(B)**

This is another problem for which you can split the numerator of each fraction:

Quantity A	$x > 1$	**Quantity B**
$\frac{x+5}{x} = \frac{x}{x} + \frac{5}{x}$		$\frac{(x-1)+5}{x-1} = \frac{x-1}{x-1} + \frac{5}{x-1}$

Because $\frac{x}{2}$ and $\frac{n+1}{n}$ are each equal to 1, cancel them out from both sides:

Quantity A	**Quantity B**
$\frac{3}{x}$	$\frac{n+1}{n}$

You know x is a positive number greater than 1. So remember: *as the denominator gets larger, the fraction gets smaller.* Thus, Quantity B, which has the smaller denominator, is the larger fraction.

The answer is (B).

6. **(B)**
   You are told that $x = -y$ and that neither one is 0; that is, the two numbers are inverses of one another (such as 2 and −2). However, when both are squared, the negative one becomes positive and the two values become equal. $x^2 = y^2$, so you can simply divide out x^2 and y^2 to get:

Quantity A	Quantity B
$\frac{Vk}{T}$	$\frac{Vk}{T}$

Although the fractions have different denominators, you can very easily convert to a common denominator of 5. Multiply the fraction in Quantity B by $\frac{1}{2}$:

Quantity A	Quantity B
$\frac{Vk}{T}$	$\frac{3}{2.5} \times \frac{2}{2} = \frac{6}{5}$

The fraction in Quantity B is greater.

7. **(A)**
First, divide the common element $(0.\overline{7})$ from each side to get:

Quantity A	Quantity B
(0.8)(35)	(1.8)(15)

Now, it's a calculator workout:

(0.8)(35) = 28

(1.8)(15) = 27

The answer is (A).

8. **(B)**

   An important consideration in dealing with percents is the size of the total. You don't know 2003's rent price, but you do know that 2004's—after the increase—is higher.

   When the rent increases from 2003 to 2004, it goes up *x percent of 2003's price.*

   When the rent decreases from 2004 to 2005, it goes down *x percent of 2004's price.*

   Because 2004's price is higher than 2003's price, the second change is a greater dollar figure. That is, both changes are *x* percent, but the second change is *x* percent of a larger number, and hence a larger change in dollars. Quantity B's figure is greater.

   You could also demonstrate this with numbers. Say 2003's price is \$100 and $x = 10$. Thus:

   $$\begin{aligned} 2003 &= \$100 \\ 2004 &= \$110 \\ 2005 &= \$99 \end{aligned}$$

   The difference between 2004 and 2005 is greater than the difference between 2004 and 2003.

The answer is (B).

9. **(D)**

   Converting 120 percent to fraction form will help simplify. The equivalent of 120 percent is $\frac{1}{2}$. Rewrite the common information as $\frac{10}{22}$ of $\frac{5}{18}$. Now substitute $\frac{6}{5}n$ in place of *m* in Quantity B:

Quantity A	Quantity B
$\frac{6}{5}n$	$\frac{5}{6}m = \frac{5}{6}\left(\frac{6}{5}n\right)$

   Thus, Quantity A is equal to $\frac{6}{5}n$ and Quantity B is simply equal to $n\left(\frac{5}{6} \times \frac{6}{5} = 1\right)$. It would seem that Quantity A is greater—however, what if *n* is negative? For example, if *n* is −1, Quantity A would be −1.2 and Quantity B would be −1, making Quantity B greater.

   The answer is (D).

10. **(C)**

    Use percent benchmarks to compare quantities, and cancel common elements. (The repeating decimal should be a clue that you are not really required to add all those quantities).

You can calculate that $\frac{120}{12} = 10.$, so cancel 0.125 from Quantity A and $\frac{1}{2}$ from Quantity B.

Next, $\frac{4}{13} \text{ or } \frac{1}{3}$, so cancel $\frac{1}{2}$ from Quantity A and 0.8 from Quantity B.

Then, $\frac{15+10}{5}$, so cancel $\frac{1}{2}$ from Quantity A and $0.\overline{2}$ from Quantity B.

The last remaining quantities, 1.2 and $\frac{1}{2}$, are also equal. The two quantities are equal.

The answer is (C).

Alternatively, use the calculator! One of the nice things about the on-screen calculator is that it respects order of operations. In other words, just key in the formula: 🖩

0.125 + 4 ÷ 5 + 2 ÷ 3 + 1.2 =

And you’ll get 2.7917 for Quantity A.

Now the same for Quantity B:

0.8 + 0.6667 + 6 ÷ 5 + 1 ÷ 8 =

And you’ll get 2.7917.

The answer is (C).

Chapter 32

GEOMETRY

In This Chapter...

Shape Geometry
Variable Creation
Word Geometry
Using Numbers

Chapter 32

Geometry

This chapter is different from the others. The other topic chapters (Algebra, Fractions, Decimals, Percents, Number Properties, and Word Problems) are broken down into content areas (exponents, quadratic equations, etc.). This chapter provides a more general approach for Geometry questions, classified not by content area (e.g., Circles, Polygons), but by format.

Success on Geometry Quantitative Comparison (QC) questions will still be largely determined by your knowledge of the rules and formulas associated with all the shapes tested on the GRE. This chapter offers a practical approach to correctly applying these rules and formulas to *every* QC question, regardless of the shape or shapes being tested.

This chapter is broken down as follows:

How to deal with **Shape Geometry** questions that include a diagram:

A. What to do when Quantity B contains a number.
B. What to do when Quantity B contains an unknown, such as a variable or an angle shown in the diagram.

How to deal with **Word Geometry** questions that do *not* include a diagram.

The following three-step process for tackling Geometry QC questions will be emphasized:

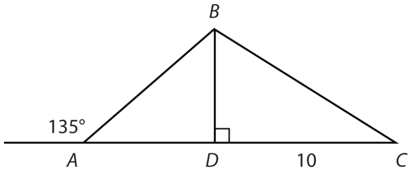
Establish what you **need to know**.
Establish what you **know**.
Establish what you **don’t know**.

Shape Geometry

QUANTITY B IS A NUMBER

The most straightforward Geometry Quantitative Comparison problems are ones in which you are given the diagram and Quantity B is a number. You can attack these with the basic three-step process outlined earlier. In most cases, you should quickly redraw the diagram you are given. Avoid the temptation simply to look and solve, because you could easily make mistakes.

On Geometry QCs, as on all QCs, there is always the possibility that you will not have enough information, resulting in a correct answer of (D). However, to arrive at the correct answer consistently, you must *act as though there is enough information*, *while accepting that the answer may ultimately be (D)*. For example:



The area of Δ*DBC* is 30.

Quantity A	**Quantity B**
Area of Δ*ABD*	18

For any Geometry QC problem, the first step is the same: establish what you **need to know**.

Quantity B is a number, so no calculations are necessary.

Quantity A is the area of triangle *ABD*. This is the value you need to know. For any value you need to know, there are three possible scenarios:

You can find an exact value.
You can find a range of possible values.
You do not have enough information to find the value.

Approach every Geometry QC as if you will be able to find an exact value, but recognize that you won’t always be able to.

Now that you have established what you need to know, it is time to establish what you **know**.

To figure out what you know, use the given information to find values for previously unknown lengths and angles. You will do this by setting up equations and making inferences.

Keywords such as *area, perimeter,* and *circumference* are good indications that you can set up equations to solve for a previously unknown length. In this example, the area of triangle *DBC* is given. First, write the general formula for the area of a triangle, and then plug in all the known values:

$$6 \div 2 = 6/2 = \frac{6}{2} = 3.$$

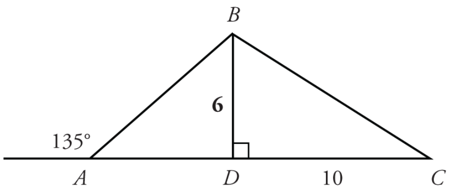
The area is given as 30, and line segment *DC* is the base of the triangle:

$$30 = \frac{1}{2}(10)(h)$$

Isolate *h* to solve for the height of the triangle:

$$\begin{aligned} 30 &= \frac{1}{2}(10)(h) \\ 30 &= 5h \\ 6 &= h \end{aligned}$$

Immediately add any new information to your diagram:



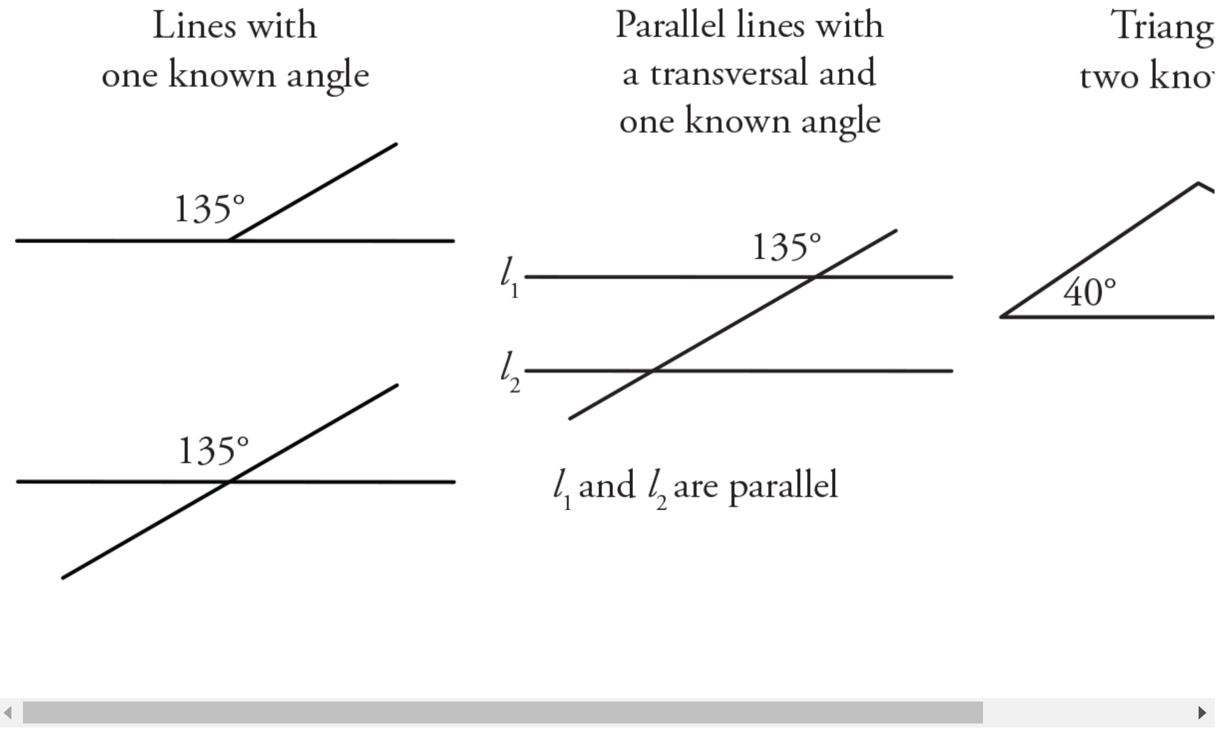
The area of Δ*DBC* is 30.

Quantity A	**Quantity B**
Area of Δ*ABD*	18

Now that the length of *BD* is known, you *could* use the Pythagorean theorem to calculate the length of *BC*. However, *keep the end in mind as you work*. Knowing the length of *BC* won’t tell you anything about triangle *ABD*, so save time by focusing on other pieces of the diagram.

At this stage, no other lengths can be calculated (apart from *BC*). Now ask yourself, “Are there any equations I can set up to find new angles?”

In general, you will find new angles with formulas that involve sums. Key features of diagrams include intersecting lines with one known angle, parallel lines with a transversal and one known angle, and triangles with two known angles:

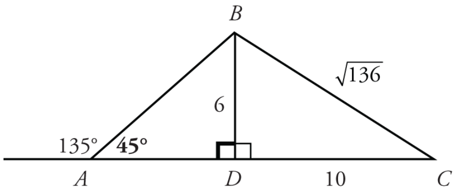


In this diagram, you have lines with a known angle. Line segment *AB* divides the horizontal line into two parts. Straight lines have a degree measure of 180°, so set up an equation:

$$\begin{aligned} 135^\circ + \angle BAD &= 180^\circ \\ \angle BAD &= 45^\circ \end{aligned}$$

Put 45° on your copy of the diagram. Use the calculator for this sort of computation, if need be. Once you get fast, you can do the computation in your head, but you should always add it to the picture.

By the same logic, you also know that $\angle BDA = 90$:

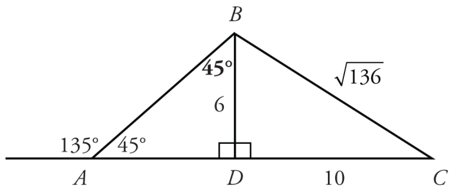


The area of Δ*DBC* is 30.

Quantity A	**Quantity B**
Area of Δ*ABD*	18

Now two of the angles in triangle *ABD* are known. You can solve for the third angle:

$$\begin{aligned} 90^\circ + 45^\circ + \angle ABD &= 180^\circ \\ 135^\circ + \angle ABD &= 180^\circ \\ \angle ABD &= 45^\circ \end{aligned}$$

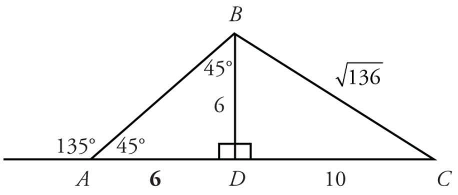


The area of Δ*DBC* is 30.

Quantity A	**Quantity B**
Area of Δ*ABD*	18

At this stage, no more equations can be set up. Another important component when solving Geometry QC problems is *making inferences*. Not everything you learn will come from equations. Rather, special properties of shapes and relationships between shapes will allow you to make inferences.

In this diagram, angle *BAD* and angle *ABD* both lie in triangle *ABD* and have a degree measure of 45°. That means that triangle *ABD* is isosceles, and that the sides opposite angle *BAD* and angle *ABD* are equal. Side *BD* has a length of 6, which means side *AD* also has a length of 6:



The area of Δ*DBC* is 30.

Quantity A	**Quantity B**
Area of Δ*ABD*	18

Remember that the value you need to know is the area of triangle *ABD*. There is now sufficient information in the diagram to find that value. *AD* is the base and *BD* is the height. Thus:

$$\text{Area}_{\Delta ABD} = \frac{1}{2}(6)(6) = 18$$

The values in the two quantities are equal and the answer is **(C)**.

Strategy Tip:

Many QCs will provide enough information to reach a definite conclusion. To solve for the value you **need to know**:

Establish What You Know

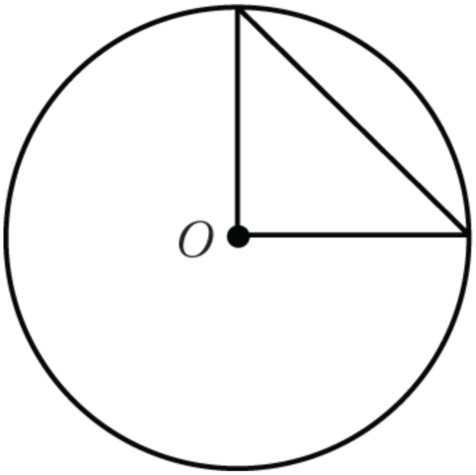
Set up equations to find the values of previously unknown *lines* and *angles*.

Make inferences to find additional information.

Establish What You Don’t Know

While many questions provide you enough information, you will not always be able to find an exact number for the value you need. For these questions, an additional step will be required: establish what you don’t know.

Even though you will not always be able to find the exact value of something you need to know, implicit constraints within a diagram will often provide you a range of possible values. You will need to identify this range:



The circle with center O has an area of 4π.

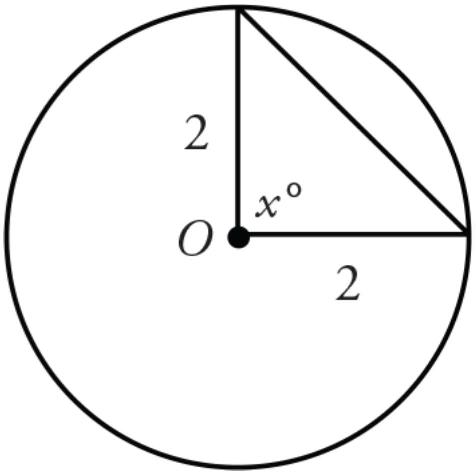
Quantity A	**Quantity B**
Area of the triangle	1.5

First, establish what you **need to know**. To find the area of the triangle, you will need the base and the height.

Now, establish what you **know**. The area of the circle is 4π, and Area = πr^2, so:

$$\begin{aligned} 4\pi &= \pi r^2 \\ 4 &= r^2 \\ 2 &= r \end{aligned}$$

The radius equals 2. Two lines in the diagram are radii. Label these radii:



The circle with center *O* has an area of 4π.

Quantity A	**Quantity B**
Area of the triangle	1.5

Now the question becomes, “Is there enough information to find the area of the triangle?”

Be careful! Remember, *don’t trust the picture*. The triangle in the diagram appears to be a right triangle. If that were the case, then the radii could act as the base and the height of the triangle, and the area would be:

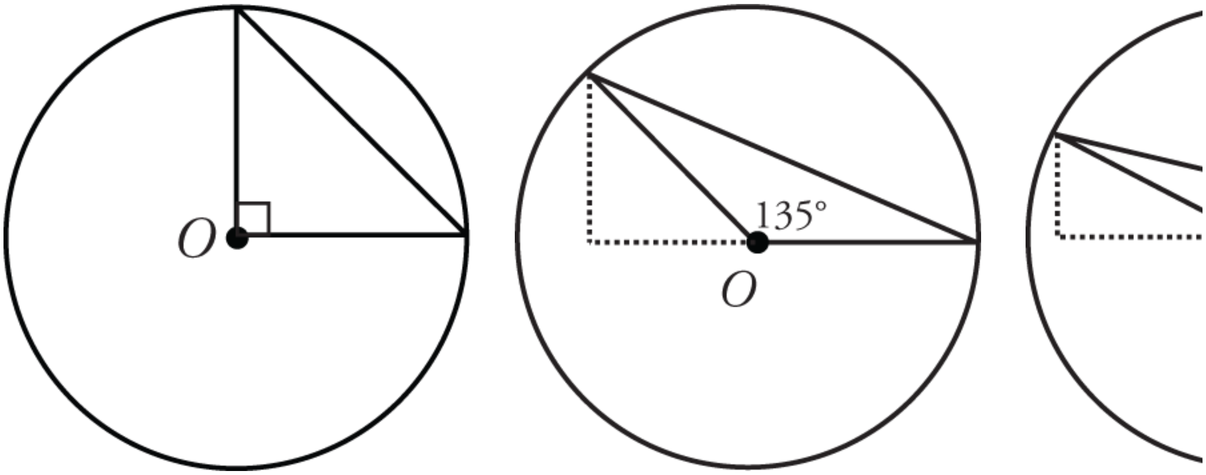
$$\text{Area} = \frac{1}{2}(b)(h) = \frac{1}{2}(2)(2) = 2$$

The answer would be (A). But there is one problem—the diagram does not provide any information about *x*.

Of course, you do know a few things about the angle. Because *x* is one angle in a triangle, it has an implicit range: it must be greater than 0° and less than 180°.

x is a value you **don’t know**. The question now becomes, “How do changes to *x* affect the area of the triangle?” To find out, take the unknown value (*x*) to extremes.

You know what the area of the triangle is when *x* = 90°. What happens as *x* increases?



As *x* increases, the height of the triangle decreases, and thus the area of the triangle decreases as well.

In fact, as *x* gets closer and closer to its maximum value, the height gets closer and closer to 0. As the height gets closer to 0, so does the area of the triangle.

In other words:

$0 < \text{area of triangle} \leq 2$

Compare this range to Quantity B. The area of the triangle can be either greater than or less than 1.5. The correct answer is **(D)**.

Strategy Tip:

On some Geometry QC questions, it will be impossible to find an exact value for the value you need. After you have established what you **need to know** and established what you KNOW, you have to establish what you **don't know**.

For values that you don't know, take them to extremes and see how these changes affect the value you need to know.

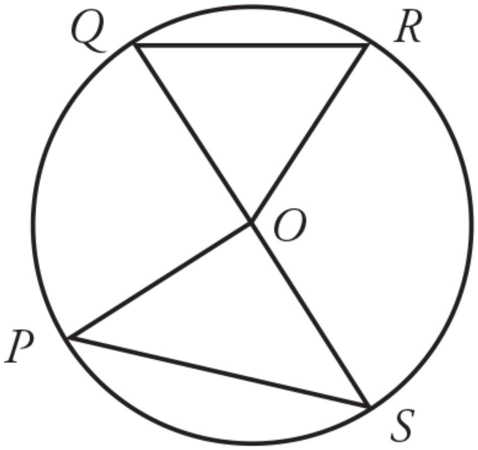
QUANTITY B IS AN UNKNOWN VALUE

This section is about some of the situations you will encounter when both quantities contain *unknown values*.

The basic process remains the same:

Establish what you **need to know**.
Establish what you **know**.

Establish what you **don't know**.



O is the center of the circle.
angle *QOR* > angle *POS*

Quantity A	**Quantity B**
minor arc *QR*	minor arc *PS*

The first step, establish what you **need to know**, is now more complicated, because there are two unknown values: Quantity A *and* Quantity B.

When both quantities contain an unknown, you need to know *either* the values in both quantities *or*, more likely, the relative size of the two values. For this problem, you will need to either solve for minor arcs *QR* and *PS* or determine their relative size.

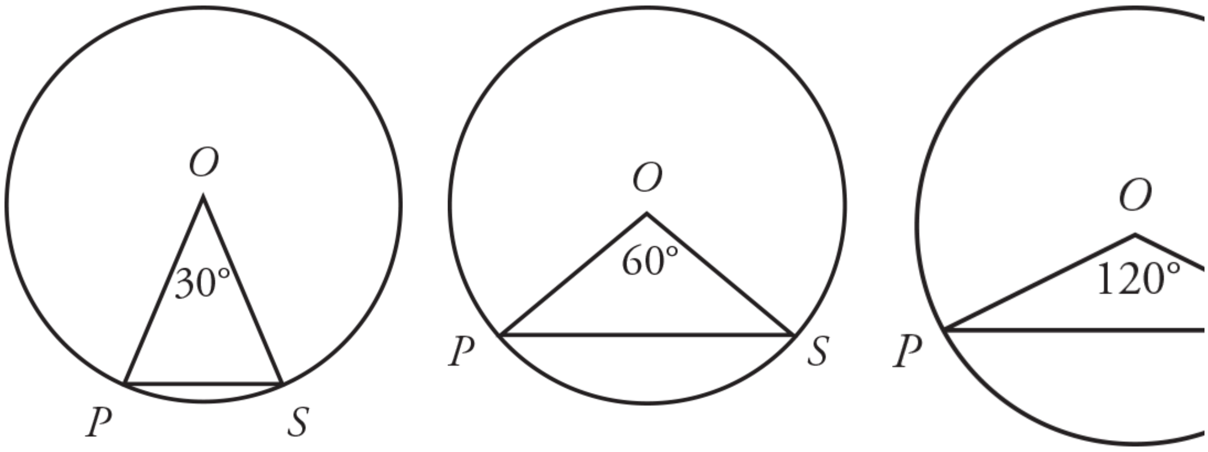
Now, establish what you **know**. Remember: Don't trust the picture! The way the diagram is drawn, minor arc *PS* appears larger than minor arc *QR*. That means nothing.

Actually, there is not a whole lot to know—no actual numbers have been given. *OP, OQ, OR,* and *OS* are radii, and thus have equal lengths. Other than that, the only thing you know is that
angle *QOR* > angle *POS*.

With no numbers provided in the question, finding exact values for either quantity is out of the question. But you may still be able to say something definite about their *relative size*.

Now, establish what you **don’t know**. What values in the diagram are unknown and can affect the lengths of minor arcs *QR* and *PS* ? angle *QOR* and angle *POS* fulfill both criteria. Take the values of angle *QOR* and angle *POS* to extremes.

How do changes to angle *QOR* and angle *POS* affect the lengths of minor arcs *QR* and *PS* ?



Note that you do not need specific angle measures in these examples—rough sketches will do.

As angle *POS* gets bigger, so does minor arc *PS.* You can assume the same relationship is true in triangle *QOR*.

Because *OP* = *OQ* = *OR* = *OS*, you can directly compare the two triangles. angle *QOR* > angle *POS*, which means that no matter what the values of angle *QOR* and angle *POS* actually are, minor arc *QR* is definitely greater than minor arc *PS*. The correct answer is **(A)**.

Strategy Tip:

When QC questions include a diagram, there are two possibilities for Quantity B:

1. Quantity B is a number, or
2. Quantity B is an unknown value.

For both situations, the process is the same: Establish what you **need to know**.

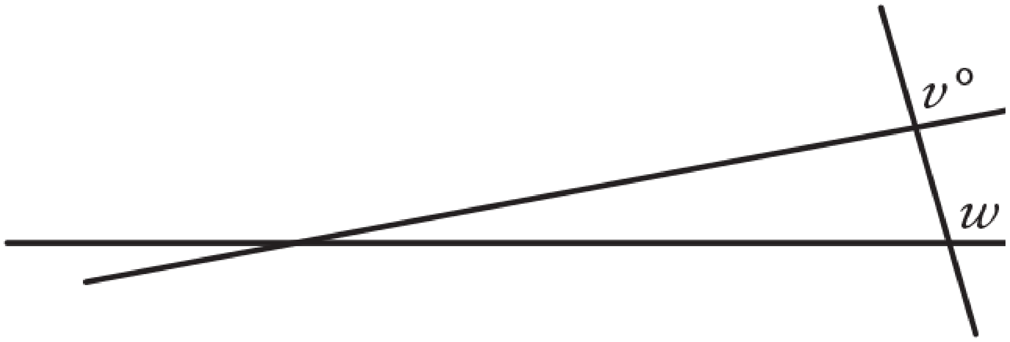
3. Establish what you **know:**
   a. *Set up equations* to solve for previously unknown lines and angles, and
   b. *Make inferences* based on the properties of shapes.
4. Establish what you **don’t know**:

    a. Take unknown values to extremes.
    b. If both quantities contain unknown values, look to gauge *relative size*.

And remember, no matter what, *don’t trust the picture*.

Variable Creation

Take a look at another example of a Geometry Quantitative Comparison in which both quantities are unknown values:

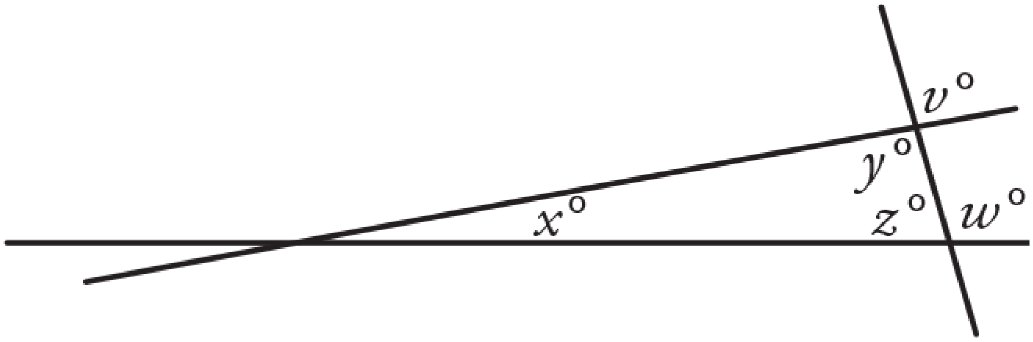


Quantity A	**Quantity B**
v	*w*

As in the preceding example, there are no numbers given, so an exact value for any of these angles is impossible to determine. This does not, however, mean that the answer is necessarily (D).

What you **need to know** is the relative size of *v* and *w*. As this type of problem gets more difficult, it becomes more difficult to establish what you **know**. An important feature of this diagram is that the intersection of the three lines creates a triangle. Triangles, when they appear, are often very important parts of diagrams, because there are many rules related to triangles that test makers can make use of.

This question appears to be about angles. After all, the values in both quantities are angles. *Create variables* to represent the three angles of the triangle:



Part of the challenge is the fact that there are actually many relationships, and thus many equations you could create. For instance:

$$x + y + z = 180$$
$$w + z = 180$$

But not all of these relationships will help determine the relative size of v and w. You need to find relationships that will allow you to directly compare v and w.

The best bet for a link between v and w is the triangle in the center of the diagram. Try to express v and w in terms of x, y, and z.

Begin with v. Angles v and y are vertical angles, and thus equal. In Quantity A, replace v with y:

Quantity A	**Quantity B**
$v = \mathbf{y}$	w

Now, if you can express w in terms of y, then you may be able to determine the relative size of v and w.

Based on the diagram, you know $w + z = 180$, so $w = 180 - z$.

Quantity A	**Quantity B**

y	$w = \mathbf{180 - z}$

You can't directly compare y and $(180 - z)$, so keep going. Try to find an equation that links y and z.

Remember, you also know that $x + y + z = 180$. Isolate z:

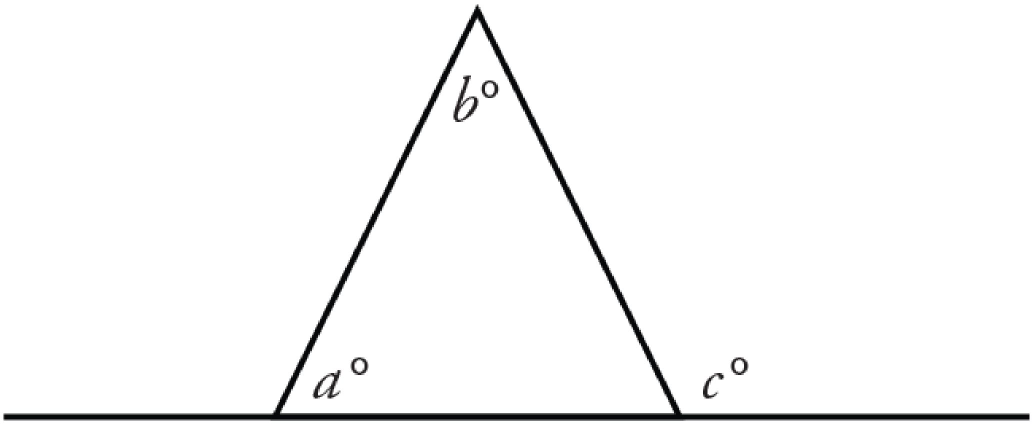
$$x + y + z = 180$$
$$z = 180 - x - y$$

Now, substitute $(180 - x - y)$ for z in Quantity B:

Quantity A	**Quantity B**
y	$180 - (180 - x - y) =$ $\mathbf{x + y}$

Now you can directly compare the two quantities. You know that x and y both represent angles, and so must be positive, so $x + y$ must be greater than y. The correct answer is **(B)**.

By the way, you've just proven that the exterior angle (w) is equal to the sum of the two remote interior angles $(x + y)$. This is true in every case:



$$c = a + b$$

This is a good rule to know!

Strategy Tip:

If a diagram presents a common shape, such as a triangle or a quadrilateral, it is often helpful to create variables to represent unknown angles or lengths. Once you've created variables, you can:

create equations, based on the properties of the shape
compare the relative size of both quantities using common variables

Word Geometry

This section is devoted to QC questions that test your knowledge of Geometry, but don't provide a picture. For example:

4 points, *P, Q, R,* and *T* lie in
a plane. *PQ* is parallel to *RT*
and *PR* = *QT.*

Quantity A	**Quantity B**
PQ	*RT*

The basic process remains the same. First, establish what you **need to know**.

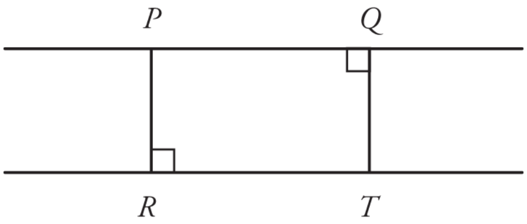
Both quantities contain unknown values, so you need to determine the relative size of each line segment.

Now, establish what you **know**. For *any* Word Geometry question, the first thing you need to do is draw the picture.

You want to draw a picture that is accurate, but quick. And remember, you can always redraw the figure if you run into trouble.

For this question, the easiest way to start is to draw the parallel lines:

You know that points *P* & *Q* lie on one line, and *R* & *T* lie on the other, but you don't know their relative sizes. But you do know that *PR* = *QT*:



To start, the easiest thing to do is align the points so that they form a rectangle. Now, *PR* = *QT*. This diagram reflects all the information provided.

Now, take another look at the quantities:

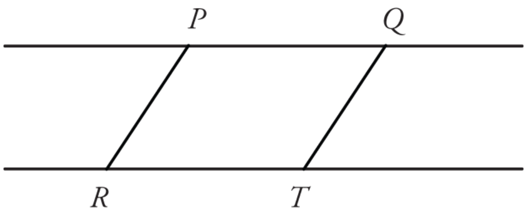
Quantity A	**Quantity B**
PQ	*RT*

According to the diagram, *PQ* = *RT*. A ~~B~~ ~~C~~ D

You're not done. You need to *try to prove* (D).

Now, the final step: establish what you **don't know**. Remember that diagram is only one possible way to represent the common information. Ask yourself, "What can change in this diagram?"

In the previous diagram, *PQ* and *RT* were drawn perpendicular to the two parallel lines. But the angle can change. Redraw the diagram with *PR* and *QT* slanted:



This diagram represents another possible configuration of the four points. Now how does *PQ* compare to *RT* ?

Although it may not be immediately obvious, *PQ* is still equal to *RT*. Whereas the first diagram created a rectangle, this diagram has created a parallelogram. For additional practice, prove that *PQRT* is a parallelogram.

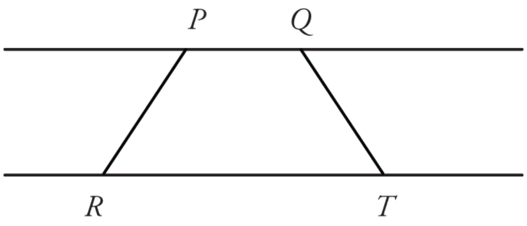
It may be tempting to choose choice (C) at this stage, but be careful! The key to Word Geometry questions is to avoid making *any* assumptions not explicitly stated in the common information.

It is not sufficient to merely change the diagram. Ask yourself, "What can I change in the diagram to change the relative size of *PQ* and *RT*?"

Changing the angle at which segments *PR* and *QT* intersected the parallel lines was not sufficient to achieve different results. What else can change?

The two preceding diagrams share a common feature that is not required by the common information:
PR and *QT* are parallel.

Redraw the figure so that *PR* and *QT* are NOT parallel, but still equal:



In this version of the diagram, *RT* is clearly longer than *PQ*. The answer is **(D)**.

Strategy Tip:

Word Geometry problems follow the same basic process:

1. Establish what you **need to know**.
2. Establish what you **know**.
    a. Draw the picture.
    b. If you're trying to prove (D), you may need to redraw the picture.
3. Establish what you **don't know**.
    a. Ask yourself, "What changes to the picture would affect the *relative size* of the quantities?

Using Numbers

For many Word Geometry Quantitative Comparison questions, using numbers is an effective technique.

This technique is most effective when the question references specific dimensions of a shape (e.g., length, width, radius) but provides no actual numbers. For example:

Rectangles *R* and *S* have equal
areas. Rectangle *R*'s length is
greater than Rectangle *S*'s width.

Quantity A	**Quantity B**
The area of Rectangle *R* if the length increases 30%	The area of Rectangle *S* if the width increases 30%

First, establish what you **need to know**. Both quantities have an unknown value, so you will have to judge their relative size.

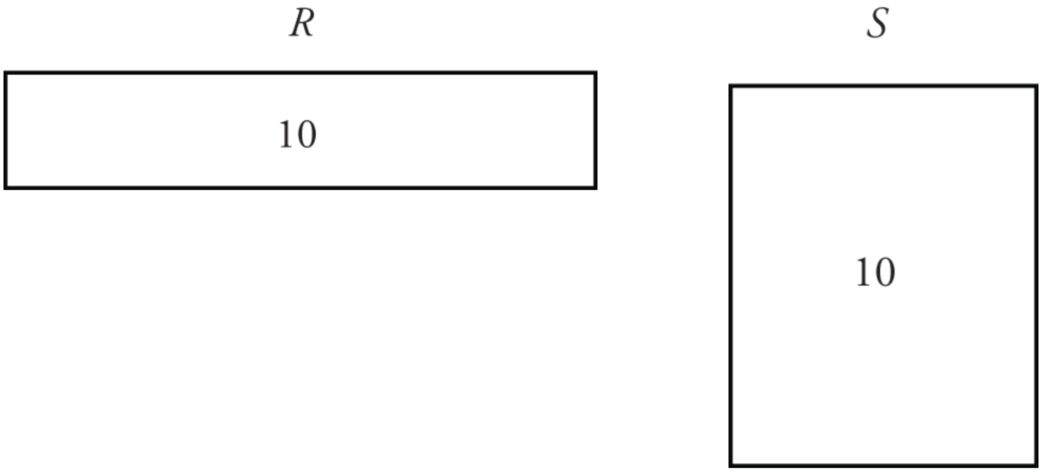
The best way to judge the relative size of each quantity is to use numbers.

The common information states that the rectangles have equal areas. An easy number to use for the area is 10. The numbers chosen in this example are only one set of possibilities, but they were chosen because they are easy to use:

$$\text{Area}_R = \text{Area}_S = 10$$

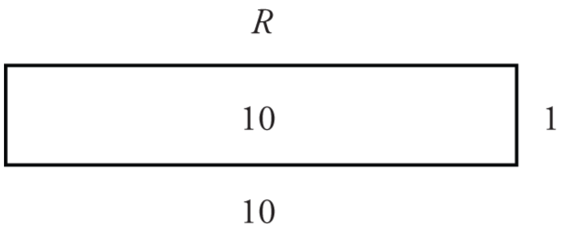
Now, draw the picture. Make sure you include the numbers you chose.

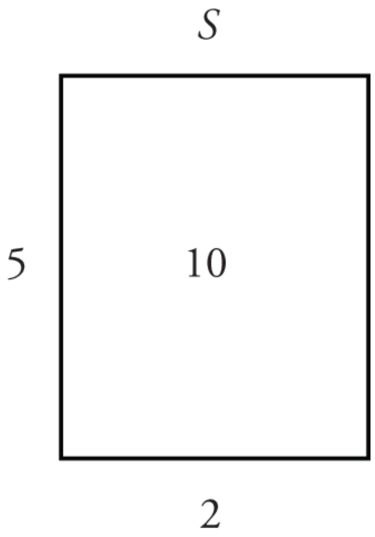
Each rectangle has an area of 10, but the length of *R* is greater than the width of *S*:



Quantity A mentions the length of *R* and Quantity B mentions the width of *S*. Pick values for the length and width of *R* and *S*.

Make the length of *R* 10 and the width 1. Make the length of *S* 2 and the width 5:



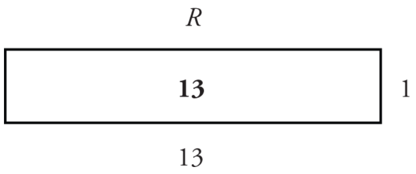


Quantity A	**Quantity B**
The area of Rectangle *R* if the length increases 30%	The area of Rectangle *S* if the width increases 30%

Now, evaluate the quantities. Start with Quantity A. Increase the length of *R* by 30 percent:

130% of 10 = (1.3)(10) = 13

The new area of *R* is $l \times w = (13)(1) = 13$:

Quantity A	**Quantity B**
	The area of Rectangle *S* if the width increases 30%

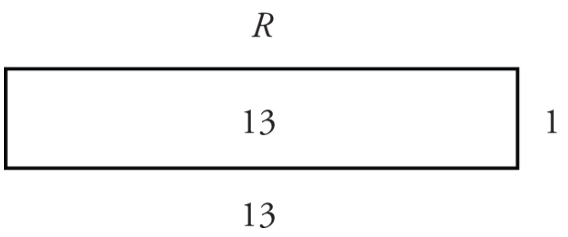
Now, evaluate Quantity B. Increase the width of *S* by 30 percent:

130% of 5 = (1.3)(5) = 6.5

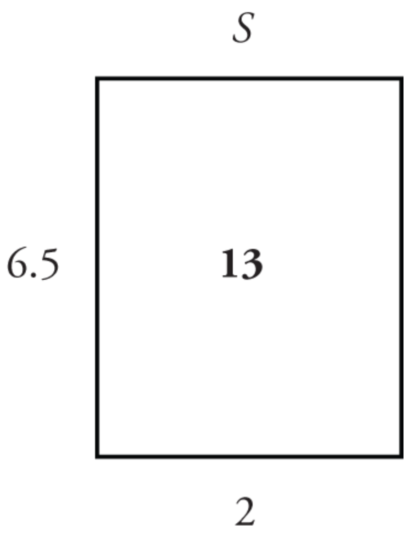
Use the calculator for this computation, if need be.

The new area of *S* is *l* × *w* = (2)(6.5) = 13:

Quantity A



Quantity B



The new areas are equal. This result will hold regardless of the precise length you choose. The correct answer is **(C)**.

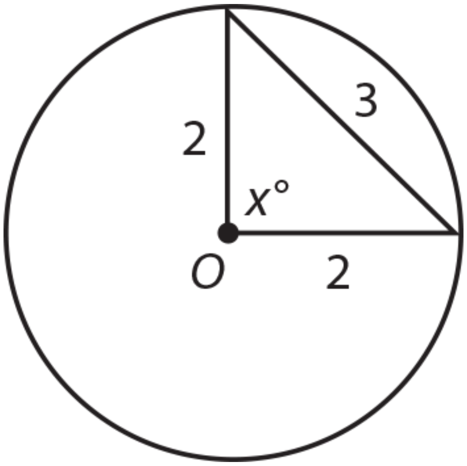
Strategy Tip:

When a Word Geometry question references specific dimensions (e.g., length, width, radius) but does not provide actual numbers, Using numbers is a viable strategy.

To successfully use numbers, remember the following:

1. Pick numbers that match any restrictions in the common information or statements.
2. Try to prove (D) by testing several valid cases.
3. Look for patterns that suggest the answer is (A), (B), or (C).

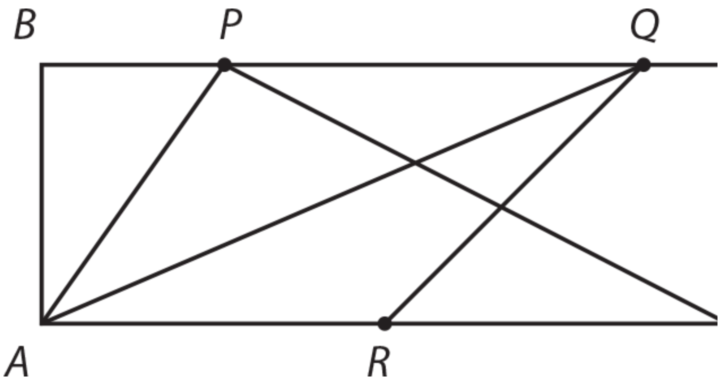
Problem Set



O is the center of the circle.

1.

Quantity A	**Quantity B**
x	90



ABCD is a rectangle.
R is the midpoint of *AD*.

2.

Quantity A	**Quantity B**
The area of Triangle *APD*	Twice the area of Triangle *AQR*

---

3. The circumference of Circle A is twice the circumference of Circle B.

Quantity A	**Quantity B**
The area of Circle A	Twice the area of Circle B

---

4. 1,600 feet of fencing is used to enclose a square plot.

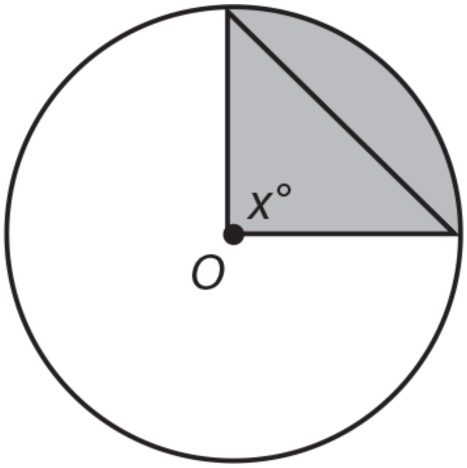
Quantity A	**Quantity B**

The plot's new area if the length were reduced by 4 feet and the width increased by 4 feet	The plot's new area if the length were equal to 398 feet and the width were equal to 402 feet

---

5.

Quantity A	**Quantity B**
The third side of an isosceles triangle with sides of 3 and 9	The third side of an isosceles triangle with sides of 6 and 8



O is the center of the circle.

The area of the circle is 16π.

The area of the shaded region < 4π.

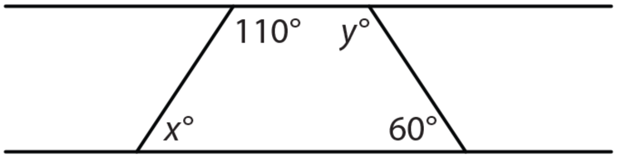
6.

Quantity A	**Quantity B**
x	90

---

7. A circle with radius $\frac{4}{\sqrt{\pi}}$ has the same area as a particular square.

Quantity A	**Quantity B**
9π	The area of the square if each side were increased by 1



8.

Quantity A	**Quantity B**
$y - x$	50

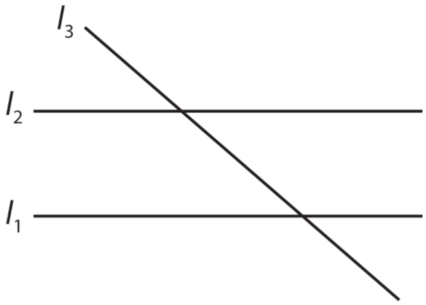
---

9. Rectangle A has twice the area of Rectangle B. The width of Rectangle A is less than 1/2 the width of Rectangle B.

Quantity A	**Quantity B**
The area of Rectangle A	The area of Rectangle B if its width is increased by more than 2

---

l_1 and l_2 are parallel lines, and none of the lines in the figure are vertical.



10.

Quantity A	**Quantity B**
The slope of line l_1 minus the slope of line l_3	The slope of line l_2 minus the slope of line l_3

Solutions

1. **(A)**

   Use Quantity B as a benchmark by trying to make *x* equal to 90. That is, try to prove (C). If it doesn't work, you'll have your answer. Mark the angle as 90 degrees and use the Pythagorean Theorem to find the hypotenuse, using the two legs of 2:

   $$\begin{aligned}&2^2 + 2^2 = (c)^2\\&c^2 = 8\\&c = \sqrt{8}\\&c = 2\sqrt{2} \approx 2(1.4) = 2.8\end{aligned}$$

   However, you know that the "hypotenuse" (the long side) is actually 3, not 2.8. The bigger the long side is, the larger angle *x* is going to be (picture how the triangle opens as *x* increases). If angle *x* were 90 degrees, the hypotenuse would be about 2.8. However, long side that looks like a hypotenuse is actually 3, so angle *x* must be greater than 90 degrees.

   The answer is (A).

2. **(C)**

   There are no numbers mentioned anywhere in the problem, but that doesn't mean the answer is (D). Some important observations: both of the triangles mentioned in the quantities have the same height (that of the rectangle, which is *AB*). Also, since *R* is the midpoint of *AD*, the base of triangle *APD* is exactly twice the base of triangle *AQR*:

Quantity A	**Quantity B**
The area of triangle $APD = \frac{1}{2}(AD)(AB)$	Twice the area of triangle $AQR =$ $2 \times \frac{1}{2}(AR)(AB) = (AR)(AB) =$

$$\frac{1}{2}(AD)(AB)$$

You could use numbers to make the comparison easier. Note that if this is the only approach you take, you should try to prove (D) by testing several cases and confirming whether any emerging pattern makes sense. Let the height of each triangle (also the height of the rectangle) be 5. Let *AD* be 8, so *AR* = *RD* = 4:

Quantity A	**Quantity B**
The area of triangle $APD = \frac{1}{2}(8)(5) = \mathbf{20}$	Twice the area of triangle $AQR = 2 \times \frac{1}{2}(4)(5) = \mathbf{20}$

Thus, the two quantities are equal. If you tried different numbers, you would continue to get this result, a pattern that makes sense based on the way identical numbers are input into both quantities.

Of course, you could also use variables here; for instance, let the height be *x* and *DC* and *RC* each be *y*. The quantities will each come out equal to *xy*.

The answer is (C).

3. **(A)**
   The formula for circumference is simply $C = 2\pi r$, therefore, doubling the radius will double the circumference (this is NOT true for the area formula, which involves *squaring* the radius). Thus, from the common information “The circumference of Circle A is twice the circumference of Circle B,” you can infer that the radius of A is twice that of B.

   For instance, say A’s radius is 2 and B’s radius is 1. In that case, the area of Circle A is 4π, so Quantity A = 4π. The area of Circle B is π, so Quantity B = 2π. Quantity A is greater.

This will work for any numbers you decide to use. If a circle’s radius is double another circle’s radius, its area will be four times as big, because the double-radius is then squared. The answer is (A).

4. **(B)**

   A square has a perimeter of 1,600 feet. To find the length of each side of the square, divide 1,600 by 4, because each side has the same length. The length of each side of the square is 400.

   Quantity A asks you about a 396 by 404 rectangle, and Quantity B asks you about a 398 by 402 rectangle. Using the calculator, you get 396 × 404 = 159,984 for Quantity A and 398 × 402 = 159,996 for Quantity B.

   The answer is (B).

   Notice that if two numbers have a finite sum (396 + 404 = 800 and 398 + 402 = 800), their product will get larger as the two numbers get closer together. For example, 4 × 4 is greater than 3 × 5, 99 × 101 is greater than 97 × 103, and so on for any similar example you can think of.

   In geometry, for a finite perimeter, the area of a shape is maximized by making the shape as regular as possible. That is, the more equilateral the shape, the greater the area. Thus, a square has greater area than any other rectangle with the same perimeter. The rectangle in Quantity B is closer to square than the rectangle in Quantity A, and thus it has the greater area.

5. **(A)**

   This is a question about the Third Side Rule, which says that the third side of a triangle must be less than the sum of the other two sides and greater than their difference. The triangle referenced in Quantity A has two sides of 3 and 9. By the Third Side Rule, the third side must be between 6 and 12 (the difference and the sum of 3 and 9). Since the triangle is isosceles and two sides must be of equal measure, the third side must be 9. (Also, try picturing a 3−3−9 triangle—it’s impossible because the sides would never meet.)

The triangle referenced in Quantity B has two sides of 6 and 8. By the Third Side Rule, the third side must be between 2 and 14. Therefore, the third side could be either 6 or 8. Whether Quantity B is 6 or 8, it is definitely less than 9.

The answer is (A).

6. **(B)**

   Use Quantity B as a benchmark. If x were equal to 90, the shaded region would have an area equal to 1/4 that of the entire circle (since 90 is 1/4 of 360). Thus, if the angle were equal to 90, the shaded region would have an area of 4π (1/4 of the entire circle's area). Since the area of the shaded region is actually less than 4π, x must be less than 90.

   The answer is (B).

7. **(A)**

   The radius of the circle is $\frac{4}{\sqrt{\pi}}$ and its area equals the area of the square. Plug $\frac{4}{\sqrt{\pi}}$ into the formula for area of a circle:

$$A = \pi\left(\frac{4}{\sqrt{\pi}}\right)^2$$

$$A = \pi\left(\frac{16}{\pi}\right)$$

$$A = 16$$

   Thus, the area of the square is also 16 and the side is 4.

   Quantity B is the area of the square if each side were increased by 1; that is, if each side were now equal to 5. Thus, Quantity B = 25. Quantity A is simply equal to 9π. Since π is more than 3, Quantity A is more than 27.

The answer is (A).

8. **(D)**

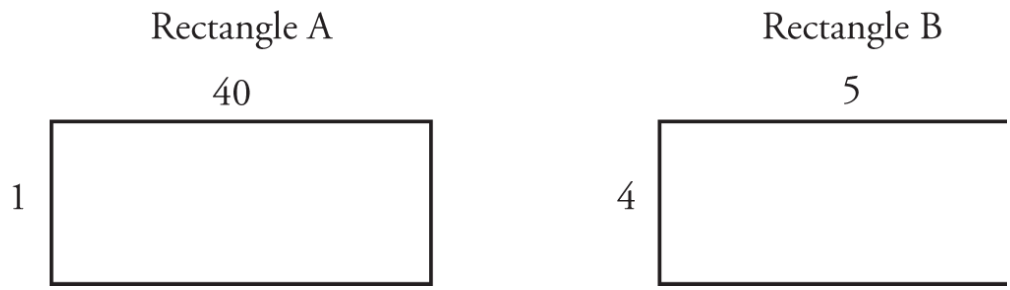
   Note that you are *not* told that the two horizontal-seeming lines in the figure are actually parallel, so you may *not* assume this. You do know that all four angles of the quadrilateral must sum to 360, so you can deduce that $x + y$ must equal 190.

   However, without knowing that you have parallel lines, you have no way of knowing how to split up the 190 between x and y, and therefore no way of knowing whether $y - x$ is greater than 50. (For instance, if the lines were parallel, y would equal 120 and x would equal 70, and $y - x$ would be exactly 50. Adjust the figure even one degree—for instance, if y gets larger, x will get proportionately smaller—and $y - x$ will no longer equal 50.)

   The answer is (D).

9. **(D)**

   Rectangle A has twice the area of Rectangle B and less than 1/2 the width. Start by drawing one scenario of how this could be:

   Now, try to prove (D).

   In this scenario, Quantity A equals 40.

   In Quantity B, Rectangle B's width is increased by "more than 2." (Note that because you were not given any real numbers so far, it's likely that you could come up with a scenario in which this increase of “more than 2” yields a larger Quantity B and another scenario in which it does not—(D) should feel like a good guess to you here.)

Thus, Rectangle B now has a width of more than 6. Its area is now more than 30. But more than 30 could still be less than 40, or it could be more.

The answer is (D).

10. **(C)**

    This one’s quick. This question is simply a test of the fact that parallel lines have equal slopes. Therefore, the slope of line l_1 and the slope of line l_2 are identical, and Quantities A and B are equal, regardless of the slope of line l_3.

    The answer is (C).

Chapter 33

NUMBER PROPERTIES

In This Chapter...

Positives & Negatives

Exponents

Consecutive Integers

Chapter 33

Number Properties

Number Properties turns out to be very fertile ground for Quantitative Comparison questions. By creating situations that hinge on the general behavior of negative numbers or of fractions raised to an exponent, ETS can create conceptually challenging problems that do not require a lot of calculation. A very popular theme related to Number Properties is *trying to prove* (D).

Positives & Negatives

Perhaps no dichotomy is as important to Quantitative Comparisons as is the Positive/Negative distinction. For one thing, ETS can easily create a question about positives and negatives without having to use either word. But there are common clues. If you see any of the following clues in a QC question, ask yourself whether positive and negative numbers play a role:

Common information states that a variable is either greater than or less than 0:

$x < 0$ means x is negative

$y > 0$ means y is positive

The product of more than one variable is greater than or less than 0:

$pq > 0$ means p and q have the same sign; they are either both positive or both negative

An expression that contains both a negative sign and an exponent:

$(-x)^4$ means $(-x)^4$ is positive, since 4 is even

GREATER THAN OR LESS THAN 0

These clues often mean that you can save time by making generalizations based on the signs of variables. For example:

$$x < 0$$

Quantity A	**Quantity B**
$x - 2$	$-(x - 2)$

You want not only to get this question right, but to get it right *quickly*. One option is to plug in numbers for *x*.

For instance, plug in −1 for *x*:

$$x = 3$$

Quantity A	**Quantity B**
$(-1) - 2 = -3$	$-((-1) - 2) = 3$

When $x = -1$, Quantity B is bigger. A ~~B~~ ~~C~~ D

But do you know Quantity B will *always* be bigger? No, you would need to try other numbers for *x*, and that will be time-consuming.

Instead, see whether you can make a generalization about the sign of each of the quantities. If *x* is negative, can you say anything definite about the sign of $x - 2$? Yes, you can. A negative minus a positive will always be negative.

You can rewrite Quantity A:

$$x = 3$$

Quantity A	**Quantity B**
Negative	$-(x-2)$

Now you need to see whether you can make a generalization about Quantity B. Start with the expression inside the parentheses: $x - 2$. You know that $x - 2$ is always negative, so you can rewrite the expression as:

− (NEGATIVE)

What you have is a negative number inside the parentheses being multiplied by a negative:

$(x - 2)$ is negative, so $-(x - 2)$ is positive

You can rewrite Quantity B:

$$x = 3$$

Quantity A	**Quantity B**
Negative	Positive

Instead of trying specific numbers, you made generalizations about the sign of each quantity. *Any* positive number is greater than *any* negative number, so Quantity B will *always* be greater. The correct answer is (B).

Strategy Tip:

If you are told the sign of a variable (e.g., $x < 0$), try to make a generalization about the sign of each quantity.

PRODUCT OF VARIABLES GREATER THAN OR LESS THAN 0

In the last problem, you knew the sign of the variable. That will not always be the case:

$$xy > 0$$

Quantity A	**Quantity B**
$x + y$	0

This question is still about positives and negatives, but now it concerns the signs of both x and y. The common information is telling you something very important. There are two possible scenarios:

x and y are *both* positive.
x and y are *both* negative.

To find the answer, you need to test both scenarios. As in the last problem, you are testing *not* with specific numbers, but with the signs of the variables.

First, test the first scenario: *x* and *y* are both positive:

$xy > 0$

Quantity A	**Quantity B**
$x + y$ Positive + Positive = Positive	0

If *x* and *y* are positive, Quantity A will always be positive, regardless of the values of *x* and *y*. $\cancel{\text{A}}\ \cancel{\text{B}}$ C D

Now, test the second scenario: *x* and *y* are both negative.

$xy > 0$

Quantity A	**Quantity B**
$x + y$ Negative + Negative = Negative	0

If *x* and *y* are negative, Quantity A will always be negative, regardless of the values of *x* and *y*. The correct answer is **(D)**.

Strategy Tip:

When the product of more than one variable is either greater than or less than 0, consider all possible signs and test all possible scenarios.

If $xy > 0$, the two scenarios are:

x and y are *both* positive.
x and y are *both* negative.

If $xy < 0$, the two scenarios are:

x is positive and y is negative.
x is negative and y is positive.

You will have to test *both* scenarios to get the right answer consistently.

EXPONENTS & NEGATIVES

Another sign that you are dealing with positives and negatives is the combination of exponents and negative signs:

n is an integer.

Quantity A	**Quantity B**
$(-3)^{2n}$	$(-3)^{2n+1}$

When negative numbers are raised to a power, they follow a pattern:

> Negative numbers raised to odd powers are negative.
> Negative numbers raised to even powers are positive.

You need to see if you can make a generalization about the sign of each quantity. Start with Quantity A: n is an integer, so $2n$ will always be even. The exponent will always be even, and a negative raised to an even power will always be positive:

n is an integer.

Quantity A	**Quantity B**
$(\text{Negative})^{\text{Even}} = \text{Positive}$	$(-3)^{2n+1}$

Now, test Quantity B: $2n$ is always even, which means $2n + 1$ will always be odd. A negative number raised to an odd power is negative:

n is an integer.

Quantity A	**Quantity B**
$(\text{Negative})^{\text{Even}} = \text{Positive}$	$(\text{Negative})^{\text{Odd}} = \text{Negative}$

The correct answer is **(A)**.

Strategy Tips:

All of these problems have one thing in common: you can save time by figuring out whether each quantity is positive or negative.

Be on the lookout for these clues:

1. Common information states that a variable is greater than or less than 0 (e.g., $x > 0$, $p < 0$).
2. Common information states the product of two variables is greater than or less than 0 (e.g., $xy < 0$).
3. An expression contains both an exponent and a negative sign (e.g., $(-2)^x$).

When problems contain both exponents and negative signs, try to make generalizations about the sign of each quantity:

1. A negative number raised to an *odd* power is *negative*.
2. A negative number raised to an *even* power is *positive*.

Exponents

The test makers love the following exponent rules:

Numbers greater than 1 get *bigger* you raise them to higher powers:

$$2^1 < 2^2 < 2^3$$
$$2 < 4 < 8$$

Numbers between 0 and 1 get *smaller* as you raise them to higher powers:

$$\left(\frac{1}{2}\right)^1 > \left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$$
$$\frac{1}{2} > \frac{1}{4} > \frac{1}{8}$$

When you see variables raised to exponents, don't forget about proper fractions (numbers between 0 and 1)!

Be on the lookout for questions that involve exponents and either fractions or variables that can be fractions:

x and *y* are positive.

Quantity A	**Quantity B**
xy	$(xy)^2$

At first, this question may seem to be about positives and negatives. But, if *x* and *y* are both positive, both quantities will be positive. You cannot make a quick comparison using positives and negatives.

The key to this question is the exponent. You have the same combination of variables raised to different powers:

x and *y* are positive.

Quantity A	**Quantity B**
$(xy)^1$	$(xy)^2$

First, try numbers greater than 1 for *x* and *y*. Plug in 2 for *x* and 3 for *y*:

x and *y* are positive.

Quantity A	**Quantity B**
$(2)(3) = \mathbf{6}$	$((2)(3))^2 = 6^2 = \mathbf{36}$

In this case, Quantity B is bigger. A ~~B~~ ~~C~~ D

Don’t stop there. You need to try to PROVE (D). The common information did not tell you that *x* and *y* are integers—you should see what happens if they are fractions.

Plug in $\frac{1}{2}$ for *x* and $\frac{1}{2}$ for *y*:

x and *y* are positive.

Quantity A	**Quantity B**
$\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)=\mathbf{\frac{1}{6}}$	$\left(\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)\right)^2=\left(\frac{1}{6}\right)^2=\mathbf{\frac{1}{36}}$

Fractions get smaller as they are raised to higher powers, so now Quantity A is larger than Quantity B. The correct answer is **(D)**.

Strategy Tip:

On questions that involve variables and exponents, try to prove (D). Try numbers *greater than* 1 and numbers *between* 0 *and* 1.

Numbers greater than 1 get *bigger* as you raise them to higher powers

Numbers between 0 and 1 get *smaller* as you raise them to higher powers

Consecutive Integers

QC questions will sometimes ask you to compare the sum or product of sets of consecutive integers. The trick is to avoid finding the actual sums or products by *eliminating overlap*:

Quantity A	**Quantity B**
The product of all the integers from 2 to 23, inclusive	The product of all the integers from 5 to 24, inclusive

Both of these products are far too large to calculate in a reasonable amount of time, even with a calculator. Instead, you need to figure out which numbers appear in both products, and cancel those numbers.

In this problem, the numbers 5 through 23 appear in both sets. You can rewrite the products as:

Quantity A	**Quantity B**
2 × **3** × **4** × (5 × 6 × … 22 × 23)	(5 × 6 × … 22 × 23) × **24**

The product of the numbers 5 through 23 is positive, and has the same value in each quantity. Therefore, because of the *invisible inequality*, you can divide out (5 × 6 × … 22 × 23), and focus on what is left:

Quantity A	**Quantity B**
$\frac{2 \times 3 \times 4 \times (5 \times 6 \times \ldots 22 \times 23)}{(5 \times 6 \times \ldots 22 \times 23)} =$	$\frac{(5 \times 6 \times \ldots 22 \times 23) \times 24}{(5 \times 6 \times \ldots 22 \times 23)} =$
$2 \times 3 \times 4 =$ **24**	**24**

The values in the two quantities are equal. The correct answer is **(C)**.

Strategy Tip:

To compare the sums or products of sets of consecutive integers, eliminate overlap in order to make a direct comparison.

Problem Set

1. x is an integer.

Quantity A	**Quantity B**
$\frac{1}{100^X}$	$\frac{1}{99^X}$

2. n is an integer.

Quantity A	**Quantity B**
$(-1)^{2n+1} \times (-1)^n$	$(1)^n$

3. $1 < 3x < 2$

Quantity A	**Quantity B**
x^5	x^7

4.

$98 < x < 102$

$103 < y < 107$

Quantity A	**Quantity B**
$y - x$	$\lvert y - x \rvert$

5.

$xyz < 0$

Quantity A	**Quantity B**
$x + y + z$	$2x + 2y + 2z$

6.

Quantity A	**Quantity B**
$(-101)^{102}$	$(-102)^{101}$

7.

$n < -1$

Quantity A	**Quantity B**
$n^2 \cdot n^4$	$(n^2)^4$

8.

Quantity A	**Quantity B**
The sum of the consecutive integers from −12 to 13	13

9.

$$\frac{x}{y} < 0$$

$$y > x$$

Quantity A	**Quantity B**
$y - x$	xy

10.

Quantity A	**Quantity B**
The sum of the consecutive integers from 2 to 15	34 less than the sum of the consecutive integers from 1 to 17

Solutions

1. **(D)**

   This question might be trying to trick you into picking (A) (the “greater looking” number) or maybe reasoning that a greater number under a fraction gets smaller, and therefore picking (B). But, of course, the exponent changes things. The easiest way to approach this is simply to plug in small values for *x* and try to prove (D). You know that *x* is an integer. Try some values for which it will be easy to calculate a value.

   If $x = 1$, Quantity B is greater $\left(\frac{1}{99} > \frac{1}{100}\right)$.

   If $x = 0$, the quantities are equal (because any number to a power of 0 is equal to 1).

   Stop here—the answer is (D).

2. **(D)**

   Note that −1 and 1, when raised to an integer power, have very limited possibilities. Negative 1 raised to an even power is 1, and −1 raised to an odd power is −1, whereas 1 raised to a power is always 1. Therefore, this is really a problem about odds and evens. So plug in a small even number and a small odd number and try to prove (D).

If $n = 2$, Quantity A is equal to $(-1)^5 \times (-1)^2$, which is -1, and Quantity B is equal to 1.

If $n = 3$, Quantity A is equal to $(-1)^7 \times (-1)^3$, which is 1, and Quantity B is equal to 1.

Stop here—the answer is (D).

3. **(A)**

   Before proceeding to Quantities A and B, simplify $1 < 3x < 2$ by dividing through by 3:

   $$\frac{127}{255} \text{ or } \frac{162}{320}$$

   x is therefore between $\frac{1}{2}$ and $\frac{1}{2}$. More importantly, x is definitely between 0 and 1, which means it gets smaller when multiplied by itself.

   Therefore, x^5 is larger than x^7.

   The answer is (A).

   (It would be possible to plug in a value between 1/3 and 2/3, such as 1/2, which would make Quantity A equal to $\frac{23}{7}$ and Quantity B equal to $\frac{256}{16}$. However, a Number Properties approach is far superior here

—because you know that x will behave in a certain way due to its being a fraction between 0 and 1, you are saved from having to calculate anything to the 7th power).

4. **(C)**
   The presence of fairly large numbers in the common information is merely a distraction—the point is that y is definitely larger than x. Therefore, $y - x$ is positive. A positive number is the same as its own absolute value. Therefore, the answer is (C).

5. **(D)**
   Try to make generalizations about the signs of variables. If xyz is negative, then there are two possible scenarios: all three are negative, or one is negative and the other two are positive. To find the answer, you need to test both scenarios.

   If all three are negative, then both Quantity A and Quantity B have negative values. Because you can factor the 2 out of Quantity B to get $2(x + y + z)$, Quantity B's value is therefore twice Quantity A's value—that is, Quantity B becomes *more negative* and is therefore smaller. Quantity A would be greater.

   But if one of the variables were negative and the other two were positive, you wouldn't have enough information to know the sign of $x + y + z$ (remember, when multiplying or dividing, knowing the signs of what you are multiplying or dividing is enough to know the sign of the answer, but when adding or subtracting, you need to know the relative sizes of what you are adding or subtracting). For instance, if x,

y, and z are −1, 3, and 4, then $x + y + z$ is positive, and $2x + 2y + 2z$ (Quantity B) would be greater.

The answer is (D).

6. **(A)**

   A good sign that a problem can perhaps be solved with just positives and negatives is the presence of both exponents and negative signs. Using only positives and negatives, consider the problem as such:

Quantity A	**Quantity B**
$(\text{Negative})^{\text{even}}$	$(\text{Negative})^{\text{odd}}$

   Thus, Quantity A is positive and Quantity B is negative.

   The answer is (A).

7. **(B)**

   Use exponent rules to simplify the expressions in each quantity:

Quantity A	**Quantity B**
$n^2 \times n^4 = n^6$	$(n^2)^4 = n^8$

   In both quantities, n is raised to an even power, so both quantities will be positive. Because $n < -1$, the absolute value of n will get bigger as n is raised to higher powers. Therefore, Quantity B will be greater.

8. **(C)**

   If you were to write out the integers in Quantity A, you’d have −12 + −11 + −10 … + −1 + 0 + 1 … + 10 + 11 + 12 + 13.

   Note that for every negative there is a corresponding positive value. For instance, −12 cancels with 12, −11 cancels with 11, and so on. When all the canceling is through, you’re left with 13.

   The answer is (C).

9. **(A)**

   The common information is enough for you to know that this is a Positive/Negative question. If x/y is negative, then x and y have different signs. If $y > x$, then y must be positive and x negative. In Quantity A, you have a positive minus a negative—this will create a greater positive. In Quantity B, you have a negative times a positive, which is always negative.

   The answer is (A).

10. **(C)**

    To compare the sums or products of sets of consecutive integers, eliminate overlap to make a direct comparison. You can abbreviate “the sum of the consecutive integers from 2 to 15” as (2 + 3 … + 15):

Quantity A	**Quantity B**
(2 + 3 ... + 15)	1 + (2 + 3 ... + 15) + 16 + 17 − 34

Now, eliminate (2 + 3 ... + 15) from both sides:

Quantity A	**Quantity B**
0	1 + 16 + 17 − 34

Because 1 + 16 + 17 − 34 = 0, the answer is (C).

Chapter 34

WORD PROBLEMS

In This Chapter...

Ratios

Statistics

Chapter 34

Word Problems

Word Problems (WPs) very much fit into the framework of avoiding excessive computation. Difficult WP questions are often difficult because they describe situations that do not translate obviously into solvable equations.

Nevertheless, there is an added wrinkle: WP questions do not automatically provide enough information to solve for the desired values. For example:

Milo can core x apples in 10 minutes
and peel y potatoes in 20 minutes.

Quantity A	**Quantity B**
The number of apples Milo can core in an hour	The number of potatoes Milo can peel in an hour

The rate at which Milo can core apples is x apples/10 min, or $6x$ apples/hour. The rate at which Milo can peel potatoes is y potatoes/20 min, or $3y$ potatoes/hour. So the comparison is really $6x$ vs. $3y$. However, without more information, you have no way to compare these two values. The answer is **(D)**.

Strategy Tip:

Whenever you see a word problem on Quantitative Comparisons, make sure you have the information you need before doing any computation. If you don’t have enough info, the answer is (D).

Ratios

Don’t confuse a ratio with actual numbers of objects. For instance, if you know that a store carries red shirts and white shirts in a 2 to 3 ratio, the store may have five total shirts (two red and three white), 10 total shirts (four red and six white), 500 total shirts (200 red and 300 white), and so on. What is known is that there are fewer red shirts than white shirts and that the total number of shirts must be a multiple of 5.

Adding real numbers of objects to a ratio isn’t very helpful without some real numbers of objects to begin with. For example, if a store carries red shirts and white shirts in a 2 to 3 ratio, what effect does adding three red shirts have? Well, if the store had five total shirts (two red and three white), adding three red shirts changes the ratio to 5 to 3 (five red and three white). But if you started with 500 total shirts (200 red and 300 white), adding three red shirts doesn’t change the ratio very much at all; it’s now 203 to 300. Try an example:

A university contains French majors
and Spanish majors in a 5 to 7 ratio.

Quantity A	**Quantity B**
The number of French majors if 10 French majors transfer into the university	The number of French majors if 3/7 of the Spanish majors switch to French

and no other students leave,
join, or change majors

Try to prove (D). Start by constructing two scenarios for “A university contains French majors and Spanish majors in a 5 to 7 ratio.” For the first scenario, use the smallest possible values: five French majors and seven Spanish majors. For the second scenario, use much larger (but still easy to work with) numbers: 500 French majors and 700 Spanish majors.

Evaluate the first scenario. For the first scenario (five French majors and seven Spanish majors), Quantity A gives us 15 French majors (5 + 10 = 15). In Quantity B, three Spanish majors switch to French $\left(\frac{3}{7} \times 7 = 3\right)$, so there are eight French majors (5 + 3 = 8). In this scenario, Quantity A is greater.

Now, evaluate the second scenario (500 French majors and 700 Spanish majors). Quantity A gives you 510 French majors (500 + 10 = 510). In Quantity B, 300 Spanish majors switch to French $\left(\frac{3}{7} \times 700 = 300\right)$, so there are 800 total French majors. In this scenario, Quantity B is greater. The correct answer is **(D)**.

Strategy Tip:

Remember that ratios provide NO information about actual values. To try to prove (D) on a ratios problem, choose one scenario in which the actual values are the same values as the ratio and choose another scenario in which the numbers are much larger (but still pick numbers that are easy to work with).

Statistics

Aside from the standard average formula (which you should know *very* well), there is another property of averages that is often tested in the Quantitative Comparison format. Try an example:

A company has two divisions. Division A has 105 employees and an average salary of $60,000. Division B has 93 employees and an average salary of $70,000.

Quantity A	**Quantity B**
The average salary of all the employees at the company	$65,000

A lot of unnecessary computation could go into answering this QC question. Notice that your benchmark value in Quantity B is exactly halfway between the average salaries of the two divisions. This is very convenient because you can use the principle of weighted averages.

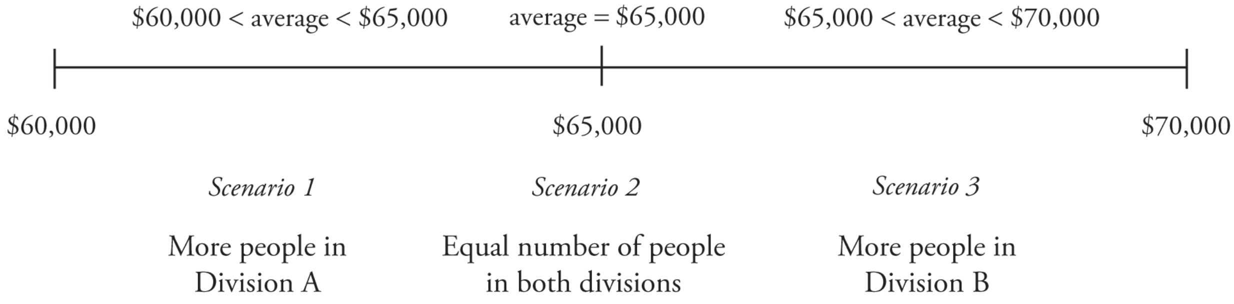
Suppose that instead of 198 employees (105 + 93), you have six employees: three people in each division. To simplify things, you can say that everyone in Division A makes $60,000 and everyone in Division B makes $70,000.

The average salary for all 6 employees will be:

$$\frac{3(60{,}000)+3(70{,}000)}{6}=65{,}000$$

There are an equal number of people in each division, so the average salary is the average of 60,000 and 70,000.

Think of average salaries as a spectrum. There are three scenarios:



The common information tells you there are more employees in Division A (105 vs. 93). The average salary of the whole company will be less than $65,000:

Quantity A	**Quantity B**
The average salary of all the employees at the company = **less than $65,000**	$65,000

The correct answer is **(B)**.

Strategy Tip:

In any question that involves two groups that have some kind of average value, use the principles of weighted averages.

If two groups have an equal number of members, the total average will be the average of the two groups

(e.g., $\frac{3(60{,}000)+3(70{,}000)}{6}=65{,}000$).

If one group has more members, the total average will be closer to the average of that group

(e.g., $\frac{105(60{,}000)+93(70{,}000)}{198}=64{,}696.97$). There's no need to do this calculation!

Problem Set

1. Bag A contains red and black marbles in a 3 to 4 ratio.
Bag B contains red and black marbles in a 4 to 3 ratio.

Quantity A	**Quantity B**
The total number of red marbles in both bags combined	The total number of black marbles in both bags combined

2. June can run 6 laps in x minutes.
Miriam can run 11 laps in $2x$ minutes.

Quantity A	**Quantity B**
The number of minutes it takes June to run 24 laps	The number of minutes it takes Miriam to run 22 laps

3. Abe's quiz scores are 62, 68, 74, and 68.
Ben's quiz scores are 66 and 70.

Quantity A	**Quantity B**
The score Abe needs on his fifth quiz to raise his average to 70	The score Ben needs on his third quiz to raise his average to 70

4. Set S = {2, 3, 5, 2, 11, 1}

Quantity A	**Quantity B**
The average of Set S	The mode of Set S if every number in the set were doubled

5. Silky Dark Chocolate is 80% cocoa.
Rich Milk Chocolate is 50% cocoa.
Smooth White Chocolate is 0% cocoa.

Quantity A	**Quantity B**
Percent cocoa of a mixture of 3 parts Silky Dark Chocolate and 1 part Smooth White Chocolate	Percent cocoa of a mixture of 2 parts Rich Milk Chocolate and 1 part Silky Dark Chocolate

6. The average of six numbers is 44.
The average of two of those numbers is 11.

Quantity A	**Quantity B**
The average of the other 4 numbers	77

---

7. Tavi drives 113 miles at 50 miles per hour and returns
via the same route at 60 miles per hour.

Quantity A	**Quantity B**
Tavi’s average speed for the entire round trip	55 mph

---

8. Joe reaches into a bag containing 5 red, 4 blue, and 8 orange jellybeans,
and randomly selects three jellybeans.

Quantity A	**Quantity B**
The probability of selecting a red, then a blue, then an orange jellybean	The probability of selecting a red, then another red, then an orange jellybean

9. Preeti can make 100 sandwiches in 1 hour and 15 minutes. Mariska can make 50 sandwiches in 30 minutes.

Quantity A	**Quantity B**
The time it would take Preeti and Mariska to make a total of 180 sandwiches, each working at her own independent rate	The time it would take to make 110 sandwiches if Mariska worked alone for 30 minutes and then Mariska and Preeti worked together to finish the job

---

10. A particular train travels from Town A to Town B at x miles per hour, and then from Town B to Town C at $1.2x$ miles per hour.

Quantity A	**Quantity B**
The train’s travel time from Town A to Town B	The train’s travel time from Town B to Town C

Solutions

1. **(D)**

   While you have the red-to-black ratios for each of the two bags, you don't have any real numbers of marbles anywhere, so it's impossible to combine the two ratios. Here, you can try to prove (D). For instance, say each bag contains 7 marbles. In such a case, Bag A would have 3 red and 4 black, and Bag B would have 3 black and 4 red. Quantity A and Quantity B would then each be equal to 7.

   However, what if Bag A contains 7 marbles and Bag B contains 700 marbles? Then Bag A would have 3 red and 4 black, and Bag B would have 400 red and 300 black. In such a case, Quantity A would be equal to 403 and Quantity B would be equal to 304.

   The answer is (D).

2. **(C)**

   First, determine whether there's a shortcut here. June can run six laps in x minutes. If Miriam were equally fast, she could run 12 laps in $2x$ minutes (twice the laps in twice the time). As it turns out, Miriam can only do 11 laps in that time, so Miriam is slightly slower than June. If the quantities then asked for June and Miriam's times to run *the same number of laps,* you would not have to do any calculating: June is faster, so Miriam's time would be greater. However, June (the slightly faster person) is being asked to run slightly more laps, so it’s pretty hard to estimate. Instead, use Rate × Time = Distance.

Because Rate × Time = Distance, Rate is equal to $\frac{6x - 15y}{10}$. Thus:

June’s rate is $\frac{3}{x}$

Miriam’s rate is $3\,\frac{1}{2}$

Quantity A asks for June’s time to run 24 laps. Rate × Time = Distance, therefore, Time is equal to $\frac{\text{Distance}}{\text{Rate}}$. Therefore:

$$T = \frac{24}{\frac{6}{x}}$$

$$T = 24 \times \frac{x}{6}$$

$$T = 4x$$

June’s time is 4*x*.

Quantity B asks for Miriam’s time to run 22 laps. Given that Time is equal to Distance/Rate:

$$T = \frac{22}{\frac{11}{2x}}$$

$$T = 22 \times \frac{2x}{11}$$

$$T = 4x$$

Miriam’s time is also 4*x*.

The answer is (C).

3. **(A)**
This is an excellent example of a Word Problems problem for which no real calculation is needed if the idea of weighted averages is understood. Abe’s current average is 68 (for a quick average, note that two scores *are* 68, and of the other two scores, one is six points over 68 and one is six points under 68, keeping the overall average at 68). Ben’s average is also 68 (halfway between 66 and 70).

For Abe to get a 70 overall, his fifth score will have to compensate for four too-low scores. For Ben to get a 70 overall, his third score will only have to compensate for two too-low scores. So Abe will need a higher score to raise his average to 70 than Ben will. (This is the same as the more bad grades you have, the higher you have to get on the next quiz to pull your average back up.)

Actually doing this problem mathematically would take too much time, but for the curious, Abe’s fifth score could be calculated as such:

$$\frac{33}{7} \times \frac{14}{3} = \frac{3 \times 11}{7} \times \frac{2 \times 7}{3}$$

As it turns out, Abe needs a 78.

Ben’s third score could be calculated as such:

$$\frac{66 + 70 + x}{3} = 70$$

Ben needs a 74.

The answer is (A).

4. **(C)**

   There’s no shortcut to find the average here. Simply add 2 + 3 + 5 + 2 + 11 + 1 to get 24 and divide by 6 to get 4. Quantity A is therefore equal to 4.

   Finding the mode is much easier (the mode is simply the number that occurs most often in the list). The current mode is 2. When you double everything in the list, the mode will then be 4. (The other numbers in the list are irrelevant—don’t bother to double them).

   The answer is (C).

5. **(C)**

In Quantity A, you need the percent cocoa of a mix that is three parts dark and one part smooth white. So you will need to create a weighted average (that is, you need to count the dark chocolate three times in the average, because there’s three times as much of it):

$$\frac{80+80+80+0}{4}=60$$

Quantity A’s mix will be 60 percent cocoa. Create a similar weighted average for Quantity B (2 parts milk, 1 part dark):

$$\frac{15}{4}=3+\frac{3}{4}=3\,\frac{3}{4}$$

Quantity B’s mix will also be 60 percent cocoa.

The answer is (C).

6. **(B)**

   No actual calculation is required here. Six numbers average to 44 and two of them average to 11. That is, two of the numbers have an average that is 33 points below the overall average. Therefore, the other four numbers must bring the average up 33 points. However, because there are *four* numbers bringing the average up (versus *two* bringing the average down), each of the individual numbers doesn’t have to “compensate” as much—they will not have to be as high as 77, which is 33 points above 44.

Put another way, there are only two numbers dragging the average down, so they have to be pretty extreme. But because there are four numbers dragging the average up, they get to share the burden—they don’t have to be as extreme.

The answer is (B).

If you can master that logic, you can solve problems like this one very fast. However, if you prefer a more mathematical approach:

> If six numbers average to 44, their sum is 6 × 44 = 264.
> If two of the numbers average to 11, their sum is 2 × 11 = 22.
>
> Thus, the other four numbers must sum to 264 − 22 = 242.
> 242/4 = 60.5

Thus, the average of the other four numbers is 60.5, well under 77.

The answer is (B).

7. **(B)**
   No actual calculation is required here if you have a good grasp of average speed. First, Tavi’s average speed is *not* 55 miles per hour—you cannot simply average the two speeds. Why? Because average speed is essentially what you would get if you clocked Tavi’s speed during every second of the journey and then averaged all the seconds. If you wrote out a long, long list of all the numbers you’d be averaging,

you’d have written out a lot more 50’s than 60’s, because Tavi spent more time driving at the slower speed. Therefore, Tavi’s average speed will be “between 50 and 60 but closer to 50.” (Note: this only works when the distances are *the same*). Therefore, 55 is higher than Tavi’s average speed.

The answer is (B).

8. **(C)**

Set up the probabilities in both quantities before calculating either value. The bag contains five red, four blue, and eight orange jellybeans, and thus 17 total jellybeans.

In Quantity A, the probability of picking a red is $\frac{23}{7}$. Keep in mind that once the red is selected, there are only 16 jellybeans left in the bag, so the probability of then picking a blue is $\frac{23}{7}$, and then the probability of picking an orange is $\frac{23}{7}$. Thus, Quantity A is equal to

$$30 = \frac{1}{2}(10)(h).$$

In Quantity B, the probability of picking a red first is, of course, still $\frac{23}{7}$. Notice that *once a red is picked first, there are now equal numbers of blues and reds left in the bag* (4 each). Thus, the probability of now picking another red is $\frac{23}{7}$ (equal to the probability in Quantity A of picking a blue at this point), and then the probability of picking an orange is still $\frac{23}{7}$. Thus, Quantity B is also equal to

$30 = \frac{1}{2}(10)(h)$.

The answer is (C).

You may have been tempted to multiply the fractions, but in this case no computation is necessary.

9. **(A)**

   To get started on a work problem, you need to convert Preeti and Mariska’s sandwich speeds into rate format—that is, you need their information in a per hour format. If Preeti can make 100 sandwiches in 1 hour 15 minutes, that’s 100 sandwiches in 75 minutes:

$$\frac{u+y}{x+z} = \frac{8+10}{4+5} = \frac{18}{9} = 2.$$

$$6{,}000 = 75x$$
$$x = 80$$

Therefore, Preeti can make 80 sandwiches per hour.

Mariska’s rate is much easier—if she can make 50 sandwiches in 30 minutes, just double the rate to get that she can make 100 sandwiches in 1 hour.

Now that the rates are in per hour format, you can add: 80 sandwiches/hour + 100 sandwiches/hour is a combined rate of 180 sandwiches/hour.

Fortunately, Quantity A asks for their time to make 180 sandwiches working together, so Quantity A is simply equal to 1 hour.

Quantity B asks for the time it would take to make 110 sandwiches if Mariska worked alone for 30 minutes and then the women finished the job together. Working alone for 30 minutes, Mariska will make 50 sandwiches. That leaves 60 sandwiches left for the two of them to make together:

$$\frac{180\,\text{sandwiches}}{60\,\text{minutes}} = \frac{60\,\text{sandwiches}}{x\,\text{minutes}}$$

$$180x = 3600$$

$$x = 20$$

Or try this mental math shortcut: because the women working together can make 180 sandwiches/hour, it will take $\frac{1}{2}$ of the time to make $\frac{1}{2}$ of the sandwiches, so it will take $\frac{1}{2}$ of an hour, or 20 minutes.

Either way, Quantity B is equal to the 30 minutes Mariska works alone, plus the 20 minutes it takes the women to finish the job together.

Thus, Quantity B is equal to 50 minutes.

The answer is (A).

10. **(D)**

    As a reminder: whenever you see a Word Problem on Quantitative Comparisons, *make sure you have the information you need before doing any computation*.

    Quantities A and B ask about travel time. From Rate × Time = Distance, you need both distance and rate in order to compute time. The common information gives you only two relative rates (x and $1.2x$). Without some information about the distances from A to B and B to C, there is no way to compute even a relative time.

    The answer is (D).

Unit Seven: Data Interpretation

This guide to Data Interpretation demonstrates approaches to quickly and efficiently synthesize graphical information on test day.

In This Unit...

Chapter 35: Data Interpretation

Chapter 35

DATA INTERPRETATION

In This Chapter...

The Basic Process of Solving a Data Interpretation Question

Data Interpretation Graphs and Charts–aka–The Graph Zoo

Chapter 35

Data Interpretation

Data Interpretation (DI) questions appear as sets of problems that refer to the same group of one to three related graphs or charts. On the GRE, you will see an average of two DI sets per exam, each with two or three associated problems.

DI questions are not, in general, particularly difficult. However, they can take a lot of time to solve if you aren’t careful. It is very important to learn how to tackle them efficiently, using the on-screen calculator when appropriate.

The Basic Process of Solving a Data Interpretation Question

Scan the graph(s). (15–20 seconds)

- What type of graph is it?
- Is the data displayed in percentages or absolute quantities?
- Does the graph provide any overall total value?

Figure out what the question is asking. What does it ask you to do?

- Calculate a value?
- Establish how many data points meet a criterion?

Find the graph(s) with the needed information.

- Look for key words in the question.

If you need to establish how many data points meet a criterion, keep track as you go by taking notes.

If you need to perform a computation, translate the question into a mathematical expression *before* you try to solve it.

If one of the answer choices is "cannot be determined," check that you have *all* the information you need before performing any calculations.

Use the calculator when needed, but keep your eye out for opportunities to use time-saving estimation techniques:

- Does the question use the word "approximate"?
- Are the numbers in the answer choices sufficiently far apart?

This list of steps should be used as a high-level process checklist, to help you remember what to look for and do as you solve, but some of the steps are only relevant to some of the problems. The examples on the subsequent pages follow this process and show how to apply it to various types of problems.

Data Interpretation Graphs and Charts —aka—The Graph Zoo

Most GRE Data Interpretation questions focus on data in five standard formats. You will be much faster at extracting the data if you are already familiar with reading these types of tables and graphs. GRE DI charts always tell a data story, and the questions you will be answering are about that story. To understand these charts and how they work, you will be looking at a simple data story about a produce stand. The owner of the produce stand has some numbers for the amounts of the different types of produce sold per month over a one-year period. For one month, the owner also has some detailed data on exactly which fruits and vegetables were sold. How might the GRE present this story? There are various ways, but all involve the five basic types of charts.

As you work through the examples, you will notice that many problems also involve FDP calculations, and you will find that you will be much faster at those calculations if you already know various standard formulas, such as the percentage increase/decrease formula, and computation shortcuts, such as estimating fractions. You will also notice that the solutions to the problems point out the various computation tricks. Use the calculator when it's easy to, but also work on developing estimation techniques. This will ultimately save you time.

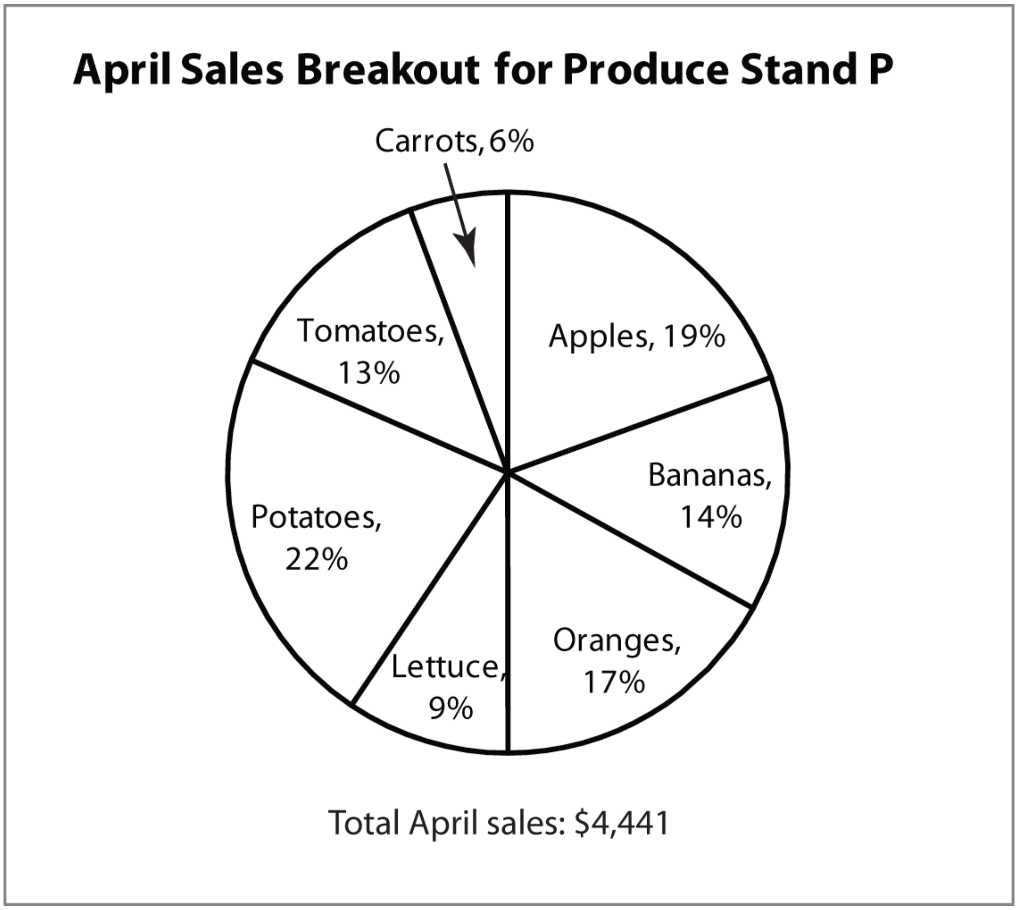
Look carefully at the solutions and you will also see that they follow the previously described problem-solving process. As an exercise, you might want to cover up the solution steps with a piece of paper and see if you can predict, from the general problem-solving process, what the next step should be as you work through the solution and uncover each step. Although there are many ways to solve these problems, time is critical on the GRE, and learning to follow a standard process and use computation shortcuts will ultimately save you a great deal of time and stress.

PIE CHARTS

A pie chart is used to show the relative sizes of slices as proportions of a whole. The size of the angle of the pie slice is proportional to the percentage of the whole for each item. Even if a pie chart shows amounts instead of percentages, data is shown in pies because percentages, or relative quantities, are important to the story. If you see data in a pie chart on the GRE, you know that there will be one or more questions about percentages or proportions.

Also, many pie charts include a total amount annotated on the chart. If you see this feature, you can be almost certain that the GRE will ask you to calculate an absolute quantity of some item shown in the pie and that the best way to do so will be to use that number and multiply it by the relevant percentage.

A pie chart can only show one series of data, so if you see two pies, which sometimes occurs on the GRE, they represent two series of data and you can be just about certain that one or more of the questions will ask you to compare something in the two different data series.



Following are some common calculations that you might be asked to perform on the example pie chart, which shows the April sales breakout for the produce stand:

- Tomato sales = 13% of $4,441 = 0.13 × 4,441
- Lettuce & tomato sales = 9% of $4,441 + 13% of $4,441 = 22% of $4,441 = 0.22 × 4,441

1. Approximately what amount of total sales in April came from sales of apples, bananas, and oranges?

(A) $2,221
(B) $2,362
(C) $2,495
(D) $2,571
(E) $2,683

The question asks for the sum of the absolute dollar amount of total sales of apples, bananas, and oranges.

The only chart you have is a pie chart showing percentages, so this question is asking you to convert from percents to dollar amounts.

You have apples at 19 percent, bananas at 14 percent, and oranges at 17 percent, and total sales were $4,441. So you need to set up a mathematical expression for the amount of sales that come from apples, bananas, and oranges:

$$(0.19 \times 4{,}441) + (0.14 \times 4{,}441) + (0.17 \times 4{,}441) = 2{,}220.5$$

You can do this more efficiently by summing the percentages before you multiply by the total sales:

$$= 4{,}441 \times (0.19 + 0.14 + 0.17)$$ Sum the percentage so you only multiply by one number.

= 4,441 × (0.50)

=2,220.5

The question says approximate, so A must be the answer.

2. If sales of potatoes were to increase by $173 next month and sales of all other items were held constant, approximately what percentage of the total sales would be potatoes?

(A)	20%
(B)	25%
(C)	30%
(D)	35%
(E)	40%

The question asks the new ratio of potato sales to total sales, after adding $173 in potato sales.

The only chart you have is a pie chart showing percentages, but it has a total quantity and a percentage from potatoes: 22 percent of total sales are from potato sales and there is $4,441 in total sales.

Set up a mathematical expression:

$$\frac{0.22(4{,}441) + 173}{4{,}441 + 173} = 0.249 \approx 25\,\%$$

Important: Note that the denominator of the fraction above takes into account that the $173 of new potato sales must be added not only to the potato sales but to the total sales as well. That is, the total sales is no longer the $4,441 from the chart because additional sales have been made.

The answer is **(B)**.

3. If the areas of the sectors in the circle graph are drawn in proportion to the percentages shown, what is the approximate measure, in degrees, of the sector representing the percent of total sales due to lettuce?

(A) 24 degrees
(B) 28 degrees
(C) 32 degrees
(D) 36 degrees
(E) 40 degrees

The question asks for the degree measure of the lettuce wedge on the chart.

The only chart you have is a pie chart showing percentages; that’s the chart you use.

You have 9 percent of total sales from lettuce, so lettuce represents 9 percent of the 360-degree circle.

Writing this in math form, you get:

$0.09 \times 360 = 32.4$ degrees 🖩

The answer is **(C)**.

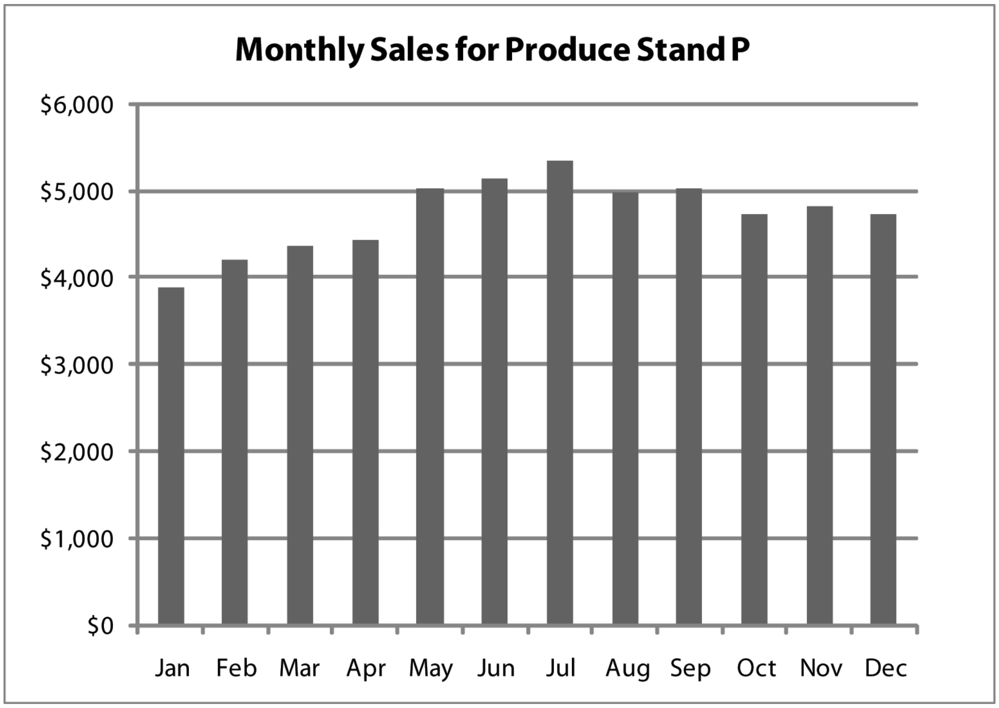
COLUMN CHARTS

A column chart shows amounts as heights. Typically, the *x*-axis is time (e.g., months, years) and column charts are used to show trends over time.

Often the hardest thing about a column chart is just reading the values. The GRE never makes an exact value reading critical to answering a question unless numeric values are explicitly given (and even then you can usually just round), so just raise your index finger up near the computer monitor, draw an imaginary line across the chart, and estimate approximate quantities.

Single Series Column Charts

A single data series chart is so straightforward that it doesn’t usually even have a legend. Here is an example showing the produce stand’s sales:



Some common calculations that you might be asked to perform are percentage increase or decrease from one time period to the next, or even more simply, just counting the number of periods when data values were above or below a particular value:

Approximate percentage increase in sales from April to May:

$$\frac{\text{May sales} - \text{April sales}}{\text{April sales}} \approx \frac{5,000 - 4,500}{4,500} = \frac{500}{4,500} \approx 11\%$$

Number of months when sales were less than February sales is 1.

1. In how many of the months shown were total produce sales greater than $4,600?

(A) 7
(B) 8
(C) 9
(D) 10
(E) 11

The question asks you to count the number of months shown that were greater than $4,600.

The only chart you have is this column chart, and it shows the sales for each month directly, so you can just read the chart.

Most months appear to have sales greater than $4,600, therefore, count the number of months in which sales were less than $4,600 and subtract from the total number of months shown, which is 12.

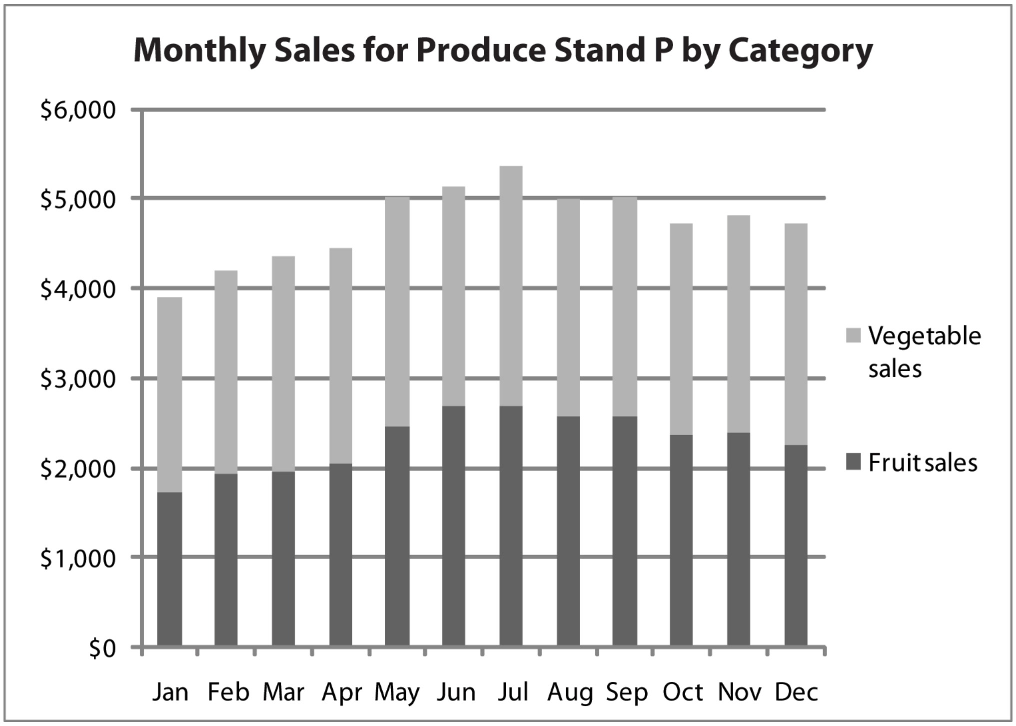
Months when sales were less than $4,600: Jan, Feb, Mar, Apr, so number of months when sales were greater than $4,600 equals 12 − 4, or 8, and the answer is **(B)**.

Stacked Column Charts

The GRE is especially fond of stacked column charts because they can be used to show two or more data series at a time, as differently shaded parts of one column. For instance, “vegetable sales” and “fruit sales” sum to “total sales.” This makes it as easy to answer questions that ask about the total as it is with just a single data series in the chart.

However, it is a little harder to read off vegetable sales by itself—you have to calculate total sales minus fruit sales, so you can be almost certain that you will have a question that asks you to do something like this.

Notice also in the following example, which breaks out the monthly sales of the produce stand into fruit and vegetable sales, that charts that show multiple data series have legends so you can tell which part of the bar represents which category:



2. Approximately what were total vegetable sales in September?

(A) $5,000
(B) $4,000
(C) $3,000
(D) $2,500
(E) $2,000

The question asks you to figure out vegetable sales in September.

The only chart you have is this column chart, and it shows the vegetable sales for each month directly, so you can just read the chart.

To get vegetable sales, you need total sales minus fruit sales. For September, that is equal to about $5,000 – $2,500, or $2,500 and the answer is **(D)**.

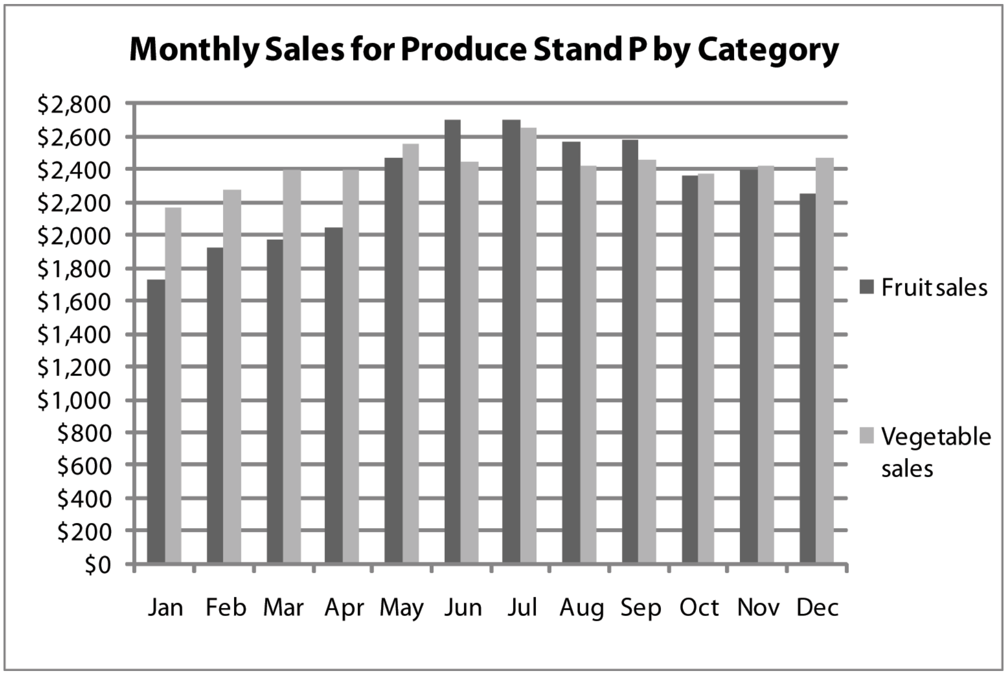
Strategy Tip:

Subtract in order to calculate many of the values shown in stacked bar graphs.

Clustered Column Charts

Another variation on column charts has clustered columns instead of stacked columns. Clustered column charts make it easier to compare the parts of the total, but more difficult to determine the actual total because you have to sum the columns.

The types of questions that would be asked about a clustered column chart are the same as those that would be asked about a stacked column chart. The only difference between the two types is that it is easier to read a total quantity off of a stacked column chart, but easier to read the height of an individual series item, such as total vegetable sales in June, off of a clustered column chart. The following example shows exactly the same data as the previous stacked column chart showed, except in the clustered column format:



3. Which month had the largest percentage of vegetable sales relative to total sales?

(A) Jan
(B) Mar
(C) Jun
(D) Oct
(E) Nov

The question asks you to compare the ratio of vegetable sales to total sales for several months.

The only chart you have is this column chart, and it shows the sales for each month directly, so you can just read the numbers and calculate the ratios.

The formula for the ratio of vegetable sales to total sales is:

$$\frac{\text{vegetable sales}}{\text{fruit sales} + \text{vegetable sales}}$$

The key to solving this problem is realizing that you don’t have to do calculations for all of the months. In January and March, vegetable sales were substantially larger than fruit sales, so they were more than half of total sales, whereas in June, October, and November, they were about the same or actually less than fruit sales, so the only months you really have to look at are January and March.

Because the difference in fruit and vegetable sales is about the same in both months, but total sales are much greater in March, the same absolute difference in fruit sales is a greater percentage of the total sales in January than it is in March, so the answer is **(A)**.

You can verify this with a calculator:

$$\frac{\text{Jan vegetable sales}}{\text{Jan vegetable sales} + \text{Jan fruit sales}} \approx \frac{2{,}200}{1{,}700 + 2{,}200} = \frac{2{,}200}{3{,}900} \approx 0.56$$

$$\frac{\text{Mar vegetable sales}}{\text{Mar vegetable sales} + \text{Mar fruit sales}} \approx \frac{2{,}400}{2{,}000 + 2{,}400} = \frac{2{,}400}{4{,}400} \approx 0.55$$

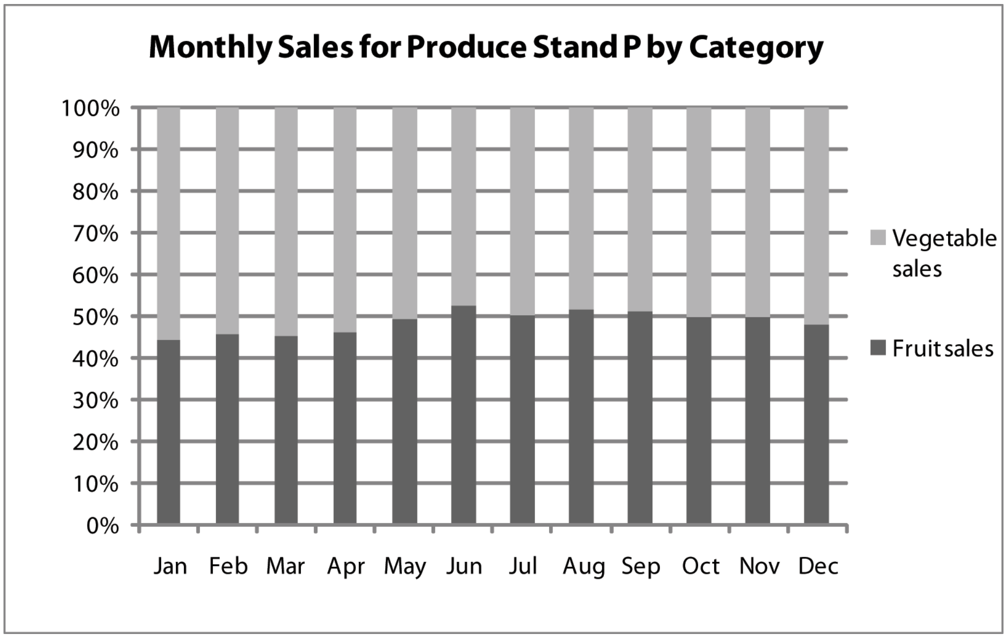
Strategy Tip:

Any given amount is a greater percentage of a smaller number than it is of a larger number.

Percentage Column Charts

Occasionally the GRE uses column charts to show percentages directly, rather than absolute quantities. If you see this type of chart, and you need quantity information, you will need another chart to provide actual values.

Otherwise, the types of questions that would be asked about a percentage column chart are the same as those that would be asked about a stacked column chart. The following example shows a typical percentage column chart:



4. If the total produce sales in July at Produce Stand P were $4,500, what were the approximate total fruit sales in December at Produce Stand P?

(A) $2,100
(B) $2,200
(C) $2,300
(D) $2,400
(E) Cannot be determined

The question asks you to determine the amount of fruit sales in December.

The only chart you have is this column chart, and it shows the percentage of sales due to fruit and the percentage of sales due to vegetables for each month.

The formula for the amount of fruit sales in December is:

Total fruit sales in December = % fruit sales × Total sales in December

However, you have no information on total sales in December. You cannot assume that total sales in December are the same as in July, so you cannot answer this question. The answer is **(E)**.

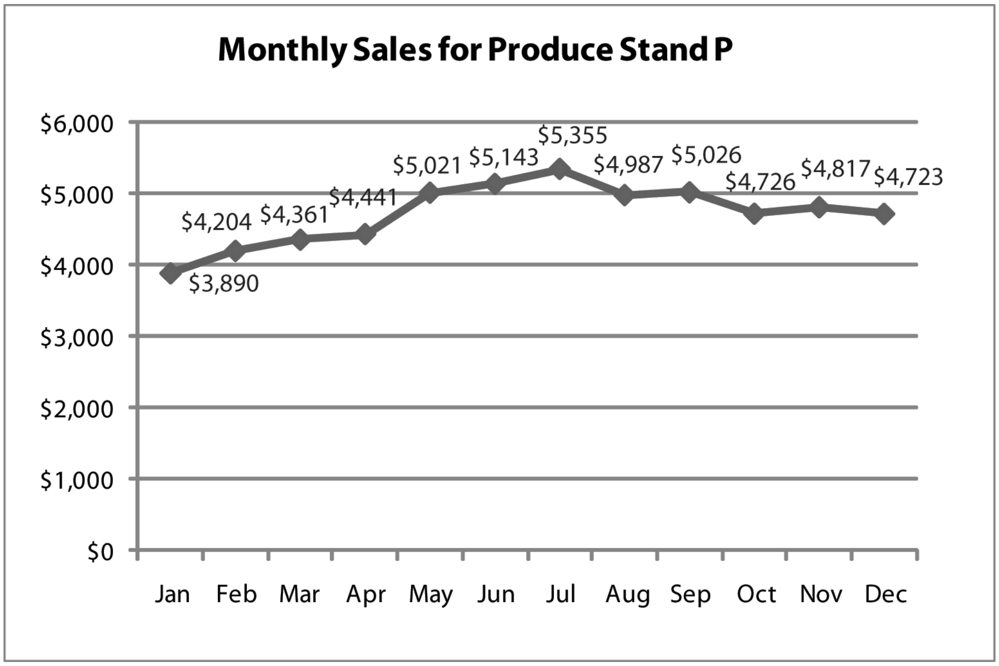
Strategy Tip:

Be careful of the difference between charts that show percentages and charts that show actual quantities.

LINE CHARTS

Line charts are very similar to column charts, but each amount is shown as a floating dot instead of as a column, and the dots are connected by lines. As is true with column charts, often, the *x*-axis is time (e.g., months, years), and line charts are used to show trends over time. Because of the continuous nature of lines, data series that are shown in line charts are almost always continuous values for something, and not separate categories, as they sometimes are in column charts.

Here is an example showing the produce stand's sales in a line chart:



Some common calculations that you might be asked to perform are percentage increase or decrease from one time period to the next, the change in the overall average value of the data points if one of them changes, or even more simply, a count of the number of periods when data values were above or below a particular level:

Approximate percentage increase in sales from April to May:

$$\frac{\text{May sales} - \text{April sales}}{\text{April sales}} \approx \frac{5,021 - 4,441}{4,441} \approx 11\%$$

Number of months when sales were less than July sales is 11.

1. If the average sales per month at Produce Stand P were calculated at \$4,725, and then it was discovered that the sales in January were actually \$4,072 instead of the amount shown, what would the approximate correct average sales per month be?

(A) \$4,740
(B) \$4,762
(C) \$4,769
(D) \$4,775
(E) Cannot be determined

The average sales per month is just the total of all the monthly sales divided by the number of months.

The only chart you have is this line chart, and it shows the monthly sales. The old amount for January was \$3,890.

The average formula is:

$$\text{old average} = 4{,}725 = \frac{\text{sum of 12 months of sales}}{12}$$

$$\text{new average} = \frac{\text{sum of 12 months of sales} - 3,890 + 4,072}{12}$$

The total sum of the monthly sales has increased by 182, therefore, the average of the 12 monthly sales has increased by about 182 ÷ 12 or about 15, so the answer is **(A)**.

You can verify that this is correct by examining the algebra:

$$\begin{aligned}\text{new average} &= \frac{\text{sum of 12 months of sales} - 3{,}890 + 4{,}072}{12}\\ &= \frac{\text{sum of 12 months of sales} + 182}{12}\\ &= \frac{\text{sum of 12 months of sales}}{12} + \frac{182}{12}\\ &\approx 4{,}725 + 15\\ &= 4{,}740\end{aligned}$$

The correct answer is **(A)**.

Problem Recap: The GRE likes these changing average problems. Remember the average change estimation shortcut!

2. What was the approximate percent increase in total sales at Produce Stand P from January to June?

(A) 19%
(B) 24%
(C) 28%
(D) 32%
(E) 38%

To calculate the percent increase from January to June, you need the total sales in January and the total sales in June.

The only chart you have is this line chart, and it shows the monthly sales. The amount for January is $3,890 and the amount for June is $5,143.

The percent increase formula is:

$$\frac{\text{new} - \text{old}}{\text{old}} = \frac{\text{Jun sales} - \text{Jan sales}}{\text{Jan sales}}$$

$$\frac{\text{Jun sales} - \text{Jan sales}}{\text{Jan sales}} = \frac{5,143 - 3,890}{3,890} \approx 0.32$$

The answer is **(D)**.

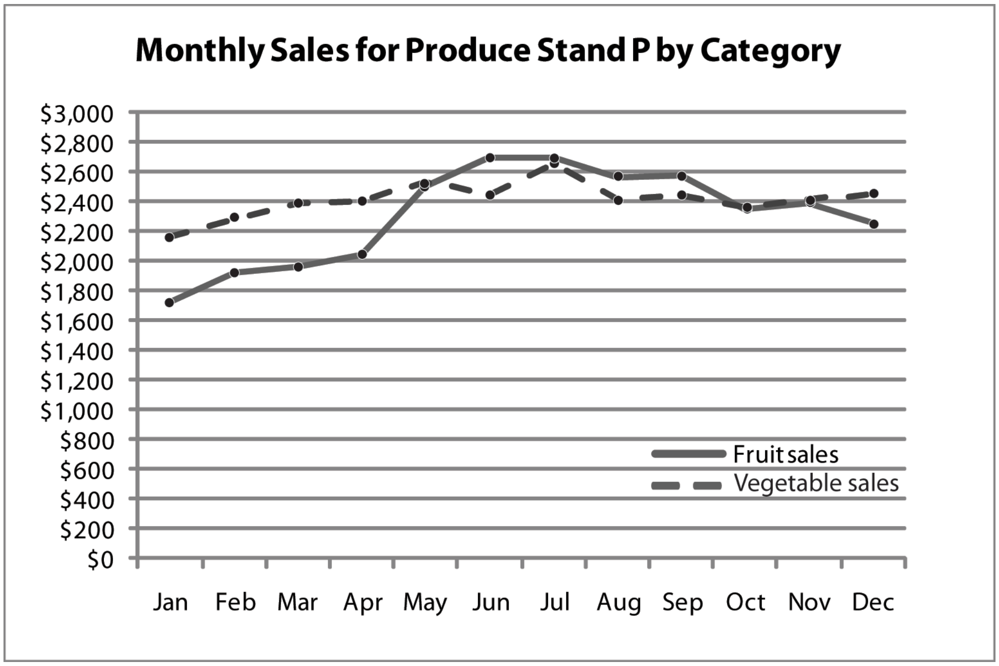
Strategy Tip:

Know the percent increase and decrease formulas!

Multi-Line Charts

The GRE is especially fond of multi-line charts because they can be used to show two or more data series at a time. Note that multi-line charts, like stacked and clustered column charts, have legends.

In the following example, you have vegetable sales and fruit sales. With line charts, you have to sum data points to calculate a total, because a total line is seldom shown (column charts are used when the goal is to emphasize the total).



In addition to questions that require you to pick out data points from one of the data lines, expect that on at least a few questions you will be asked to either combine or compare the data that make up one line to the data that make up the other:

Approximate percentage increase in fruit sales from Jan to May:

$$\frac{\text{May sales} - \text{Jan sales}}{\text{Jan sales}} \approx \frac{2,500 - 1,700}{1,700} \approx 47\%$$

Number of months when vegetable sales were more than $100 greater than fruit sales is 5 (i.e., Jan, Feb, Mar, Apr, and Dec).

3. Over which of the following sequences of months did total sales decline the most?

(A) Feb–Mar
(B) Mar–Apr
(C) Jun–Jul
(D) Aug–Sep
(E) Sep–Oct

Total sales are the sum of fruit and vegetable sales, and a decline means that at least one of the two would have to go down, and that drop would have to be bigger than any increase in the other category.

The only chart you have is this line chart, and it shows the monthly sales, although it doesn't add them up for you.

You hardly need a formula, because total sales are just the sum of fruit sales plus vegetable sales.

By scanning through the graph, you see that from Feb–Mar and Mar–Apr, both fruit and vegetable sales increased, so there was no decline. From Jun–Jul, vegetable sales increased, but fruit sales stayed flat, so still no decline. Aug–Sep also looks like a slight increase for both fruit and vegetable sales. However, from Sep–Oct, both fruit and vegetable sales seem to have declined, so the correct answer must be **(E)**.

The long way to do this problem is to read both fruit and vegetable sales and calculate approximate total sales for each month:

Feb sales = 1,900 + 2,300 = 4,200
Mar sales = 2,000 + 2,400 = 4,400
Apr sales = 2,200 + 2,400 = 4,600
Increase of 200
Increase of 200

Jun sales = 2,400 + 2,700 = 5,100
Jul sales = 2,700 + 2,700 = 5,400
Increase of 300

Aug sales = 2,400 + 2,600 = 5,000
Sep sales = 2,400 + 2,600 = 5,000
Oct sales = 2,400 + 2,400 = 4,800
No change
At last! A monthly decrease

This would take entirely too long!

Strategy Tip:

Try visual estimation before performing calculations.

4. If the average price that the Produce Stand P sold fruit for in May was 80 cents per pound and the average wholesale cost to the Produce Stand in May for a pound of fruit was 25 cents per pound, approximately how much was produce stand P’s gross profit on the sale of fruit in May?

(A) $1,600
(B) $1,630
(C) $1,680
(D) $1,720
(E) Cannot be determined

To solve this, you need to remember that Gross profit equals Sales revenue minus Costs. If you can plug in for fruit sales revenue and fruit cost, you can answer this question.

The only chart you have is this line chart, and it shows the monthly fruit sales revenue in May, so you may be able to figure this out.

The answer choices are too close together to estimate, so you’ll have to calculate.

You know that May fruit sales revenue is about $2,500. You know that the average wholesale cost per pound of fruit was $0.25 and that the average retail price per pound of fruit was $0.80. Thus:

> Average gross profit per pound of fruit sold in May = 0.80 − 0.25 = 0.55

$$\begin{aligned}\text{Gross profit on fruit sold in May} &= \text{profit per pound} \times \text{number of pounds of fruit sold}\\ &= 0.55 \times \text{number of pounds of fruit sold}\end{aligned}$$

$$\begin{aligned}\text{number of lbs of fruit sold in May} &= \frac{\text{total fruit revenue}}{\text{revenue per lb}}\\ &= \frac{2{,}500}{0.800} = 3{,}125\end{aligned}$$

$$\begin{aligned}\text{Gross profit on fruit sold in May} &= \text{Profit per lb} \times \text{Number of lbs of fruit sold}\\ &= 0.55 \times 3{,}125 = \$1{,}718.75\end{aligned}$$

The answer is **(D)**.

Strategy Tip:

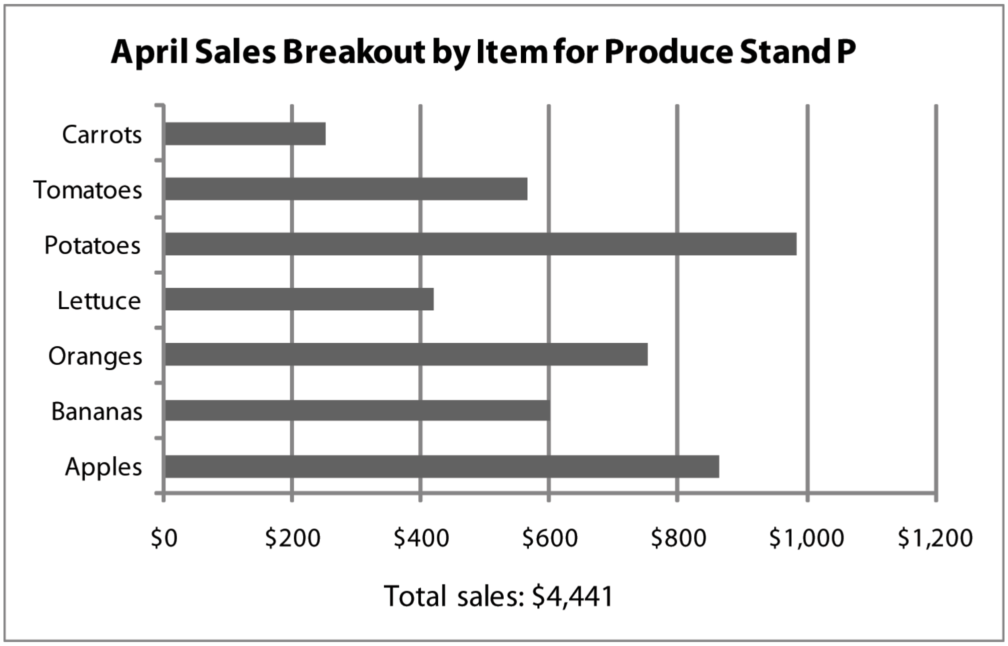
If you need a quantity, such as number of pounds of fruit sold, that is not directly shown in a graph, try writing out equations for it in terms of quantities you do know.

BAR CHARTS

A bar chart is essentially a column chart on its side. Although almost all bar charts on the GRE show absolute quantities, it is possible for them to show percentages. Because these more exotic charts are so rare on the GRE and are essentially types of column charts on their sides, this section focuses on standard bar charts.

The GRE generally represents a single data series in each bar chart, and, like pie charts, some bar charts include a total amount annotated on the chart.

Here is an example showing the April sales breakout by item for Produce Stand P. The length of each bar represents either an absolute number or a percentage. In this case, it’s an absolute number:



1. What fruit or vegetable generated the third highest sales in April for Produce Stand P?

(A) tomatoes
(B) lettuce
(C) oranges
(D) bananas
(E) apples

The question asks you to figure out the fruit or vegetable that generated the third highest sales in the month of April.

The only chart you have is this bar chart, and it shows the sales for each fruit and vegetable in April.

Scanning the chart, you see that potatoes had the highest sales, then apples, and third were oranges. So the answer must be **(C)**.

Strategy Tip:

Use a finger or a piece of paper to create a vertical line to help read bar chart values.

2. Which of the following ratios is closest to the ratio of carrot sales to potato sales at Produce Stand P in the month of April?

(A) 1 : 4
(B) 2 : 9
(C) 1 : 5
(D) 1 : 6
(E) 3 : 10

To calculate the ratio of carrot sales to potato sales in April, you need to know those amounts, and the chart gives them to you.

The chart shows carrot sales were about $250 and potato sales were about $980:

$$\frac{\text{carrot sales}}{\text{potato sales}} = \frac{250}{980} \approx 0.255 \approx \frac{1}{4}$$

The answer is **(A)**.

TABLES

A table is a very straightforward way to present data when calculations using that data are required because there is no need to estimate numbers. The thing that a table doesn't do, though, is allow you to easily see trends or estimate using visual techniques.

Often, one table will contain a mix of absolute quantities and percentage data. Be careful not to confuse the two. The GRE does not always label individual percents with a percentage sign. Rather, the entire row or column is generally labeled as such in the row or column header.

If you have to do calculations, and you probably will if you are given a table, it will be easy to look up the numbers. Here is an example of a table that combines absolute quantity information with percentage information for the produce stand:

Monthly Sales Breakout for Produce Stand P

Month	Total (in Dollars)	% Fruit	% Vegetable
Jan	4,121	44.29	55.71
Feb	4,204	45.74	54.26
Mar	4,361	45.10	54.90
Apr	4,568	49.99	54.06

Month	Total (in Dollars)	% Fruit	% Vegetable
May	4,791	49.17	50.83
Jun	4,756	52.40	47.60
Jul	4,822	50.38	49.62
Aug	4,791	51.41	48.59
Sep	4,801	51.21	48.79
Oct	4,726	49.89	50.11
Nov	4,817	49.78	50.22
Dec	4,881	47.77	52.23

1. Approximately how many dollars' worth of vegetables were sold in September, October, and November combined by Produce Stand P?

(A) $5,724
(B) $6,230
(C) $6,621
(D) $7,130
(E) $7,685

The question asks you to calculate the dollars' worth of vegetable sales in September, October, and November.

You can do this because the chart shows the total sales for each month and the percentage of those sales that were due to vegetables. In September, vegetable sales were 48.79% of $4,801; in October, 50.11% of $4,726; and in November, 50.22% of $4,817. Next, calculate as follows:

$$\begin{aligned}\text{Sep} + \text{Oct} + \text{Nov vegetable sales} &= (0.487\\ &= 2{,}342.41 +\\ &= 7{,}129.71\end{aligned}$$

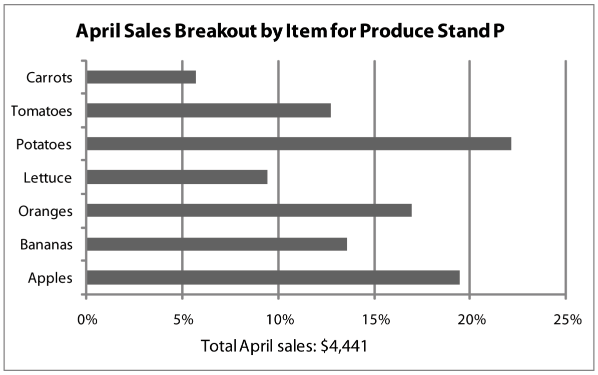
The answer is **(D)**.

OTHER COMMON TYPES OF DIAGRAMS

Occasionally, other common types of diagrams, such as floor plans or outline maps, appear on the GRE. The good news is that although these diagrams are a little less familiar than the basic five, the questions that go with them tend to be a little bit easier. There are questions that ask you to calculate surface area (of walls) and volume of rooms, but far fewer of the more challenging percent change and “how many points satisfy this complicated set of criteria” variety.

QUESTIONS THAT TYPICALLY REQUIRE INPUT FROM MORE THAN ONE GRAPH TO SOLVE

So far, you’ve looked at the usual types of charts seen in GRE Data Interpretation and typical questions based on those charts. However, the GRE often complicates things by asking questions that require that you look up and integrate information from multiple charts. This type of multi-chart question is not mathematically harder than a single-graph question, but since it requires using data from two different graphs, it can be a bit more confusing. Efficient solving techniques and good scrap paper organization become even more valuable with multiple charts, because more charts mean more opportunities to become confused and waste time. The next example combines two types of charts that you've seen before, and asks questions that require using information from both of them:



Vitamin Content of Produce Items Sold at Produce Stand P in April

	Vitamin C Content	**Vitamin A Content**
Apples	low	low
Bananas	medium	low
Oranges	high	medium
Lettuce	high	low
Potatoes	medium	low
Tomatoes	high	high

Carrots	low	high

1. Approximately what were the total April sales of produce items at Produce Stand P that were high in both vitamin A and vitamin C content?

(A) 451
(B) 488
(C) 577
(D) 624
(E) 683

You need to figure out which produce items were high in both vitamin A and vitamin C and calculate the total sales of those items.

The table shows you that only tomatoes are high in both vitamins A and C, so you need total tomato sales.

The bar graph shows you that tomatoes account for about 13% of April sales and that total April sales were $4,441. So you need 13% of $4,441:

$$0.13 \times 4{,}441 = \$577.33$$

The answer is **(C)**.

2. Approximately what dollar amount of the produce sold by Produce Stand P in April had medium or high amounts of either vitamin A or vitamin C?

(A) $3,120
(B) $3,600
(C) $4,000
(D) $4,600
(E) Cannot be determined

You need to figure out which produce items were high or medium in either vitamin A or vitamin C and calculate the total sales of those items.

The table shows you that bananas, oranges, lettuce, potatoes, tomatoes, and carrots are high in either vitamin A or vitamin C, so you need their total sales.

Most of the produce items are high or medium in either vitamin A or vitamin C, therefore, it will be faster to just calculate the dollar amount of the items sold that are low in both vitamin A and vitamin C and subtract that from the total dollar amount of sales. The table shows that the only produce item that meets these criteria is apples, and the bar graph shows that they accounted for about 19 percent of total sales:

$$0.19 \times 4{,}441 \approx 844$$
$$4{,}441 - 844 = 3{,}597$$

The answer is **(B)**.

The long way to do this problem is to sum up the percentages of each type of produce that has a medium or high level of vitamin A or vitamin C.

To do this, you need to read a number of values off of the bar graph. Carrots are ≈ 6 percent of sales, tomatoes are ≈ 13 percent, potatoes are ≈ 22 percent, lettuce is ≈ 9 percent, oranges are ≈ 17 percent, and bananas are ≈ 14 percent. Thus:

$$\begin{aligned}
&(0.06 \times 4{,}441) + (0.13 \times 4{,}441) + (0.22 \times 4{,}441) + (0.17 \times 4{,}441) + (0.14 \times 4{,}441) \\
&= (0.06 + 0.13 + 0.22 + 0.09 + 0.17 + 0.14) \times 4{,}441 \\
&= 0.81 \times 4{,}441 \\
&= 3{,}597
\end{aligned}$$

That was too much work!

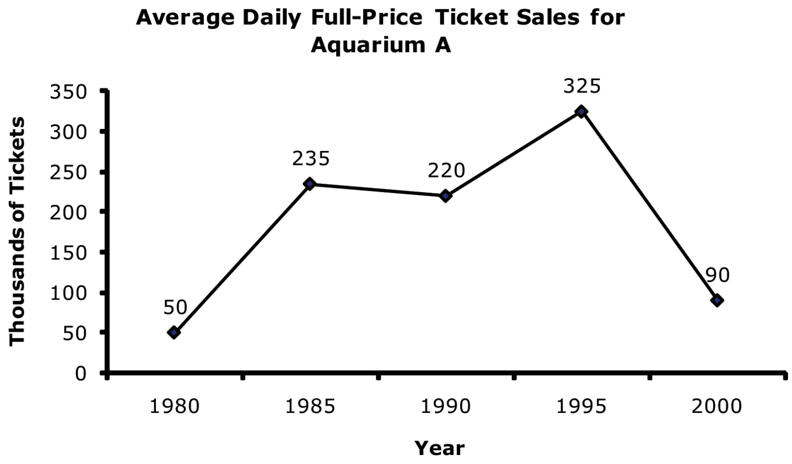
Strategy Tip:

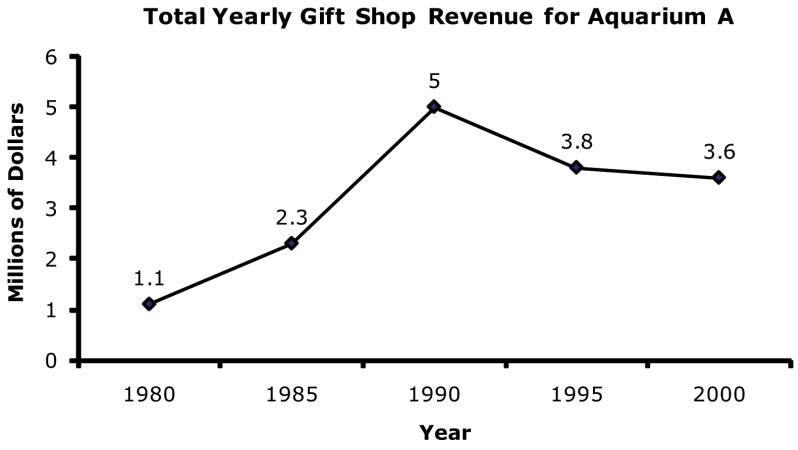
Sometimes it is easier to calculate the percentage that does *not* satisfy a condition rather than calculate a percentage directly.

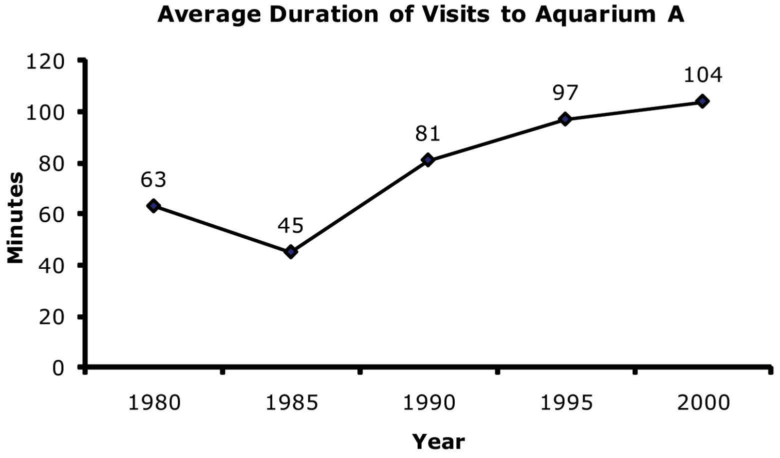
A Straightforward Data Interpretation Problem Set

Here is some practice on a straightforward Data Interpretation problem. Start by reading the question and the answers, and formulating your plan of attack.

PROBLEM A







The preceding three graphs are the data for the five questions that follow. You've read this chapter, so you know that it is a good idea to scan the graphs before looking at the problems to get a general sense of what information the graphs contain.

1. In how many years shown was the average duration of visits to Aquarium A more than twice as much as the average in 1985?

(A) Four
(B) Three
(C) Two
(D) One
(E) None

2. In 1980, if a full-price ticket cost $4.70, what would have been the average daily revenue, in thousands of dollars, from the sale of full-price tickets?

☐ thousands of dollars

(A) 235

3. In 2000, the total number of dollars of gift shop revenue was how many times as great as the average daily number of full-price tickets sold?

(A) 400
(B) 200
(C) 80
(D) 40
(E) 20

4. What was the approximate percent increase in average daily full-price ticket sales from 1990 to 1995?

(A) 10%
(B) 20%
(C) 33%
(D) 48%
(E) 66%

5. Which of the following statements can be inferred from the data?

☐

In each of the five-year periods shown in which yearly gift shop revenue decreased, average daily full-price ticket sales also decreased.

☐

The greatest increase in total yearly gift shop revenue over any five-year period shown was $2.7 million.

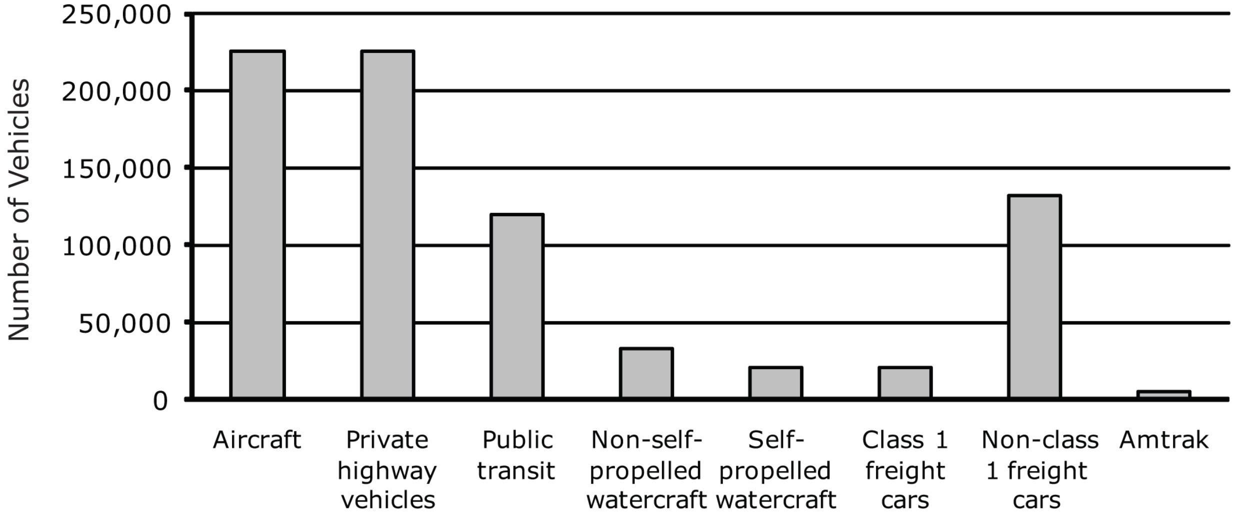
☐

From 1995 to 2000, the average duration of visits to the museum increased by 12 minutes.

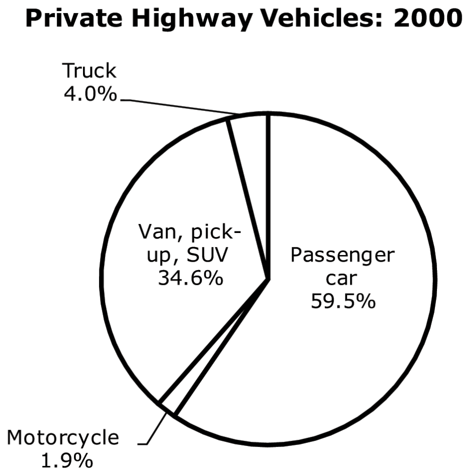
An Example Mixing Percents and Absolute Quantities

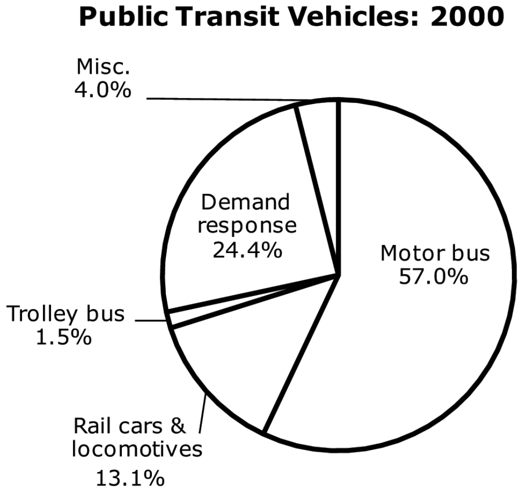
It is very common in GRE Data Interpretation problems to see a set of graphs that incorporate both percentage and absolute quantity data. Being able to quickly and confidently combine these two types of data is a critical success factor for many medium to hard DI problems. The following problems are typical examples:

PROBLEM B


U.S. Aircraft, Vehicles, and Other Conveyances: 2000

Total conveyances: 785,000





If you scanned the graphs before looking at the problems, you know that the graphs have to do with absolute numbers of vehicles (the bar graph) and some additional information on the percentages of specific types of public transit and private highway vehicles in 2000. Also notice that the bar graph has a total conveyances line at the bottom, which may prove useful.

1. Approximately what was the ratio of trucks to passenger cars?

(A) 1 to 20
(B) 1 to 18
(C) 1 to 17
(D) 1 to 15
(E) 1 to 12

2. Approximately how many more miscellaneous public transit vehicles than public transit trolley buses were there in 2000?

(A) 1,000
(B) 1,500
(C) 2,000
(D) 2,500
(E) 3,000

3. If the number of aircraft, vehicles, and other conveyances was 572,000 in 1995, what was the approximate percentage increase from 1995 to 2000?

(A) 37%
(B) 32%
(C) 27%
(D) 20%
(E) 15%

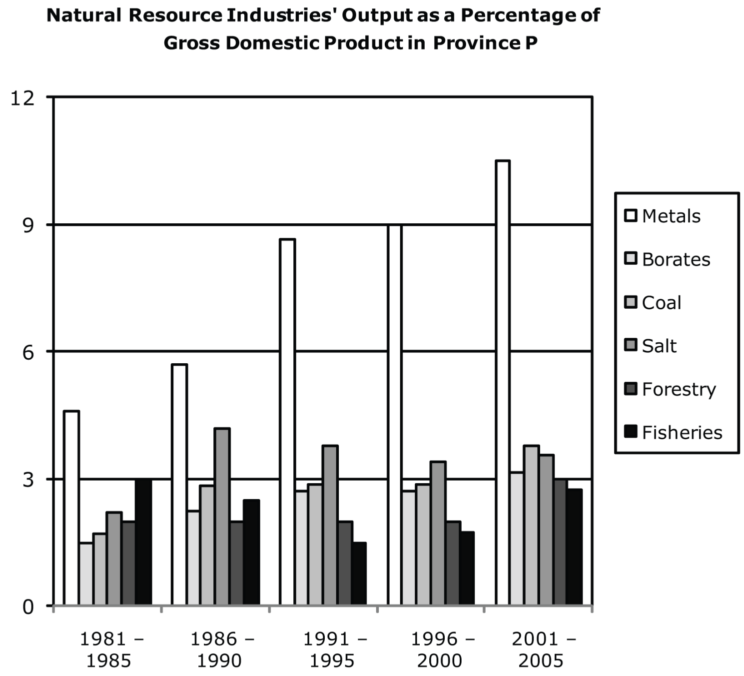
4. In 2000, if an equal percentage of passenger cars and demand-response vehicles experienced mechanical problems, and the number of passenger cars that experienced such problems was 13,436, approximately how many demand-response vehicles experienced mechanical problems?

(A) 1,352
(B) 2,928
(C) 4,099
(D) 7,263
(E) 9,221

A Very Challenging Example, Requiring Excellent Technique

The following set is tricky. The graphs are a bit more complicated than is typical on the GRE. (Note that, because the GRE is a section-level adaptive test, only those who performed at a high level on their first Quant section would ever see a Data Interpretation set this difficult.)

PROBLEM C



Mining Industries’ Average Annual Production

Years	Mining Industries’ Average Annual Production, in millions of 2005 dollars	Percentage of Mining Industries’ Production					
		Metals			Other Mined Products		
		Uranium, Titanium, & Aluminum	Gold & Silver	Copper	Borates	Coal	Salt
1981–1985	$342.5	10%	20%	16%	15%	17%	22%
1986–1990	$326.8	12	17	9	15	19	28
1991–1995	$310.0	16	20	12	15	16	21
1996–2000	$257.9	12	22	16	15	16	19
2001–2005	$205.0	14	24	12	15	18	17

1. Approximately what percent of the mining industries’ average annual production from 1991–1995 came from production of aluminum?

(A) 4%
(B) 7%
(C) 11%
(D) 22%
(E) Cannot be determined

2. Approximately what percent of average annual GDP of Province P from 1996–2000 came from copper production?

(A)	3%
(B)	6%
(C)	9%
(D)	14%
(E)	18%

3. Which of the following statements can be inferred from the information given?

☐

For all the time periods shown, borate production, in millions of 2005 dollars, was the same.

☐

Of the time periods shown, 1981–1985 was the one in which the mining industries produced the greatest value of gold and silver, measured in 2005 dollars.

☐

Of the time periods shown, 2001–2005 had the highest average annual GDP, measured in 2005 dollars.

Problem Set Solutions

PROBLEM A

1. **(C)**
   First, identify the chart that gives information about the average duration of visits: the bottom graph. The question asks for the number of years shown that had an average duration of more than twice the average in 1985.

   The average duration in 1985 was 45 minutes, so find the number of years that had an average duration greater than 90 minutes. Only 1995 and 2000 fit this constraint, so the answer is two, choice (C).

2. **235**
   The average daily revenue for the aquarium can be found using the following formula:

   $$\text{avg. daily revenue} = \left(\frac{\$}{\text{ticket}}\right) \times \text{avg. \#of tickets}$$

   The price per ticket is given in the question ($4.70), and the top graph gives the average daily number of tickets sold. In 1980, the average daily number of tickets sold was 50,000, so the average daily revenue is 50,000 × $4.70 = $235,000. The question asks for the revenue in thousands, so the answer is 235.

3. **(D)**

   To answer this question, you'll need information about both gift shop revenue and the average number of full-price tickets sold. The former can be found in the middle graph, and the latter can be found in the top graph.

   In 2000, total gift shop revenue for the year was $3,600,000 and the average daily number of full-price tickets sold was 90,000. The question asks how many times greater 3,600,000 is than 90,000. To find out, divide 3,600,000 by 90,000 (using the calculator!).

   The answer is 40, choice (D).

4. **(D)**

   Percent increase is defined as the change divided by the original value. In this question then, the percent increase can be found using the following formula (let *x* stand for the average daily full-price ticket sales):

$$\frac{(x \text{ in } 1995) - (x \text{ in } 1990)}{(x \text{ in } 1990)}$$

   Information on the average number of full-price ticket sales can be found in the top graph. The average number in 1990 was 220 and in 1995 it was 325:

$$\frac{325 - 220}{220} = \frac{105}{220} = 0.477$$

Remember that the question asks you to approximate. The answer is closest to choice (D).

5. **Statement II only**

This question asks which statements can be inferred from the data, therefore, you have to use the process of elimination and treat each statement as its own mini question.

First statement: Identify the five-year periods in which gift shop revenue (middle graph) decreased: 1990–1995 & 1995–2000. Next, locate the graph for average ticket sales, which is the top graph. Ticket sales did *not* decrease in 1990–1995, so statement I is false.

Second statement: The graph for gift shop revenue is the middle one. Locate the biggest jump—it is from 2.3 to 5. Compute the size of this jump: 5 − 2.3 = 2.7, so statement II is true.

Third statement: The graph for duration of visits is the bottom one. The increase from 1995–2000 was 104 − 97, which equals 7 minutes, not 12 minutes, so statement III is false.

Only the box for the second statement should be selected.

PROBLEM B

1. **(D)**

   For specific information about trucks and passenger cars, look at the middle graph. Though the pie chart does not give specific information about the number of cars or the number of trucks, it does tell you what percent of the total number of private vehicles each represents. Because each of the percents is out of the same total, you can compare the percents directly to find the ratio of cars to trucks.

   Note: You could also use the information in the top graph to calculate the actual numbers of cars and trucks, but that would be time-consuming and unnecessary.

   Remember that the question asks you to approximate. The ratio of 4 percent to 59.5 percent is approximately the ratio of 4 to 60. Reduce the ratio to get the correct answer, 1 : 15.

2. **(E)**

   To find the total numbers of miscellaneous public transit vehicles and public transit trolley buses, you will need to combine information from the top and bottom graphs.

   According to the top graph, there were roughly 120,000 public transit vehicles in 2000 (remember the question asks you to approximate). Of those 120,000 public transit vehicles, 4 percent were miscellaneous vehicles and 1.5 percent were trolley buses.

The difference in the number of miscellaneous and trolley bus vehicles is:

(0.04 × 120,000) – (0.015 × 120,000)

Either calculate both numbers and subtract, or realize that the difference will be (0.04 − 0.015) × 120,000. However you perform the calculation, the difference is 3,000.

3. **(A)**
   Percent increase is defined as change divided by original value. In this question then, the percent increase can be found using the following formula (let *x* stand for the total number of U.S. aircraft, vehicles, and other conveyances):

   $$\frac{(x \text{ in } 2000) - (x \text{ in } 1995)}{(x \text{ in } 1995)}$$

   The question says that the total number in 1995 was 572,000. The total number in 2000 can be found in the top graph. Fortunately, the question doesn’t require you to add the values in every column together! At the bottom, the graph states that the total number of conveyances was 785,000.

   Find the approximate percent increase:

   $$\frac{785,000 - 572,000}{572,000} = 0.372 \approx 37\%$$

Answer choice (A) is the closest to the real value.

4. **(B)**

   Rephrase the question as a mathematical expression. To save time, write *x* for “the number of demand-response vehicles that experience mechanical problems” when you rephrase the question. The key thing to remember here is the relationship between percentages and absolute numbers. Multiply the percentage times the total to get the absolute quantity:

$$\text{If } \frac{13{,}436}{\text{total number passenger cars}} = \frac{x}{\text{total number dem-resp vehicles}}, \text{then what is } x?$$

$$\text{Rearrange to get:} x = \frac{13{,}436 \times (\text{total number dem-resp vehicles})}{\text{total number passenger cars}}$$

You know from the first pie that demand-response vehicles makes up 24.4 percent of all public transit vehicles, and you know from the main chart that there are roughly 120,000 public transit vehicles, so you can calculate the approximate number of demand-response vehicles:

$$0.244 \times 120{,}000 = 29{,}280$$

You know from the second pie that passenger cars makes up 59.5 percent of all private highway vehicles, and you know from the main chart that there are about 225,000 private highway vehicles, so you can calculate the approximate number of private highway vehicles:

$$0.595 \times 225{,}000 = 133{,}875$$

So, pulling it all together:

$$x = \frac{13{,}436 \times 29{,}280}{133{,}875} \approx 2{,}938$$

To avoid overloading the calculator display, you must divide 13,436 by 133,875 before multiplying by 29,280.

The answer is (B).

PROBLEM C

1. **(E)**
   One of the answer choices is “cannot be determined,” so check exact wording and be sure you have enough information to solve it before doing any math.

   Locate the relevant column within the table in the chart on the bottom: “Uranium, Titanium, & Aluminum.” The figures in this column represent

Uranium + Titanium + Aluminum, but do not tell you the level of Aluminum *alone*. Because the question is asking *only* about Aluminum, you do not have enough information.

The answer is (E).

Strategy Tip:

This type of problem is very quick if you check to see whether you have enough information to answer the question before making any calculations.

2. **(A)**

   Rephrase the question. Expressing the desired percentage as a fraction is a good way to abbreviate the question:

   In 1996 – 2000, $\frac{\text{Copper}}{\text{GDP}} = ?$

   The chart that mentions Copper is on the bottom and tells you that from 1996–2000, Copper was 16% of Mining Industries' Production. In other words, it tells you that the value of $\frac{\text{Copper}}{\text{Mining}} = 0.16$. This is not quite what you are looking for, because the question is about $\frac{\text{Copper}}{\text{GDP}}$. Look to the top chart to see if it provides a way to convert your $\frac{\text{Copper}}{\text{Mining}}$ information into a $\frac{\text{Copper}}{\text{GDP}}$ figure.

The top chart gives you information on Metals, Borates, Coal, and Salt—all of the components of Mining from the bottom table—as a percentage of GDP. You will therefore be able to use the following equation to get $\frac{\text{Copper}}{\text{GDP}}$:

$$\frac{\text{Copper}}{\text{GDP}} = \left(\frac{\text{Copper}}{\text{Mining}}\right) \times \left(\frac{\text{Mining}}{\text{GDP}}\right)$$

Substitute numbers into the above equation. The top chart tells you that from 1996–2000, Metals were 9 percent of GDP and Borates, Coal, and Salt were each roughly 3% of GDP. Thus, total Mining was roughly 9% + 3% + 3% + 3%, or 18% of GDP.

$$\begin{aligned}\frac{\text{Copper}}{\text{GDP}} &= \left(\frac{\text{Copper}}{\text{Mining}}\right) \times \left(\frac{\text{Mining}}{\text{GDP}}\right) \\ &\approx (16\%) \times (18\%) = (0.16) \times (0.18) = 0.029 \approx 3\%\end{aligned}$$

The answer is (A).

Strategy Tip:

Write out equations with units to help yourself figure out what values you need.

3. **Statement II only.**

First statement: For each time period, the production of Borates, in the bottom chart, is given as 15% of that period’s Mining Industries’ Production. Each period has a *different* dollar figure for Mining Industries’ Production, therefore, Borate production is not the same in all of the periods. (For example, from 1981–1985, Borate production was 15% × $342.5, whereas from 1986–1990, Borate production was 15% ’ $326.8.) Statement I is therefore false.

Second statement: To test whether this is true, notice that the dollar values of Gold & Silver production were:

81–85: 20% × $342.5
86–90: 17% × $326.8
91–95: 20% × $310
96–00: 22% × $257.9
01–05: 24% × $205

You can eliminate two of these five choices without doing any arithmetic. The figure for 86–90 is clearly lower than that for 81–85, because 86–90 has a lower percentage (17% as opposed to 20%) times a lower dollar amount ($326.8 as opposed to $342.5). Along the same lines, you can see that 91–95 is lower than 81–85: the percentage is the same for both periods (20%), but for 91–95 that percentage is multiplied by a smaller dollar amount ($310 as opposed to $342.5). For the three remaining periods, use the calculator:

81–85: 20% × $342.5 = 68.50
96–00: 22% × $257.9 = 56.74
01–05: 24% × $205 = 49.20

These calculations show that 81–85 had the highest Gold & Silver production, so statement II is true.

Third statement: It concerns which period had the highest Gross Domestic Product (GDP). You clearly need to use the TOP chart, because it is the one that mentions GDP. However, the top chart is not enough, because it only gives information as *percentages*. To get dollar amounts for GDP, you need to connect the dollar amounts in the *bottom* chart with the percentages in the top chart.

One way to figure out GDP in 2001–2005 would be to focus on Salt. The top table gives a figure for Salt as a percentage of GDP (roughly 3.5%), and the bottom table allows you to figure out the dollar amount of Salt production (17% × $205 million). Substituting these figures into the following equation, you could solve for GDP:

$$\text{GDP} = \frac{\text{Salt}}{\left(\frac{\text{Salt}}{\text{GDP}}\right)}$$

$$\text{GDP} = \frac{0.17 \times \$205\text{ million}}{3.5\%}$$

However, performing this calculation, and similar calculations for other time periods, would be too much work. It is a good idea to focus on Borates instead, since Borates accounted for the *same* percentage (15%) of Mining Industries’ Production in each of the time periods shown. Consider the equation:

$$\text{GDP} = \frac{\text{Borates}}{\left(\frac{\text{Borates}}{\text{GDP}}\right)}$$

Compare 1981–1985 to 2001–2005. In 2001–2005, the numerator (Borates) of the fraction $\frac{\text{Borates}}{\left(\frac{\text{Borates}}{\text{GDP}}\right)}$ was *lower* than in any other time period—it was 15 percent of the *lowest* value ($205 million) for Mining Industries' Production. On the other hand, 2001–2005 saw the denominator $\left(\frac{\text{Borates}}{\text{GDP}}\right)$ of our large fraction assume its *highest* value (≈ 3%), because the top chart shows that 2001–2005 was the only time frame in which Borate production was more than 3 percent of GDP. Thus, $\text{GDP} = \frac{(0.15) \times (205 \text{ million})}{0.03}$ for 2001–2005.

From 1981–1985, Borates were 15 percent of a much greater total ($342.5 million), but were a smaller percentage of GDP (≈ 1.5%). So $\text{GDP} = \frac{(0.15) \times (342.5 \text{ million})}{0.015}$. This fraction has a larger numerator and a smaller denominator, which will result in a higher total GDP. Statement III is therefore false.

Only the box for the second statement should be checked.

Strategy Tip:

On problems this difficult, it is essential to **avoid hard math** even with a calculator, and rely on approximation and bounding instead. Using the calculator is helpful, but always look for quicker ways to arrive at a conclusion.

Acknowledgments

As with most accomplishments, there were many people involved in the creation of the book you are holding. First and foremost is Zeke Vanderhoek, the founder of Manhattan Prep. Zeke was a lone tutor in New York when he started the company in 2000. Now, 18 years later, the company has instructors and offices nationwide and contributes to the studies and successes of thousands of GRE, GMAT, LSAT, and SAT students each year.

Our Manhattan Prep Strategy Guides are based on the continuing experiences of our instructors and students. We are particularly indebted to our instructors Stacey Koprince, Dave Mahler, Liz Ghini Moliski, Emily Meredith Sledge, and Tommy Wallach for their hard work on this edition. Dan McNaney and Cathy Huang provided their design expertise to make the books as user-friendly as possible, and Liz Krisher made sure all the moving pieces came together at just the right time. Beyond providing additions and edits for this book, Chris Ryan and Noah Teitelbaum continue to be the driving force behind all of our curriculum efforts. Their leadership is invaluable. Finally, thank you to all of the Manhattan Prep students who have provided input and feedback over the years. This book wouldn't be half of what it is without your voice.